Hedge Fund Fee Structure and Risk Exposure: Theory and Empirical Evidence

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Abstract

We solve in closed form the optimal investment strategy of an infinitely lived risk neutral hedge fund manager compensated by a management fee and a high water mark (HWM) contract. The fraction of asset under management allocated in equity is a convex increasing function of the distance to the HWM; this effect is enhanced by the size of the incentive fee rate. Frequently beating the HWM by a small amount is optimal as it mitigates the ratchet feature of the HWM. Data support our theoretical predictions: returns’ volatility and the time elapsed between hits are strongly related to the distance to the HWM.

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1 Introduction

As of the third quarter of 2018, the hedge fund industry was managing an estimated wealth of $3.065 trillion compared to an estimated $2.137 trillion before the 2008 financial crisis. Hedge funds are exempt from many regulatory rules to which the financial industry in general must abide such as, for the case of the United States, the Investment Company Act of 1940, which is an extensive regulatory code\(^1\). Increased investment opportunities allow managers to implement more flexible strategies and make full use of their talent to deliver profits, which should entitle them to high rewards. Hedge fund managers’ compensation exhibits two key features: a management fee, usually, a fraction of the assets under management (AUM) and an incentive fee, typically, a fraction of the fund profits is paid to the fund manager when profits exceed a target value, the high water mark (HWM). The incentive fee intends to align the interests of managers with those of investors: the HWM aims at ensuring that the fund managers’ reward is commensurate to performances while keeping track of the history of the fund profits, more specifically, its all time high value ever reached. It can be adjusted to incorporate a minimum return required on the fund, for instance to account for inflation as well as each time some fund inflows or outflows take place. The HWM is a specific feature of the hedge fund industry\(^2\): The standard remuneration for hedge funds is so called the “2/20-rule”, 2% per year of the AUM and 20% of the profits whenever the fund is above its HWM\(^3\).

In this paper, we study the optimal investment strategy chosen by an infinite lived risk neutral fund manager who earns a management fee as well as an incentive fee as previously described. Our baseline model is essentially an extension of the work of Panageas and Westerfield (2009) that accounts for the impact of a management fee rate. In practice, the management fee plays an important role: it will be very difficult for a fund to operate on a daily basis by only relying on bumpy and infrequent hikes in income earned when the HWM is hit\(^4\). Our motivation is twofold. First we are interested in the combined effect of a management and an incentive fee on the level of risk exposure of the fund, in particular how the latter varies with the distance to the HWM, and, on the size of the incentive fee

\(^1\)The Security and Exchange Commission (SEC) limits the use of short sales, derivative contracts, asset concentration for mutual funds in an attempt to protect investors from high risk investment strategies.

\(^2\)Eighty four percent of the funds tracked by the HFR Database have a HWM provision.

\(^3\)Fees have been coming down; the fee structure of the typical fund in our data is 1.479 % / 18.309 % and 20 percent of profits in excess of the HWM. Other common fund fee structures include: “1/50-rule”.

\(^4\)Lan, Wang and Yang (2013) point out that management fees contribute to the majority of total management compensation and report calibrations in which three quarters of the fund manager’s compensation are due to management fees. Calibrations performed in Goetzmann, Ingersoll and Ross (2003) reveal that total lifetime fees could represent up to 30 percent of the value of the AUM, with nearly two third of the cost being due to the management fee.
earned each time the HWM is surpassed. Second, using data from the Hedge Fund Research database of monthly observations of returns of both active and liquidated hedge funds over the 1976-2013 period, we test the empirical implications of our model.

Our main result is to show an increasing convex relationship between risk exposure of the AUM and the distance to the HWM. The farther away the fund drifts from the HWM, the smaller the management fee earned and the present value of the incentive fee, which exacerbates the risk taking behavior by the manager. The economic magnitude of the effect in the data is large: being 20% underwater is associated with 192 bps increase in the standard deviation of the next 12-month returns.

Both theoretical and empirical findings coincide to assert that an increase in the incentive rate enhances the positive relationship between risk exposure and distance to the HWM as the fund manager looks forward and intends to mitigate the ratchet effect. The estimate for the effect is economically relevant: the slope of risk to distance is 20% higher for a fund with a 20% incentive fee compared to one with a 15% charge. The intuition for this result is straightforward: ceteris paribus, when the incentive fee is large, the fund manager chooses a small step strategy that consists in reducing the fund volatility when approaching the HWM. As a result, the HWM is pushed up by a small amount but more often.

Numerical simulations reveal that an increase in the management fee rate triggers a more aggressive investment strategy. The intuition is the following: A higher management fee rate allows the fund manager to insure part of her compensation, which fosters a risk seeking behavior. The data corroborates this intuition. Then, we compute the expected time until the HWM is reached as a function of the distance to the HWM. It is found to be increasing, validating the intuition that the farther away of the HWM, the longer it takes to collect the next incentive fee. The extent to which the fund surpasses its HWM is smaller and the frequency with which it happens higher when the fund is close to the HWM, as it will be optimal to beat the high-water mark frequently by a small amount to mitigate the ratchet effect. The data supports this prediction.

The closest paper to our baseline model is Drechsler (2014) who extends Panageas and Westerfield (2009) setting by granting the fund manager the option to walk away. A solution is derived by assuming that the management fee is proportional to the HWM rather than being proportional to the value of the AUM. Under such an approximation, the optimal investment strategy has an identical pattern to that in absence of management fee. Our model differs from Dreschler (2014) in several dimensions: first of all, a management fee is seen from the client point of view as a loss, so the HWM is not adjusted downward. Second, we derive an exact analytical solution that allows us to investigate the impact of the management and incentive fee rates on the optimal investment strategy, the relative
size of earned fees by the fund manager and the frequency of consecutive hits of the HWM.

At the theoretical level, Goetzmann, Ingersoll and Ross (2003) use a contingent claim approach to derive the implied market value of the lifetime fees earned by a manager who has no discretion on portfolio allocations. In our paper, the optimal investment strategy is endogenous. We compute the market values of the two sources of the fund manager’s revenues, seen as a couple of claims whose payoffs are the management (for the first claim) and incentive fees (for the second claim) collected overtime. Janeček and Sirbu (2012) examine the case of an agent whose optimally chooses to consume and invest but must pay a fee to a fund manager whenever her asset holdings exceed a high-watermark. Guasoni and Oblój (2015) study the case of a CRRA preference fund manager who maximizes the long term certainty equivalent of the cumulated fees paid by the fund. The fee structure is identical to the one considered in our paper; earned fees are required to be invested in the riskfree money market account. The optimal investment strategy consists in allocating a constant fraction of the AUM in the risky asset, whose level depends on the management fee rate and fund manager’s risk aversion5.

This paper is also related to a growing literature on portfolio allocations under wealth performance relative to an exogenous benchmark such as in Browne (1999) and Tepla (2001) or subject to growth objectives required by the decision maker as in Hellwig (2004). In Carpenter (2000), the fund manager is compensated with a call option on the wealth process with a benchmark index as strike price. As in Ross (2004), the author shows that the option compensation does not necessarily lead to more risk seeking. In a similar setting, Buraschi, Kosowski, and Sritrakul (2014) obtain that investment in the risky asset decreases as the AUM approaches the HWM and exceeds the latter up to an extent after which it starts to increase.

An extension to the baseline model introduces an early termination by the investor should the AUM experience a sufficiently large drawdown, measured as fraction of the HWM6. Essentially, the presence of the liquidation floor introduces a put option component into the optimal investment strategy in order to restrain and hedge drawdowns of the AUM. Although significantly more complex than the baseline model, we are still able to solve the problem in closed form. We find that the impact on the optimal investment strategy is significant. The closer the AUM gets to the minimum floor, the higher the fund manager’s lifetime risk aversion, which curbs down risk exposure. Our empirical findings are

5 In a companion paper, Guasoni and Wang (2015) analyze the optimal investment strategies of a risk averse fund manager in charge of either a mutual fund (no HWM) or a hedge fund (with HWM) who is free to invest her own wealth in equity. Investing a constant fraction of the AUM in the stock is still optimal whereas the fund manager shall invest her cumulative earned fees into the riskfree asset and a constant fraction of the rest of her own asset into the stock.

6 Grossman and Zhou (1993) argue that when “leverage is used extensively, […] an essential aspect of the evaluation of investment managers and their strategies is the extent to which large drawdowns occur. It is not unusual for the managers to be fired subsequent to achieving a large drawdown (typically above 25 percent).”
consistent with this latter result: risk in funds with high probability of being liquidated is lower and increases less rapidly with the distance to HWM.

There exists an extensive empirical literature regarding the interplay between compensation contracts with convex payoffs and risk taking behavior that focuses on hedge funds. Results are not always consistent. Brown, Goetzmann and Park (2001) do not find evidence of excessive risk taking behavior when below the HWM. In fact, they argue that fund managers are mainly concerned about their reputation and future in the industry. Studying returns of more than 900 funds over the period 1988-1995, Ackermann, McEnally and Ravenscraft (1999) report that the fear of excess risk taking behavior triggered by the incentive fee seems unfounded. Nevertheless, the incentive fee is a key variable at explaining a risk-adjusted returns (measured by the Sharpe ratio). They also establish a strong positive link between the management fee and the volatility of returns (agency problem). Elton, Gruber, and Blake (2003) report that mutual funds with incentive-fees raise risk exposure after poor performance. Aragon and Nanda (2012) find that funds that perform poorly in absolute terms, relative to others and relative to their HWM tend to increase risk. The effect is stronger with funds with incentive pay but tends to disappear for funds that have HWM provision. The threat of losing AUMs or being liquidated appears to be relevant and even change the direction of the effects. Agarwal et.al (2002) document a convex flow-performance relation and suggest that, in addition to explicit incentives, managers also face significant implicit incentives to risk taking. Since funds charging higher incentive fees exhibit higher money flows, this would induce those to moderate their risk taking behavior. Buraschi et al. (2012) show that funds that have experienced large deviations from their HWM actually reduce volatility.

In our paper, by studying the data in light of a more structured theoretical model we can uncover the mechanism through which the various fees affect risk taking behavior of fund managers. This also allows to test ancillary implications such as the frequency of the hits, the extent to which the fund surpasses the HWM, among others. We are able to explore in a unified empirical framework the role of distance to HWM, relative and absolute performance, the structure of fees, the frequency and extent of HWM surpasses, and the impact of the threat of liquidation. Our results are robust and we can accommodate some seemingly inconsistent previous results.

The paper is organized as follows. Section 2 describes the baseline setting and presents a heuristic derivation of the optimal solution and its properties. Section 3 presents a verification theorem that formally proves the validity of the heuristic solution. In section 4, we discuss an extension of the baseline model that introduces the possibility of early termination of the fund by investors. Section 5
presents the empirical evidence. Section 6 concludes. All proofs are contained in the appendix.

2 Baseline Model

Time is continuous. An infinitely lived risk neutral hedge fund manager has to optimally allocate the AUM of her fund between a risk-free bond and a risky asset (index) in order to maximize her lifetime compensation. We assume that the fund manager does not have a private account.

2.1 Financial Markets

There are two securities available in the financial market: a risk-free bond whose price $B$ evolves according to

$$dB_t = \hat{r}B_t dt,$$

where $\hat{r}$ is the interest rate and a stock index whose price $S$ follows a geometric Brownian motion

$$dS_t = S_t(\hat{\mu}dt + \hat{\sigma}dw_t),$$

with $S_0 > 0$, where $dw_t$ is the increment of a standard Wiener process $w$. Finally, let $\pi$ denote the fraction of the AUM invested in the risky asset. In order to have a well-defined problem, we require $\int_0^T \pi_t^2 dt < \infty$, for all $T > 0$.

2.1.1 AUM Dynamics and High Watermark

Let $c_I > 0$ denote the (constant) withdrawal rate by the investor from the fund. At each period, a management fee is charged that is proportional to the AUM with rate $c_F > 0$; usually $c_F$ is around 2%. Thus, total withdrawals take place at (continuous) rate $c = c_F + c_I > 0$.

For $\lambda > 0$, define

$$\hat{M}_t = \sup_{0 \leq s \leq t} \max \{\hat{M}_0 e^{(\lambda - c_I)t}, \hat{W}_s e^{(\lambda - c_I)(t-s)}\},$$

$\lambda$ is the (minimum) growth rate of the returns required by the investor. Then set $\hat{W}_t = \hat{W}_t e^{-(\lambda - c_I)t}$ and $M_t = \hat{M}_t e^{-(\lambda - c_I)t}$. Observe that $M_t = \sup_{0 \leq s \leq t} \{M_0, W_s; 0 \leq s \leq t\}$ and

$$d\hat{M}_t = (\lambda - c_I)\hat{M}_t dt + e^{(\lambda - c_I)t}dM_t.$$

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7This assumption is key for our results. Panageas and Westerfield (2009) show that within a finite horizon framework the volatility of the fund becomes unbounded as the terminal date approaches.
As long as \( dM_t = 0 \), the HWM \( \hat{M} \) is growing at rate \( \lambda - c_I \). Whenever \( dM_t > 0 \), the fund manager earns an incentive fee\(^8\) equal to \( ke^{(\lambda-c_I)t}dM_t \), with \( k \in (0,1) \). Another important force driving the dynamics of the fund is attracting new money\(^9\). Following Lan, Wang and Yang (2013), we assume that whenever \( dM_t > 0 \), this triggers some new money inflows \( d\tilde{I}_t \) that are proportional to the fund proceeds in excess of the HWM, i.e.,

\[
d\tilde{I}_t = ie^{(\lambda-c_I)t}dM_t,
\]

where \( i > 0 \). Significantly beating the HWM signals the hedge fund manager’s asset management skills. Finally, the dynamics of the AUM process are given by

\[
d\hat{W}_t = (\tilde{r} - c)\hat{W}_tdt + \pi_t(\tilde{\mu} - \tilde{r})\hat{W}_tdt + \pi_t\sigma\hat{W}_tdw_t - (k-i)e^{(\lambda-c_I)t}dM_t,
\]

so that the dynamics of discounted AUM process \( W \) are given by

\[
dW_t = (r - c_F)W_tdt + \pi_t(\mu - r)W_tdt + \pi_t\sigma W_tdw_t - (k-i)dM_t, \tag{1}
\]

with \( r = \tilde{r} - \lambda \), \( \mu = \tilde{\mu} - \lambda \) and \( \sigma = \tilde{\sigma} \).

### 2.2 Hedge Fund Optimization Problem

A risk neutral hedge fund manager maximizes the expected value of her management and incentive fees

\[
\max_{\pi} E_0 \left[ \int_0^\infty e^{-\theta + \delta}t (c_F\hat{W}_tdt + e^{(\lambda-c_I)t}dM_t) \right]
\]

\[
d\hat{W}_t = (\tilde{r} - c)\hat{W}_tdt + \pi_t(\tilde{\mu} - \tilde{r})\hat{W}_tdt + \pi_t\tilde{\sigma}\hat{W}_tdw_t - (k-i)e^{\lambda t}dM_t,
\]

with \( 0 < W_0 \leq M_0 \) given. Equivalently the fund manager’s objective function can be written

\[
F(W_0, M_0) = \max_{\pi} E_0 \left[ \int_0^\infty e^{-\theta + \delta}t (c_FW_tdt + kdM_t) \right] \tag{P}
\]

s.t. \( dW_t = (r - c_F)W_tdt + (\mu - r)\pi_tW_tdt + \sigma\pi_t W_tdw_t - (k-i)dM_t, \)

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\(^8\)Some authors, see for instance Hodder and Jackwerth (2007) assume that the fund manager is periodically evaluated at a discrete time, such as the end of the year. An incentive fee is paid at the that time should the AUM be above the HWM.

\(^9\)Asset growth remains the focus of a majority of managers, in particular, for mid-size fund managers whereas the largest managers have already their growth strategy in place and prefer to concentrate on talent (EY 2015).
with \( \theta = \hat{\theta} - \lambda + c_I \) is the (adjusted) manager’s subjective time discount rate. We also impose a transversality condition
\[
\lim_{T \to \infty} E_t \left[ e^{-(\theta+\delta)(T+t)} F(W_{t+T}, M_{t+T}) \right] = 0. \tag{2}
\]
Termination is exogenous and follows a Poisson process with constant intensity \( \delta \) that is independent of the fund returns. We assume that \( \theta + \delta > 0 \).

2.2.1 Conditions for a Well-Defined Problem

Let \( \beta_1 \) and \( \beta_2 \) be respectively the positive and negative roots of the quadratic \( Q \) with
\[
Q(y) = \frac{1}{2} \left( \frac{\mu - r}{\sigma^2} \right)^2 y^2 + (\theta + \delta - r + c_F - \frac{1}{2} \left( \frac{\mu - r}{\sigma^2} \right)) y - (\theta + \delta),
\]
We make the following assumptions:

A1. Growth condition: \( \beta_2(k - i) + 1 + i < 0 \).

Condition A1 can be seen as a non-Ponzi game or transversality condition that ensures that \( F(W, M) \) is finite (see Appendix).

A2. \( \mu \neq r \).

A3. \( r > c_F \).

Condition A3 guarantees that investing continuously (and infinitesimally) all the AUM into the riskless asset increases the HWM. Worth observing is the fact that \( r > c_F \) implies that \( \beta_1 > 1 \).

Finally, whenever \( W_t \geq M_t \) we have
\[
F(W_t, M_t) = kdM_t + F(W_t + (i - k)dM_t, M_t + (1 + i)dM_t)).
\]
Taking a Taylor expansion and letting \( dM_t \) goes to zero leads to
\[
(k - i)F_1(M_t, M_t) = k + (1 + i)F_2(M_t, M_t).
\]

In what follows, we provide a heuristic derivation of the fund manager optimization problem (P).

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10When \( \mu = r \) optimization problem (P) is ill-posed as the optimal investment strategy \( \pi^* \) is unbounded; for more details, see Panageas and Westerfield (2009).

11For the baseline model, this condition is not required for the existence of optimization problem (P) but nevertheless simplifies the analysis and the exposition of the results. In the general case, results can be derived relying on the confluent hypergeometric functions and their properties. However, the condition \( r \geq c_F \) needs to hold so for the extension of the baseline model, the optimization problem (P') is well-defined.
We construct a solution of the Hamilton Jacobi Belman (HJB) equation associated to the optimization problem that satisfies the appropriate boundary conditions at $W = 0$ and $W = M$. A verification theorem is presented in the Appendix to formally establish the optimality of the proposed solution.

2.2.2 Value Function

Due to the homogeneity of degree 1 in variables $(W, M)$ of the hedge fund manager’ compensation contract and the wealth dynamics equation (1), $F$ is homogeneous of degree 1 so we can write

$$F(W, M) = Mf(u),$$

where $u = \frac{W}{M}$, for some smooth function $f$. In the rest of the paper, we shall refer to $u$ as “the distance to the high water mark”. Also note that $F_1 \geq 0$, so $f' \geq 0$. Clearly, we shall have $f(0) = 0$ and the boundary condition at $u = 1$ is

$$(1 + k)f'(1) = k + (1 + i)f(1).$$

(3)

For $u < 1$, the reduced Hamilton Jacobi Bellman (HJB) equation satisfied by $f$ is:

$$(\theta + \delta)f(u) = c_F u + (r - c_F)uf'(u) + \max \pi(\mu - r)uf'(u) + \frac{\sigma^2}{2}u^2f''(u).$$

(4)

Assuming that $f$ is a (strictly) concave function (we prove this claim in the sequel), it follows that

$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{f'(u)}{uf''(u)},$$

and the reduced HJB is

$$(\theta + \delta)f(u) = c_F u + (r - c_F)uf'(u) - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{(f'(u))^2}{f''(u)}.$$ 

(5)

2.2.3 Dual Value Function

We introduce the dual value function $J$ as the Legendre transform of value function $f$. Primal and dual variables $(u, x)$ satisfy

$$x = f'(u) \text{ and } u = -J'(x),$$

9
and $f(u) = J(x) - xJ'(x)$. Set $\Lambda = -\frac{c_F \beta_1 \beta_2}{\theta + \delta} > 0$. The dual (reduced) HJB satisfies:

$$x^2 J''(x) + [(1 - \beta_1 - \beta_2)x - \Lambda]J'(x) + \beta_1 \beta_2 J(x) = 0.$$  \hspace{1cm} (6)

The general solution of (6) is given by:

$$J(x) = K_1 H_1(x) + K_2 H_2(x),$$

with

$$H_1(x) = x^{\beta_2} e^x U(1 + \beta_1, 1 + \beta_1 - \beta_2, \frac{\Lambda}{x}),$$

$$H_2(x) = x^{\beta_2} M(-\beta_2, 1 + \beta_1 - \beta_2, -\frac{\Lambda}{x}),$$

where $M$ and $U$ are respectively the Kummer and the Tricomi functions\textsuperscript{12}. Some useful properties of functions $H_1$ and $H_2$ are provided in the Appendix.

We are looking for a solution of (6) $J$ defined on some interval $I \subseteq \mathbb{R}_+$ such that: (i) $J$ is non-negative on $I$ (see Appendix), (ii) $J'$ is negative on $I$, (iii) $J''$ is positive on $I$ and, (iv) at one extremity of interval $I$, both $J$ and $J'$ are equal to zero.

**Proposition 1** The reduced dual value function $J$ is defined on the interval $[x^*, \infty)$, is decreasing and strictly convex and is given

$$J(x) = -\frac{H_2(x)}{H_2'(x^*)},$$

where $x^* > 1$ is uniquely defined by $(k - i)x^* - k + (1 + i)\frac{H_2(x^*)}{H_2'(x^*)} = 0$.

**Proof.** See the Appendix. \hfill \blacksquare

The strict convexity of $J$ implies the strict concavity of $f$, so the interior solution for maximization problem in (4) is justified. The definition of $x^*$ is implied by condition (3). The next proposition summarizes the properties of the reduced value function $f$.

**Proposition 2** The value function $F$ is homogeneous of degree 1, strictly increasing in $W$ and $M$ and strictly concave in $W$ and in $M$. For $(c_F, c_I, i)$ given, if $k_1 < k_2$, then for all $u \in [0, 1]$, we have $f_1(u) > f_2(u)$. For $(c, k)$ given, if $i_1 < i_2$, then for all $u \in [0, 1]$, we have $f_1(u) < f_2(u)$. For $(k, i, c_I)$ given, if $c_{F_1} < c_{F_2}$, then for all $u \in [0, 1]$, we have $f_1(u) > f_2(u)$. Finally, a representation of the

\textsuperscript{12}Representations of the Kummer and the Tricomi functions are provided in the Appendix. These functions can be calculated in most mathematical software packages.
(optimal) reduced wealth process $u$ inside the open interval $(0, 1)$ is given by

$$u_t = -J'(x_t),$$

where

$$x_t = x_0 + \int_0^t ((\theta + \delta - r + c_F)x_s - c_F) ds - \frac{\mu - r}{\sigma} \int_0^t x_s dw_s,$$

(7)

with $x_0 > x^*$ satisfying $u_0 = -\frac{H'(x_0)}{H'(x^*)}$. Process $x$ is mean reverting if and only if $\beta_1 + \beta_2 < 1$.

**Proof.** See the Appendix. ■

The higher either the management or the incentive fee rate, the lower the lifetime manager compensation as a high fee rate thwarts the growth of the AUM. Overall, this effect overcomes the positive effect for the manager of collecting a larger fraction of the AUM as well as a larger fraction of the incentive reward. In the presence of a HWM, a higher management fee rate reduces all the more the AUM, making it harder to beat the target.

When $c_F = 0$, the reduced wealth process can be represented as a function of a geometric Brownian motion. In presence of a management fee $c_F > 0$, a similar representation is possible but the drift of process $x$ is now of the affine type. In the sequel, we make use of this representation to compute the frequency of hits of the HWM.

### 2.2.4 Baseline Parameter Set

We calibrate the baseline model for the following values of the model parameters: $\theta + \delta = 0.16$, $\mu - c_I = 0.07$, $r - c_I = 0.03$, $\sigma = 0.25$, $c_F = 2\%$, $k = 20\%$ and $i = 0\%$. One can check that under this set of parameters, condition A3 is indeed satisfied.

Unless specified otherwise, all the simulations are performed for this set of parameters and we shall investigate the quantitative impact of parameters $c_F$ and $k$.

We now examine the fund manager compensation decomposition between, on the one hand earned fees for managing the fund and, on the other hand earned fees based on performance.
2.3 Fund Manager Compensation Decomposition

Let

\[ F_c(W, M) = E_0 \left[ \int_0^\infty c_F e^{-(\theta + \delta)t} W_t dt \right] \]

\[ F_k(W, M) = E_0 \left[ \int_0^\infty k e^{-(\theta + \delta)t} dM_t \right] , \]

be the present value of the management and incentive fees respectively. By homogeneity, we can write

\[ F_c(W, M) = Mf_c(u) \]

\[ F_k(W, M) = Mf_k(u) \]

In what follows, we focus on the derivation of function \( f_k \). Function \( f_k \) satisfies the same boundary conditions than function \( f \) at \( u = 0 \) and \( u = 1 \). Recall that \( u = -J'(x) \) and define function \( g_k \) such that

\[ g_k(x) \triangleq f_k(-J'(x)) \].

For \( x > x^* \), function \( g_k \) satisfies the following HJB

\[ (\theta + \delta)g_k(x) = [(\theta + \delta - r + c_F)x - c_F]g_k'(x) + \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} x^2 g_k''(x), \]

and note that \( \lim_{x \to \infty} g_k = 0 \).

**Proposition 3** The present value of the incentive fee \( F_k \) is increasing and strictly concave in \( W \) and is given by

\[ F_k(W, M) = A_k M \left[ f(u) - uf'(u) \right] , \]

where

\[ A_k = \frac{k}{-(k + 1) \frac{H_2(x^*)}{H_1(x^*)} + (1 + i) \frac{H_2(x^*)}{H_1(x^*)}} > 0. \]

**Proof.** See the Appendix. ■

Interestingly, the present value of the incentive fees is proportional to the marginal value of the total earned fee with respect to the HWM. Next, we look at the share of the incentive fee in the total compensation, i.e. the ratio

\[ \frac{f_k(u)}{f(u)} = A_k (1 - \frac{uf'(u)}{f(u)}) . \]

In Appendix D1, we establish that \( \lim_{u \to 0} \frac{f_k(u)}{f(u)} = \frac{A_k}{1 + i} \) and \( \frac{f_k}{f} \) is increasing in \( u \). This is fairly intuitive as the closer to the HWM, the larger the option value associated with surpassing the HWM. Numerical simulations are performed for the baseline parameter case with an exception for parameter \( c_F = 3\% \).
Figure 1 depicts the share of the total compensation due to the incentive fee as a function of the distance to the HWM. It reveals that the fund manager derives most of her revenues from the incentive fee part and, the closer to the HWM, the larger the fraction of the fund manager’s compensation due to the incentive fee. For instance, at \( u = 0.8 \), the management fee only accounts for 15% of the total compensation. Also worth noticing is the concave shape of the curve: as the AUM moves away from the HWM, the probability of surpassing the HWM drops sharply (convexity effect) while the loss associated with a reduction in the management fee only decreases in a linear manner. These results are in sharp contrast with the results obtained in Lan, Wang and Yang (2013) where the fund manager being concerned with downside liquidation risk selects a (much) more conservative leverage level than in our setting.

To understand the magnitude of the incentive fee, one needs to investigate the mechanism between risk taking and exceeding the HWM, in particular the impact of portfolio holdings in the neighborhood of the HWM. As derived in Grossman and Zhou (1993)

\[
E_t [M_{t+h} - M_t \mid W_t = M_t] = \sqrt{\frac{2}{\pi} \left| \pi_t^* \right| M_t \sqrt{h}} + O(h). \tag{8}
\]

The expected increase in the HWM over the time interval \([t, t+h]\) is proportional to the fraction \(\pi^*\) of the AUM invested in equity and \(\sqrt{h}\). As \(\sqrt{h}\) dominates \(h\), clearly an increase in the HWM has a significant (instantaneous) impact on the manager’s compensation. In section 5, we report that on
average, the HWM is surpassed by a margin of 12.7%.

2.4 Optimal Investment Strategy

**Proposition 4** The fraction of AUM invested in the risky asset $\pi^*$ is decreasing in the ratio $\frac{\mu - r}{\sigma^2}$ with $\pi^* \leq \frac{\mu - r}{\sigma^2}(1 - \beta_2)$.

**Proof.** See the Appendix. ■

The interpretation of proposition 4 is quite intuitive. Recall that there is no penalty for depleting the fund: the deeper out of the money the incentive contract of the fund manager, the higher the risk exposure in order to hasten the appreciation of the AUM. Even though the fund manager is assumed to be risk neutral, the position taken in the risky asset is finite: the ratio $R_R(u) = \frac{-uf''(u)}{f'(u)}$ can be interpreted as a measure of her lifetime relative risk aversion; $R_R$ is increasing in $u$, i.e. the fund manager’s lifetime utility exhibits increasing relative risk aversion (IRRA). This is in line with an extensive literature that argues that the convex payoff structure in hedge fund fees creates incentives for the manager to take excess risk and, in particular when the contract reward is deep out of the money (see Carpenter (2000) and Ross (2004)). Interestingly, the maximum value of the fraction of the AUM invested in the risky asset admits an upper bound as $u$ approaches 0 that is equal to $\frac{\mu - r}{\sigma^2}(1 - \beta_2)$ and independent of $c_F$. Recall that when $c_F = 0$, we always have $\pi^* = \frac{\mu - r}{\sigma^2}(1 - \beta_2)$ (Panageas and Westefield (2009)). Thus, we can claim that the optimal investment strategy does exhibit excess risk behavior; in fact, it corresponds to the case where the HWM is always seen as “infinitely” far away. Finally, note that the minimum value of the fraction of wealth is reached when the HWM is hit and is equal to $-\frac{\mu - r}{\sigma^2}x^H_H'(x^*)$. Unlike when $c_F = 0$, this ratio depends on the incentive fee rate $k$.

We now examine the impact of the fee structure on the optimal investment strategy.

2.4.1 Impact of the Incentive Fee Rate

**Proposition 5** For $(c, i)$ given, if $k_1 < k_2$, then for all $u \in [0, 1]$, we have $\pi^*_1(u) \geq \pi^*_2(u)$. For $(c, k)$ given, if $i_1 < i_2$, then for all $u \in [0, 1]$, we have $\pi^*_1(u) \leq \pi^*_2(u)$.

**Proof.** See the Appendix. ■

The manager has all the more incentives to inflate the fund volatility near the HWM as the incentive fee rate $k$ is small. This reflects the intertemporal trade-off faced by the manager between

---

13Drechsler (2014) obtains a similar result when the outside payoff of the fund manager is sufficiently large with respect to the continuation value at the liquidation threshold, even though no management fee is charged. In our setting (baseline case), there is no liquidation threat: excess risk behavior is solely induced by the presence of the management fee.
(i) her short term objective, namely earning a (high) incentive fee whenever she beats the HWM, and
(ii) her long term objective, i.e. the continuation value. To illustrate this intertemporal trade-off, assume that the HWM is surpassed by a margin of $q$ percent. The fund manager receives an incentive fee equal to $kqM$, the AUM level is now $M(1 + (1 + i - k)q)$ and the new HWM level is $M(1 + (1 + i)q)$. This implies that the AUM will have to grow by $\frac{kq}{1+(1+i-k)q}$ percent to hit again the HWM; this ratio is indeed increasing in $q$. Thus, one of the main predictions of the model is that the optimal strategy consists in often beating the HWM by small amounts rather than beating the HWM by large amounts infrequently. Recall that the model assumes that the fund has no incentive to leave the fund so the impact of the continuation value is large; in practice, fund managers may have short term concerns (if periodically evaluated for instance) or personal motives to quit, which may provide them with additional reasons to increase the risk exposure of the AUM, in particular if $k$ is large.

Finally, the larger parameter $i$, the larger the inflow of new money that is, by assumption, proportional to the performance of the fund manager at exceeding the HWM, which provides additional incentives to take risk.

![Figure 2: Fraction of the AUM invested in stocks as a function of $\frac{W}{M}$](image)

Worth mentioning is the fact that the slope of the curves gets steeper as parameter $k$ raises. This indicates that the excess risk taking behavior is all the more significant as the incentive fee rate becomes larger.
2.4.2 Impact of the Management Fee Rate

![Fraction of the AUM invested in stocks as a function of \( \frac{W}{M} \)](image)

Figure 3: Fraction of the AUM invested in stocks as a function of \( \frac{W}{M} \).

Numerical simulations suggest that the fund manager’s appetite for risk is growing as the management fee rate increases. The management fee acts as an insurance: ceteris paribus, a fund manager earning a hefty management fee is more keen on increasing risk exposure as her revenues are smoothed out across time. Similar to what we observed for the incentive fee, we note that the slope of the curves gets steeper as parameter \( c_F \) raises. This indicates that the excess risk taking behavior is exacerbated as the management fee rate is large.

2.4.3 How Often is the HWM Hit?

Define the stopping time until next hit

\[
\tau = \inf_{t \geq 0} \{ u_\tau \geq 1, \ u_0 < 1, \text{ given} \},
\]

where \( u_0 = -\frac{H_2'(x_0)}{H_2(x_0)} \). Then, let us introduce the auxiliary function \( A \) with

\[
A(x) = \int_0^1 e^{-\frac{A}{x}(1-t)^{\beta_1+\beta_2}} \left[ 1 + \left( \frac{A(1-t)}{x} + \beta_1 + \beta_2 + 1 \right) \ln t \right] dt.
\]

In the Appendix, we show that \( A \) is an increasing function. Finally, we assume that \( \beta_1 + \beta_2 > 0 \) so that \( E[\tau] < \infty \).
Proposition 6  For an initial condition $u_0 < 1$, the expected time until the HWM is hit is given by

$$E[\tau] = \frac{1}{(\beta_1 + \beta_2)\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}} \left( \ln \frac{x_0}{x^*} + A(x^*) - A(x_0) \right),$$

where $u_0 = -\frac{H'(x_0)}{H'_2(x^*)}$.

Proof. See the Appendix. □

The expression obtained in Proposition 6 for $E[\tau]$ is a natural extension of the usual geometric Brownian motion (GMB) case. In absence of management fee $c_F = 0$, i.e., $\Lambda = 0$, the reduced wealth is a function of the GMB $x$; in this case, it is well known that the expression of $E[\tau]$ is given by

$$E[\tau] = \ln \frac{x_0}{x^*} \left( \frac{1}{\beta_1 + \beta_2} \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} \right).$$

Then, clearly, since starting farther away from the HWM delays the next hit and $u_0$ is a decreasing function of $x_0$, $E[\tau]$ must be increasing in $x_0$. We deduce that $E[\tau]$ is decreasing in $x^*$ and as $\frac{\partial x^*}{\partial k} < k$, we can claim that given $u_0$, the expected time until the HWM is hit is increasing in the incentive rate $k$. This does not come as a surprise: when $k$ raises the fund manager’s appetite for risk is tamed so on average, it takes more time to hit again the HWM. However, we cannot claim that as $k$ goes up, the time elapsed between hits shrinks since the probability to surpass the HWM by a large margin decreases, which in turn reduces the average time until the next hit.

Then, we note that $E[\tau]$ is not always finite as there may be a positive probability that the HWM is never hit. Condition $\beta_1 + \beta_2 > 0$ and condition A.1 are not always jointly met under the baseline parameter set as it is not easy to have the two conditions satisfied for a reasonable set of parameters. We perform simulations with $\theta + \delta = 0.01$, $\mu = 0.04$, $r = 0.03$, $\sigma = 0.4$, $c_F = 2\%$, $k = 40\%$. 

Figure 4 : Expected Time until Hitting the HWM as a function of $\frac{W}{M}$

Figure 4 depicts the expected time until the HWM is hit as a function the distance to the HWM for several values of the management fee rate. To infer the impact of the management fee on the expected time, we have to investigate the impact of $c_F$ on the law of motion of process $x$. Recall that process $x$ remains above $x^* > 1$ and only its drift $\mu_x = (\theta + \delta - r + c_F)x - c_F$ depends on parameter $c_F$. An increase in $c_F$ raises the drift of $x$, thwarting any decrease in process $x$. Therefore, we should expect that, being at any given distance to the HWM, the higher the management fee rate, the more infrequently the HWM is hit.

Figure 4 reveals that the expected time is increasing and convex as a function of the distance to the HWM making the HWM increasingly difficult to surpass as one moves away from it. This result is indeed in line with the previously developed intuition.

3 Extension to the Baseline Model

We extend our analysis by incorporating an endogenous termination threat of the fund. More specifically, we assume that the AUM cannot experience a too large drawdown otherwise clients will withdraw all their money. Grossman and Zhou (1993) argue that when “leverage is used extensively, [...] an essential aspect of the evaluation of investment managers and their strategies is the extent to which
large drawdowns occur. It is not unusual for the managers to be fired subsequent to achieving a large drawdown (typically above 25 percent).” In this section, we assume that the AUM must satisfy

\[ W_t \geq \alpha M_t \text{ for all } t \geq 0, \]

(9)

with \( \alpha \in [0, 1) \), otherwise the fund is liquidated with no severance paid. Goetzman, Ingersoll and Ross (2003) and Lan, Wang and Yang (2013) use a similar termination condition. We would like to emphasize that condition (9) differs from the one imposed in the two aforementioned papers as at \( W_t = \alpha M_t \), liquidation does not take place. In fact, we shall see that it is never optimal to liquidate the fund, which has interesting implications on the optimal investment strategy.

Define stopping time

\[ \tau_L = \inf_{t \geq 0} \{ W_t < \alpha M_t \}, \]

so that the fund manager’s optimization problem now is

\[ F(W_0, M_0) = \max_{\pi} E_0 \left[ \int_{\tau_L \wedge \infty} e^{-(\theta+\delta)t} (c_F W_t dt + k dM_t) \right], \]

(P’)

subject to (1).

We expect several implications on the optimal investment strategy. First, when the AUM approaches its termination floor, the fund manager’s risk aversion should rise, which will curb her position in risky asset, in sharp contrast with the baseline model. Second, the manager has now additional reason to mitigate the growth of the HWM since the ratchet effect is now twofold, at the lower \( u = \alpha \) and at the upper level \( u = 1 \).

First, we examine the special case where no management fee is charged \( c_F = 0 \).

### 3.1 No Management Fee \( c_F = 0 \)

#### 3.1.1 Value Function

Let \( f_\alpha \) denote the (reduced) value function. For all \( u \in [0, 1) \), \( f_\alpha \) satisfies the following HJB

\[ (\theta + \delta) f_\alpha (u) = ru f'_\alpha (u) - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{(f'_\alpha (u))^2}{f''_\alpha (u)}. \]

(10)

The general solution \( J_\alpha \) of the associated dual HJB is given by \( J_\alpha (x) = K_1 x^{\beta_1} + K_2 x^{\beta_2} \), so that

\[ u = -\beta_1 K_1 x^{\beta_1 - 1} - \beta_2 K_2 x^{\beta_2 - 1}. \]

(11)
Denote $x_\alpha^* = f'_\alpha(1)$ and $x_{\alpha^*} = f'_\alpha(\alpha)$. At $u = \alpha$, in order not to violate the drawdown constraint with some positive probability in the future, all the AUM must be invested in the bond. This implies that we must have $J''_\alpha(x_{\alpha^*}^*) = 0$ or equivalently $\alpha x_{\alpha^*}^*(1 - \beta_1 - \beta_2) = \beta_1 \beta_2 J_\alpha(x_{\alpha^*}^*)$. Note that this condition is different from the one imposed by Lan, Wang and Yang (2013) - namely $f_\alpha(\alpha) = 0$ - as in our setting, $f_\alpha(\alpha) = J_\alpha(x_{\alpha^*}^*) + \alpha x_{\alpha^*}^* \neq 0$. In our setting, the fund is never terminated. In the Appendix, we show that if the condition $f_\alpha(\alpha) = 0$ were to be imposed instead of $\pi^*(\alpha) = 0$, a leverage constraint would be necessary to have a well-defined optimization problem.

In the Appendix, we show that $(K_1, K_2, x_\alpha^*, x_{\alpha^*}^*)$ is the unique solution of a non-linear system of four equations and $\imath = \frac{x_{\alpha^*}^*}{x_\alpha^*}$ in $(0, 1)$ is implicitly (and uniquely) defined by

$$\alpha [(1 - \beta_2) \imath^{\beta_1 - 1} + (\beta_1 - 1) \imath^{\beta_2 - 1}] = \beta_1 - \beta_2.$$ 

Set $\imath^* = \left[\frac{\beta_1 ((k-1)\beta_2 + 1+i)}{\alpha (1+i)(1-\beta_2)} \vee 0\right]^{1/(1-\beta_2)} \geq 0$. Since imposing a drawdown constraint limits the growth of the AUM, assumption A1. can be weakened as follows:

A1′. Growth condition: $\imath > \imath^*$. 

Full details on the existence and uniqueness of a solution of system $(S_0)$ are reported in the Appendix. Note that condition A3. is required in this setting to ensure that the wealth process bounces back after $W$ hits the floor $\alpha M$.

3.1.2 Optimal Investment Strategy

Proposition 7 The optimal fraction of the AUM $\pi_\alpha^*$ invested in the risky asset is increasing in $u$ and (uniformly) decreasing in the drawdown parameter $\alpha$.

Proof. See the Appendix. ■

Clearly, imposing a threat of termination triggered by a large drawdown of the AUM has a major impact on the optimal investment policy. First, the fear of termination overcomes the fund manager’s appetite for risk shifting. The lifetime manager’s relative risk aversion becomes decreasing in wealth (DRRA) and consequently the optimal fraction of the AUM invested in risky asset $\pi_\alpha^*$ is now increasing in $u$. Second, as shown in Figure 5, the level of risk exposure is also sharply affected: the lifetime manager’s relative risk aversion is (globally) magnified by the fear of liquidation and consequently the fraction of AUM invested in the risky asset is all the more (uniformly) reduced as the termination threat becomes more stringent (larger value for $\alpha$).
3.2 General Case

3.2.1 Value Function

Recall that the general solution for the reduced dual HJB (6) is given by

\[ J_\alpha(x) = K_1 H_1(x) + K_2 H_2(x). \]

It follows that

\[ u = -J'_\alpha(x) = -K_1 H'_1(x) - K_2 H'_2(x). \]

The boundary conditions at \( u = \alpha \) and \( u = 1 \) (resp. at \( x^*_\alpha \) and \( x^{**}_\alpha \)) lead to the following system (S)

\[
\begin{align*}
\alpha & = -K_1 H'_1(x^{**}_\alpha) - K_2 H'_2(x^{**}_\alpha) \\
0 & = K_1 H''_1(x^*_\alpha) + K_2 H''_2(x^*_\alpha) \\
1 & = -K_1 H'_1(x^*_\alpha) - K_2 H'_2(x^*_\alpha) \\
k - i x^*_\alpha & = \frac{k}{1+i} + K_1 H_1(x^*_\alpha) + K_2 H_2(x^*_\alpha).
\end{align*}
\]

**Proposition 8** For all \( \alpha \in (0,1) \), system (S) admits a unique solution with \( x^*_\alpha < x^{**}_\alpha \). The reduced dual value function \( J_\alpha \) defined on \([x^*_\alpha, x^{**}_\alpha]\) is decreasing and strictly convex and is given by

\[ J_\alpha(x) = K_1 H_1(x) + K_2 H_2(x), \]

with \( K_1 < 0 \) and \( K_2 > 0 \).

**Proof.** See the Appendix.

The optimal (reduced) wealth process is now given by

\[ u = -K_1 H'_1(x) - K_2 H'_2(x), \]

where process \( x \) is defined in (7). The first term encapsulates hedging motives to ensure that AUM does fall below the liquidation floor. Typically, this term has a put option flavor, similar to portfolio insurance strategies such as in Black and Perold (1992) and Tepla (2001). As in the baseline model, the second term regulates the growth rate of the AUM to mitigate the ratchet effect of the HWM.
3.2.2 Optimal Investment Strategy

The optimal investment strategy \( \pi^*_\alpha \) is given by

\[
\pi^*_\alpha = -\frac{\mu-r}{\sigma^2} x J'_{\alpha}(x), \quad x^*_\alpha \leq x \leq x^{**}_\alpha.
\]

As in the no management fee case, \( \pi^*_\alpha \) may be increasing in \( u \) but alternatively it can be hump-shaped in \( u \); a sufficient condition for the latter to occur is \( \frac{\partial \pi^*_\alpha}{\partial x} \bigg|_{x=x^*_\alpha} > 0 \), i.e.

\[
J''_{\alpha}(x^*_\alpha) + x^*_\alpha J'''_{\alpha}(x^*_\alpha) + x^*_\alpha (J''_{\alpha}(x^*_\alpha))^2 < 0.
\]

Even though we do not report further results, numerical simulations for the baseline case parameters reveal that this condition is always satisfied for sufficiently (very) small values of the drawdown coefficient \( \alpha \). The intuition for such a result is straightforward: The appetite for risk of the hedge fund manager is greatly reduced when the AUM approaches the termination floor but as soon as the AUM is moving away from the floor, the optimal investment strategy exhibits excessive risk taking behavior although at a decreasing rate.

![Graph](image)

Figure 5: Fraction of the AUM invested in stocks as a function of \( \frac{W}{M} (c_F = 0) \)
Figure 6: Fraction of the AUM invested in stocks as a function of $\frac{W}{M}$ ($c_F > 0$)

Comparing figure 5 and figure 6, we note that for sufficiently small values of $\alpha$, the risk exposure is (globally) larger when a management fee is charged whereas for sufficiently large values of $\alpha$, the converse is true.

3.2.3 Fund Manager Compensation Decomposition

We follow the same approach as in the baseline case; functions are now indexed with parameter $\alpha$. The only difference with respect to the baseline case is the boundary condition at $u = \alpha$ or equivalently at $x = x_\alpha^{**}$. Note that $g_{\alpha,c}(x) = -J'_\alpha(x)f'_{\alpha,c}(u)$, so it must be the case that $g_{\alpha,c}(x_\alpha^{**}) = 0$ as $J''_\alpha(x_\alpha^{**}) = 0$. Similarly, we must have $g'_{\alpha,k}(x_\alpha^{**}) = 0$. In particular, this last boundary condition implies that $f_{\alpha,k}(\alpha) = 0$ (but $f_{\alpha,c}(\alpha) > 0$). In the Appendix, we show that for all $x \in [x_\alpha^*, x_\alpha^{**}]$, we have

$$g_{\alpha,k}(x) = A_{k_1}H_1(x) + A_{k_2}H_2(x),$$

where the expressions of $(A_{k_1}, A_{k_2})$ are provided in the Appendix.

For the baseline set of parameters, numerical simulations (not reported here) reveal that the discounted value of cumulative incentive fee is quite low and accounts only for a small fraction of the manager total fee compensation. As risky investment is severely curbed down, so is the fund manager’s compensation share due to the incentive fee.
4 Empirical Evidence

To test the basic implications of the model, we look at data from the Hedge Fund Research database that comprises monthly observations of returns of both active and liquidated hedge funds over the 1976-2013 period. It also includes several fund characteristics, importantly regarding the compensation structure. We drop the observations with non-positive assets and age, and missing data for the basic variables. The sample excludes those funds that do not have a hurdle rate because of the complexity of computing such hurdle for different benchmarks. We look only at funds that do have incentive fees with high-water mark provisions. The final sample consists of 34,919 observations corresponding to 6,267 different funds. All the variables are winsorized at the 1% level.

As stressed by Joenvääri et al. (2016), the relevance of different potential biases vary across datasets - BarclayHedge, TASS, HFR, EurekaHedge, and Morningstar, in particular – and may alter some conclusions on the performance of funds. We feel, however, that the problem is of less importance when one explores volatility since this is not as salient as returns. For instance, if agents observe or care more about returns than the risks associated to them, funds will not self-select or misreport as much on this variable. Similarly, backfilling and survivorship biases will matter less if they are more related to returns than to volatility.

Risk-taking is not observable and thus has to be estimated. We measure the degree of risk that each fund takes with the volatility of realized monthly returns for the 12 months that follow the anniversary of the inception of the fund. Moreover, since the distance to the HWM is not reported, we follow Aragon and Nanda (2012), and compute it assuming the fund at inception is at the water mark and calculate the HWM in each period as the maximum between the one the year before and the actual value of the fund. The actual value of the fund is the cumulative after-fees return. The value of the fund relative to its HWM corresponds to 1 when the fund is at the HWM and decreases as the fund moves farther away from it.

The main simplification here is that we compute only one high-water mark for each fund and period, even though there are many such marks depending on when each investor invested in the fund. This can be an issue for funds with many investment rounds for which the errors-in-variables problem would be a greater problem. However, as those funds are probably older, our age control should ease this concern.
Our benchmark regression model is as follows:

\[
\text{risk}_{i,t+1} = \alpha + \beta_1 \times \ln(\text{age})_{i,t} + \beta_2 \times \text{US-Based}_i + \beta_3 \times \text{Return}_{i,t} + \beta_4 \times \text{Rank}_{i,t} \\
+ \beta_5 \times \text{Value relative to HWM}_{i,t} + \text{controls} + \varepsilon_{i,t},
\]

where \( t \) is one year. In different specifications we consider various controls - including strategy, fund and year fixed-effects - and also allow for the effect of distance to be non-linear and to vary with the management and incentive fees.

Proposition 4 implies that \( \beta_5 \) is negative since the farther away the fund drifts from the HWM, the smaller the management fee and the present value of the incentive fee will be. If the AUM is far away from the HWM, the infinitely-lived manager will be less worried about surpassing the HWM by too much. This will incentivize risk-taking behavior.

The specification also includes past absolute returns (\( \text{Return} \)) and returns relative to other funds (\( \text{Rank} \)). Empirically, returns relative to other funds have been shown to be of first order importance (see, for instance, Brown et al. (2001) and Aragon and Nanda (2012)). Theoretical models suggest their importance, although it is often difficult to separate absolute returns from returns relative to the high-water mark. We will be looking at the effect of the distance to HWM after controlling for these.

We also include age to account for potential career concerns and reputational effects and whether the fund is based on the U.S. to accommodate institutional differences in the capacity to take risk and other conventions (when not including fund fixed effects). We also add year fixed effects to capture changing conditions that affect all funds, such as market swings, that may influence performance, and therefore their distance to the HWM and potentially the portfolio managers’s risk aversion. Controlling for strategy is also important.

In our main specification, we use fund fixed effects to capture time-invariant fund characteristics; these include features such as strategy, localization, etc. The identification relies on the within fund variation as we are comparing funds with themselves at different moments in time. In the end, we are asking whether a particular fund behaved differently at different distances from the high watermark. More specifically, if the fund took more risk in the years in which it was farther away from its HWM.

Some previous empirical work has adopted a framework that is a bit different from ours in that it has sought to explain changes in risk-taking from the first to the second semester with the distance to the high-water mark at midyear (see, for instance, Brown et al. (2001), Agarwal et al. (2002), and Aragon and Nanda (2012)). We believe our framework is more in the spirit of our model where the
manager is infinitely lived and is not concerned just with the short-term: she will have an incentive
to take on more risk whenever she is under the HWM and not just (or especially) during the second
semester of each year. This is also consistent with funds being part of fund management companies
that outlast managers. Also, we are not forced to assume that HWM are always reset on January.

When looking at the impact of fees, many papers look at differences in risk-taking across funds
with and without incentive fees. Here we focus on funds with incentive fees and HWM provisions and
explore whether the level of fees make a difference on risk-taking and the way it responds to distance.

The downside of our specification is that we are not be able to identify the effect of time-invariant
(or nearly) fund characteristics; importantly the impact of the structure of fees. However, we begin
by presenting results with no fund fixed effects to provide suggestive evidence of the likely impact of
those factors.

As a robustness check, in Table A1 in the Appendix, we show that previous specifications provide
results that are qualitative and (for the most part) quantitatively consistent. We compute robust
standard errors to consider potential heteroskedasticity and cluster them at the management firm
level\textsuperscript{14}

Table 1. Summary Statistics

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<td>5</td>
<td>1,978</td>
<td>2,012</td>
</tr>
</tbody>
</table>

Table 1 summarizes the data. There is important variation around the mean ex-post volatility
of returns of 11.7% that we can exploit: the standard deviation is 9.7%, the minimum 0.9% and the
maximum 49.7%. The average fund size is 190 million dollars and 5.2 years old. Seventy-six percent
of them are based in the U.S. As for the fees, these are not far from the traditional 2/20 structure,
with a median of 1.5% for the management fee and 18.3% for the incentive remuneration. Overall, all
these figures are similar to the ones reported in the literature using other datasets and time frames.

\textsuperscript{14}Aragon and Nanda (2012) also cluster by strategy. Results are mainly unaffected, but the incentive fee effect turns
out to be non-significant.
When one considers all the funds, the average distance to the watermark is 2.7%. Conditional on being underwater, the mean distance is 14.3%, with a standard deviation of 13.8%.

On average, when a fund surpasses its HWM, it does so by a margin of 12.7% (standard deviation 16.9%) and once it does, it takes on average 1.22 years to reach it again. Having the fee structure of a fund, one can compute the share of income that comes from the incentive and the management component during a given period. If we consider the period between two hits, during which both kinds of fees are collected, a fund manager will get on average 48.5% of its income from the incentive component. That is, both kinds of fees are similarly important for the manager, which is why both of them should be expected to play an important role in the strategic behavior of fund managers, as our model suggests.
Table 2. Benchmark Regressions

The dependent variable is the standard deviation of monthly returns during the 12 months that follow the anniversary of each fund’s inception date. Value relative to HWM is the value of the fund divided by the HWM, that is, it corresponds to 1 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>Coefficient</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log age (years)</td>
<td>0.001</td>
<td>0.001</td>
<td>0.002</td>
<td>0.003</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>U.S. based</td>
<td>-0.005</td>
<td>0.004</td>
<td>-0.003</td>
<td>0.006</td>
<td>-0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>Value relative to HWM</td>
<td>-0.283***</td>
<td>0.012</td>
<td>-0.468***</td>
<td>0.015</td>
<td>-0.096***</td>
<td>0.010</td>
</tr>
<tr>
<td>Return t-1</td>
<td>0.173***</td>
<td>0.006</td>
<td>0.172***</td>
<td>0.005</td>
<td>0.172***</td>
<td>0.006</td>
</tr>
<tr>
<td>Rank t-1 x 10^-3</td>
<td>-0.005***</td>
<td>0.001</td>
<td>-0.006***</td>
<td>0.001</td>
<td>-0.006***</td>
<td>0.001</td>
</tr>
<tr>
<td>Management fee</td>
<td>0.011***</td>
<td>0.004</td>
<td>0.010***</td>
<td>0.003</td>
<td>0.011***</td>
<td>0.003</td>
</tr>
<tr>
<td>Incentive fee</td>
<td>0.001*</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value relative to HWM X Management fee</td>
<td>-0.018</td>
<td>0.002</td>
<td>-0.018</td>
<td>0.002</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value relative to HWM X Incentive fee</td>
<td>-0.010***</td>
<td>0.002</td>
<td>-0.010***</td>
<td>0.002</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| N | 34919 | 34919 | 34919 | 34919 | 34919 | 34919 |
| R-sqr | 0.262 | 0.336 | 0.724 | 0.341 | 0.343 | 0.724 |

Strategy fixed effects | YES | YES | YES | YES | YES | YES |
Fund fixed effects | NO | NO | YES | NO | NO | YES |
Year fixed effects | YES | YES | YES | YES | YES | YES |

Table 2 presents the main results. To explore the main effects of time-invariant fund characteristics, importantly the fee structure, we do not include fund fixed effects in some of the regressions in the table. We then corroborate that the results are not due to time-invariant omitted variable bias by adding the fund fixed effects.

The first column establishes that risk-taking increases with the distance to the HWM: the coefficient of this variable is negative and highly significant. Consistent with our model, funds that are further away from their HWM tend to take more risk when compared to others that are closer to it. Being older and being based in the U.S. does not seem to have a statistically significant impact on risk.

The second column shows that the positive relation between distance and risk is not driven by the absolute return of the fund or its return in relation to the others: the coefficient is still negative and...
significant. The negative sign for the Rank variable is consistent with the tournament behavior that has been documented before: funds that do poorly relative to others tend to take on more risk. The positive effect of Return means that funds that perform better this year tend to be riskier the next one. This is not inconsistent with previous findings such as Aragon and Nanda (2012)'s since this effect is found after controlling for performance relative to others. Consistent with what Brown et al. (2001) finds, the significance of the negative relation between relative performance and volatility, although it does not disappear completely, drops to half when one controls for the distance to HWM (not reported).

The results above come from pooled regressions, that is, are identified via comparing the same fund at different moments but also across different funds of the same strategy category. This can be problematic since funds do not only vary in terms of their distance to HWM and leaving those other characteristics aside may induce estimation bias. In the third column we add fund fixed effects and show that the relation between distance to HWM and risk still agrees with the prediction of the model when we identify the effect just by comparing the same fund at different moments: the coefficient for the value relative to HWM is still strongly negative. The economic magnitude of the effect in the data is large: being 20% underwater is associated with 192 bps increase in the standard deviation of the next 12-month returns, that is, a 16.4% rise.

In the following columns we explore the role of the fee structure on risk-taking. Column four reports that risk increases with the level of management fee since the coefficient is positive and significant. This is what we expected from the model and the simulations ever since the management fee acts as an insurance. Increasing the management fee from, say, 1% to 3% is associated with an increase in 20% of the standard deviation of returns.

The positive coefficient for the incentive fee is, however, at odds with the model’s prediction. We expected a negative coefficient because if the manager takes on more risk and beats the HWM by a larger amount, it becomes harder to beat it again in the future. In any case, the coefficient is only marginally significant (p-value 9%) and very small: increasing the fee from 15% to 20% would increase risk by only 42 bps or 5%. Moreover, as can be seen in the next column, the result is not robust either.

In column 5 we explore how the relation between the distance to the high-water mark and risk is shaped by the structure of fees. Our simulation depicted in Figure 2 suggests that the increase in risk as the fund gets farther away from the HWM is all the more severe as the incentive fee gets larger as moving farther away from the HWM implies a larger forgone value of the incentive fee.

We therefore expect the coefficient of Value relative to HWM to be more negative for funds that charge a higher incentive fee rate. This is exactly what we find. Its size is relevant: the slope of risk
to distance is 20% higher for a fund with a 20% incentive fee compared to one with a 15% charge. This expands Aragon and Nanda (2012)’s results as the relation not only gets stronger for funds with incentive pay but also increases with the level of it. On the contrary, we do not find the effect of distance being stronger with the level of management fee.

From now on we include fund-fixed effects in all regressions and therefore, the results are robust to controlling for all time-invariant fund characteristics. As far as fees have virtually no variation in time, this accounts for the potential endogeneity of fees. Of course, we are no longer able to identify the main effect of the fee structure. Column six shows that our results are robust to controlling for fund fixed effects, in the sense that most of the increase in risk as the AUM gets further away from the HWM comes from funds with high incentive fee rates.
Table 3. Further Results

Panel A

The dependent variable is the standard deviation of monthly returns during the 12 months that follow the anniversary of each fund's inception date. Value relative to HWM is the value of the fund divided by the HWM, that is, it corresponds to 1 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

<table>
<thead>
<tr>
<th></th>
<th>Equity</th>
<th>Hedge</th>
<th>Event-Driven</th>
<th>Fund of Funds</th>
<th>Macro</th>
<th>Relative Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log age (years)</td>
<td>0.003</td>
<td>0.001</td>
<td>0.001</td>
<td>0.004*</td>
<td>-0.001</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.002)</td>
<td>(0.004)</td>
<td>(0.002)</td>
<td>(0.005)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Value relative to HWM</td>
<td>-0.049**</td>
<td>-0.093***</td>
<td>-0.149***</td>
<td>-0.012</td>
<td>-0.126***</td>
<td>-0.052*</td>
</tr>
<tr>
<td></td>
<td>(0.023)</td>
<td>(0.013)</td>
<td>(0.035)</td>
<td>(0.025)</td>
<td>(0.032)</td>
<td>(0.029)</td>
</tr>
<tr>
<td>Value relative to HWM squared</td>
<td>0.117**</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.053)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return t-1</td>
<td>0.044***</td>
<td>0.039***</td>
<td>0.047***</td>
<td>0.042***</td>
<td>0.035***</td>
<td>0.030**</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.007)</td>
<td>(0.015)</td>
<td>(0.015)</td>
<td>(0.012)</td>
<td>(0.012)</td>
</tr>
<tr>
<td>Rank t-1 x 10^{-3}</td>
<td>-0.006***</td>
<td>-0.004***</td>
<td>-0.008***</td>
<td>-0.003***</td>
<td>-0.004***</td>
<td>-0.009***</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.001)</td>
<td>(0.002)</td>
<td>(0.001)</td>
<td>(0.002)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>N</td>
<td>34919</td>
<td>14950</td>
<td>3941</td>
<td>4974</td>
<td>6593</td>
<td>4461</td>
</tr>
<tr>
<td>R-sqr</td>
<td>0.724</td>
<td>0.704</td>
<td>0.7</td>
<td>0.76</td>
<td>0.718</td>
<td>0.674</td>
</tr>
<tr>
<td>Strategy fixed effects</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>Fund fixed effects</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>Year fixed effects</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>

In Table 3 we expand the results. First, we find that the distance effect is not linear but also convex: as the fund gets farther away from the HWM, the incentive risk-taking increases more rapidly. In the first column of Panel A, we show that the square of distance enters significantly positive in the regression. Figure 7 depicts this result. This is what we obtain from our simulations.
The next columns of Panel A show that, although to different degrees, the main result generally applies to all kinds of hedge funds. The coefficient for Value Relative to HWM is always negative (although not significantly so in the case of fund of funds). Furthermore, the increase in risk following poor performance relative to the HWM is more pronounced for funds with higher incentive fees in most kinds, although not always significantly so (not reported). One would expect the response of the fund managers to be larger when there is more flexibility for them to change the level of risk. We do find such an effect since the impact is stronger for funds that can lever up, as reported in the first column in Panel B, as shown by the negative and significant coefficient for the interaction between an indicator of this capacity and Value Relative to HWM.
Table 3. Further Results (continued)

Panel B

The dependent variable is the standard deviation of monthly returns during the 12 months that follow the anniversary of each fund's inception date. Value relative to HWM is the value of the fund divided by the HWM, that is, it corresponds to 1 when the fund is at the HWM and decreases as the fund moves farther away from it. Column 1 includes only the funds that are allowed to lever up. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

<table>
<thead>
<tr>
<th></th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log age (years)</td>
<td>0.003</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.002)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>Value relative to HWM</td>
<td>-0.077***</td>
<td>-0.113***</td>
<td>-0.290***</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.013)</td>
<td>(0.101)</td>
</tr>
<tr>
<td>Return t-1</td>
<td>0.045***</td>
<td>0.045***</td>
<td>0.126***</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.005)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>Rank t-1 x 10^-3</td>
<td>-0.006***</td>
<td>-0.006***</td>
<td>-0.059***</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.001)</td>
<td>(0.015)</td>
</tr>
<tr>
<td>Value relative to HWM x Leveraged</td>
<td>-0.028*</td>
<td>(0.016)</td>
<td></td>
</tr>
<tr>
<td>Value relative to HWM x Fast Redemption</td>
<td>0.037**</td>
<td>(0.017)</td>
<td></td>
</tr>
<tr>
<td>Threat of Liquidation</td>
<td>-1.876***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.528)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value relative to HWM x Threat of Liquid</td>
<td>0.600*</td>
<td>(0.312)</td>
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</tr>
<tr>
<td>N</td>
<td>34760</td>
<td>34919</td>
<td>34919</td>
</tr>
<tr>
<td>R-sqr</td>
<td>0.724</td>
<td>0.724</td>
<td>0.724</td>
</tr>
</tbody>
</table>

Strategy fixed effects: YES
Fund fixed effects: YES
Year fixed effects: YES

Theoretically, the threat of a large drawdown following poor performance has a major impact on the optimal policy. The fear of liquidation magnifies the relative risk aversion of the manager and consequently the fraction invested in the risky asset is reduced. To test this implication, we ask whether the impact of distance on risk is reduced when this is more likely. In column two, we add the interaction between Value Relative to HWM and an indicator that takes the value of 1 if the funds can
be redeemed within less than a month and 0 otherwise. The coefficient for that variable is significantly positive, as expected. We did the same exercise for funds needing less than 30-day notice in advance to withdraw the money and also found a positive and significant coefficient.

To further test for the impact of the likelihood of liquidation we took a two-step approach. First, we estimated a probit model to predict whether a fund would be liquidated at any point in time. The model has an indicator variable that takes a value of 1 if the fund was actually liquidated and zero otherwise, and two independent variables: absolute return and return relative to the other funds. In the second stage we add the predicted value for liquidation from the first step to our benchmark regression, both alone and interacted with the distance variable. We ask whether the funds that, although still alive, share characteristics with those that are liquidated behave differently in terms of risk. We get a negative coefficient for the threat of liquidation and a positive coefficient for its interaction with Value Relative to HWM. This is exactly what we expected from the model.

Table 4 explores what happens with the frequency and extent to which the high-water mark is surpassed.
Table 4. Frequency and Extent of Surpass of HWM

The dependent variable is the time between two hits in columns 1 and 2, and the percentage increase in the HWM in columns 3 and 4. Value relative to HWM is the value of the fund divided by the HWM, that is, it corresponds to 1 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

<table>
<thead>
<tr>
<th></th>
<th>Time between Hits</th>
<th>Extent of Surpass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log age (years)</td>
<td>0.060***</td>
<td>0.060***</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.007)</td>
</tr>
<tr>
<td>Value relative to HWM</td>
<td>-2.452***</td>
<td>-3.028***</td>
</tr>
<tr>
<td></td>
<td>(0.078)</td>
<td>(0.292)</td>
</tr>
<tr>
<td>Return t-1</td>
<td>-0.275***</td>
<td>-0.276***</td>
</tr>
<tr>
<td></td>
<td>(0.023)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>Rank t-1 x 10^{-3}</td>
<td>-0.107***</td>
<td>-0.107***</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>Value relative to HWM X Management fee</td>
<td>0.047</td>
<td>0.019</td>
</tr>
<tr>
<td></td>
<td>(0.132)</td>
<td>(0.016)</td>
</tr>
<tr>
<td>Value relative to HWM X Incentive fee</td>
<td>0.027*</td>
<td>0.009**</td>
</tr>
<tr>
<td></td>
<td>(0.014)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>N</td>
<td>34045</td>
<td>34045</td>
</tr>
<tr>
<td>R-sqr</td>
<td>0.551</td>
<td>0.552</td>
</tr>
</tbody>
</table>

Strategy fixed effects: YES
Fund fixed effects: YES
Year fixed effects: YES

The first column shows that the time elapsed between hits increases with the distance to the HWM, as reflected in the negative coefficient for the Value Relative to HWM. That is, when a fund is close to its HWM, it will typically be hitting the HWM often. This is what was expected from the model: as the fund falls behind the HWM it becomes increasingly difficult to surpass it in the future; taking on more risk mitigates this effect. The size of the effect is relevant: being 20% under the HWM means that the time between hits is 6 months (or 40%) higher than if the fund were at its HWM.

Also in line with our theoretical framework, is the fact that this effect is weaker as the incentive fee increases (column two). We did not find, however, that the frequency of hits decreased more rapidly.
in funds with higher management fees.

The next two columns explore the extent to which the watermark is surpassed. Since the coefficient is negative, the jump is smaller when the fund is closer to its HWM. This supports the intuition that it will be optimal to beat the high-water mark frequently by a small amount to mitigate the ratchet effect. On average, when a fund is 20% under its HWM, it surpasses it by 9.2 percentage points more (116%) than when it is at its HWM. Those considerations become less important as the fund gets farther away.

In Table A1 in the Appendix, we conduct our benchmark analysis using an alternative specification, in the line of Brown et al. (2001), and Aragon and Nanda (2012). This consists on observing the change in risk during the second semester of the year with respect to the first semester and relating it to the fund’s performance. To be consistent with our previous assumptions, in the first three columns we consider only the funds with inception in the month of January. This greatly reduces the number of observations. The results coincide with what has been documented in the literature before: on average risk increases following poor performance measured in relation to others and to the HWM. In the following three columns, we expand the sample to include all funds, regardless of their inception month. That is, we compute the change in the standard deviation of monthly returns for months \( t + 6 \) through \( t + 12 \) versus months \( t \) through \( t + 5 \) relative to each fund’s inception date. The results are qualitatively the same compared to our benchmark in Table 2: risk increases with the management fee and with the distance to HWM, especially when the incentive fee is high.

As a robustness check, in column 7 we just keep the funds that have neither and incentive fee nor a HWM provision and show that, for them, there is no effect on risk-taking of being far from what would have been their high-water mark.

5 Conclusion

We have examined how a management fee combined with an incentive fee affects the optimal investment strategy of chosen by a hedge fund manager in presence of a high water mark. Our baseline model is an extension of the work by Panageas and Westerfield (2009). One of our main finding is to highlight the important role played by the management fee as it contributes to smooth out the manager’s revenues, acting as an insurance policy. Ceteris paribus, this translates into a more aggressive optimal investment strategy with respect to the no management fee case. Second, even though the fund manager has risk neutral preference over money, the option like compensation scheme makes her lifetime utility exhibit increasing relative risk aversion (IRRA). Consequently, the fraction of the AUM
invested in equity is all the more rising the farther away the AUM moves from the HWM. Third, the degree to which the holdings of risky assets increases with distance to the HWM is negatively related to the magnitude of the incentive fee rate. This reflects the intertemporal trade-off faced by the manager: rising the AUM volatility by tilting portfolio holdings towards equity in order to significantly, although infrequently, surpass the HWM and pocket a hefty incentive fee or alternatively beating more often the HWM by a small amount by choosing a more conservative investment strategy. The latter turns out to be optimal, reflecting the manager’s preference for a policy of small steps to achieve smoother revenues. Consistently, we find that the expected time until the next HWM hit is increasing in the (relative) distance between the current value of the AUM and the HWM.

An extension to the baseline model introduces an early termination should the AUM experience a sufficiently large drawdown, measured as fraction of the HWM. The impact on the optimal investment strategy is significant. The closer the AUM gets to the minimum floor, the higher the fund manager’s lifetime risk aversion, which curbs down the risky portfolio allocation. Depending on the parameters of the model, the optimal fraction of AUM invested in the risky asset is either increasing in wealth or hump shaped. The former pattern always prevails when the management fee is small and the liquidation floor is high. Conversely, for sufficiently large management fee rate and low liquidation floor we observe a hump-shaped relationship between the volatility of the AUM and the distance to the HWM, which indicates that as soon as the termination threat is low enough as the AUM has moved away from the liquidation floor, the convex like feature of the compensation scheme induces an excess risk taking behavior as in the baseline model.

We provide empirical tests for the main implications of the model. Data seem to support the theoretical predictions: return volatility is strongly related to distance to the HWM, especially for funds with a high incentive fee rate. Also, the time elapsed between hits and the extent to which the fund surpasses the HWM both increase with distance. Finally, the threat of fund termination reduces risk and mitigates the positive relationship between risk and distance to the high-water mark.
6 Appendix

Appendix A

For all $0 \leq W_0 \leq M_0$ we have

$$F(W_0, M_0) \leq E_0 \left[ \int_0^\infty e^{-(\theta + \delta)t}(c_F M_t + k d M_t) \right] \leq \frac{c_F}{\theta + \delta} M_0 + \frac{(\theta + \delta) c_F + k}{k} \max_\pi E_0 \left[ \int_0^\infty e^{-(\theta + \delta)t} k d M_t \right].$$

Under condition A1, we have $\max_\pi E_0 \left[ \int_0^\infty e^{-(\theta + \delta)t} k d M_t \right] < \infty$ (see Panageas Westerfield (2009)).

Reduced Dual Value Function $J$. Assume that function $J$ is a solution to ODE (6). Define auxiliary function $h$ such that $J(x) = x^{\beta_2} h(-\frac{\Delta}{x})$ and set $y = -\frac{\Delta}{x} < 0$. It is easy to verify that function $h$ is a solution of the Kummer’s equation

$$y h''(y) + (b - y) h'(y) - ah(y) = 0,$$

where $a = -\beta_2$ and $b = 1 + \beta_1 - \beta_2$. Next we show that if $\varphi$ is a solution on the positive real line of (12) with parameters $(a, b)$, then $\tilde{\varphi}$ defined by $\tilde{\varphi}(y) = e^y \varphi(-y)$ with $y < 0$ is a solution on the negative real line of (12) with parameters $(b - a, b)$. Set $x = -y$, we have $\varphi'(x) = e^x [\tilde{\varphi}(-x) - \tilde{\varphi}'(-x)]$ and $\varphi''(x) = e^x [\tilde{\varphi}(-x) - 2\tilde{\varphi}'(-x) + \tilde{\varphi}''(-x)]$. It is easy to check that $\tilde{\varphi}$ satisfies:

$$y \tilde{\varphi}''(y) + (b - y) \tilde{\varphi}'(y) - (b - a) \tilde{\varphi}(y) = 0.$$

One solution of equation (12) is the Kummer (confluent geometric) function

$$M(a, b, z) = 1 F_1(a, b, z) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k} \frac{z^k}{k!},$$

where $(a)_k = a(a + 1)...(a + k - 1)$. For $b > a > 0$, we have the following integral representation

$$M(a, b, z) = \frac{1}{B(a, b - a)} \int_0^1 e^{zt} t^{a-1}(1 - t)^{b-a-1} dt,$$

where $\Gamma$ and $B$ denote the Euler Gamma and Beta functions, respectively. An independent solution for $z < 0$ of equation (12) is

$$e^z U(b - a, b, -z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{z(1+t)} t^{b-a-1} (1 + t)^{a-1} dt, \text{ with } a > 0.$$
where $U$ denotes the Tricomi (confluent geometric) function. The expression for function $J$ follows easily. Then, one can check that

$$
H_1'(x) = \beta_1 x^{\beta_2-1} \int_0^\infty e^{-\frac{\Lambda (1+t)}{x}} t^{\beta_1-1} (1+t)^{-\beta_2} dt > 0
$$

$$
H_1''(x) = \beta_1 (\beta_1 - 1) x^{\beta_2-2} \int_0^\infty e^{-\frac{\Lambda (1+t)}{x}} t^{\beta_1-2} (1+t)^{-\beta_2+1} dt > 0
$$

$$
H_2'(x) = -\beta_1 x^{\beta_2-1} \int_0^1 e^{-\frac{\Lambda}{x} t^{\beta_2}} (1-t)^{\beta_1-1} dt < 0
$$

$$
H_2''(x) = \beta_1 (\beta_1 - 1) x^{\beta_2-2} \int_0^1 e^{-\frac{\Lambda}{x} t^{1-\beta_2}} (1-t)^{\beta_1-2} dt > 0.
$$

In the paper, we make use of the following asymptotic behaviors

$$
H_1(x) \sim \Lambda^{-\beta_1-1} \Gamma(\beta_1 + 1) x^{\beta_1+\beta_2+1} e^{-\frac{\Lambda}{x}} \quad \text{and} \quad H_1'(x) \sim \Lambda^{-\beta_1} \Gamma(\beta_1 + 1) x^{\beta_1+\beta_2-1} e^{-\frac{\Lambda}{x}}
$$

$$
H_1(x) \sim \Lambda^{\beta_2-\beta_1} \Gamma(\beta_1 - \beta_2) x^{\beta_1} \quad \text{and} \quad H_1'(x) \sim \beta_1 \Lambda^{\beta_2-\beta_1} \Gamma(\beta_1 - \beta_2) x^{\beta_1-1}
$$

$$
H_2(x) \sim \Lambda^{\beta_2} \Gamma(-\beta_2) \quad \text{and} \quad H_2'(x) \sim -\beta_1 \Lambda^{\beta_2-1} \Gamma(1-\beta_2)
$$

$$
H_2(x) \sim B(-\beta_2, 1 + \beta_1) x^{\beta_2} \quad \text{and} \quad H_2'(x) \sim \beta_2 B(-\beta_2, 1 + \beta_1) x^{\beta_2-1},
$$

so that we obtain that

$$
f(u) \sim \frac{\beta_2 - 1}{\beta_2} \frac{\beta_2}{Q u^{\beta_2-1}}, \text{ with } Q = \left( \frac{\beta_2 B(-\beta_2, 1 + \beta_1)}{H_2'(x^*)} \right)^{\frac{1}{1-\beta_2}}. \quad (13)
$$

The Wronskian of ODE (6) is given by

$$
W(H_1, H_2)(x) = H_1''(x) H_1(x) - H_1'(x) H_2(x)
$$

$$
= -\Lambda^{\beta_2-\beta_1} \Gamma(-\beta_2) \Gamma(\beta_1 + 1) e^{-\frac{\Lambda}{x} x^{\beta_1+\beta_2-1}} < 0
$$

$$
W(H_1', H_2')(x) = H_2''(x) H_1'(x) - H_1''(x) H_2'(x)
$$

$$
= \beta_1 \beta_2 x^{-2} W(H_1, H_2)(x) > 0.
$$

Assume that the boundary condition at $u = 0$ is reached a finite point $x \in (0, \infty)$. We have (i) $K_1 H_1(x) + K_2 H_2(x) = 0$ and (ii) $K_1 H_1'(x) + K_2 H_2'(x) = 0$. Condition (i) implies that constants $K_1$ and $K_2$ must have opposite signs; using condition (ii), since $H_1' > 0$ and $H_2' < 0$, we deduce that constants $K_1$ and $K_2$ must have the same sign, which leads to a contradiction as both $K_1$ and $K_2$ are not equal to zero. We conclude that $x \in \{0, \infty\}$. Assume that $x = 0$; then, we must have $K_2 = 0$. Since $J' \leq 0$, we must have $K_1 < 0$. It follows that $J'' = K_1 H_1'' < 0$, which leads to a contradiction.
We conclude that $\pi = \infty$, so $K_1 = 0$ and we must have $K_2 > 0$ in order to have $u \geq 0$. We deduce that $J \geq 0$. The boundary condition for $f$ at $u = 1$ maps into:

$$
1 = -K_2 H'_2(x^*)
$$

$$
(k - i)x^* H'_2(x^*) = k H'_2(x^*) - (1 + i)H_2(x^*).
$$

We deduce that interval $I$ must be of the form $[x^*, \infty)$, with $x^* > 1$. ■

Appendix B

Existence and Uniqueness of $x^*$. We want to show that function $\varphi_2$ has a unique root $x^* > 1$, where

$$
\varphi_2(x) = k(ax - 1) \frac{H'_2(x)}{H_2(x)} + 1 + i, \quad (14)
$$

with $a = \frac{k-i}{k} \in (0, 1)$. Let $z = a - \frac{1}{x}$, so that $x = \frac{1}{a-z}$. Define $\phi_2(z) = z \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2} (1-t)^{\beta_1-1} dt = -z + z \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1-1} dt$ and observe that $\varphi_2(x) = -k\beta_1 \phi_2(z) + 1 + i$. We want to show that $\phi_2$ is increasing in $z$. We have

$$
\phi'_2(z) = -1 + \frac{\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1-1} dt}{\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1} dt} + \frac{\Lambda z \Phi_2(z)}{D^2},
$$

where $D = \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1} dt$ and

$$
\Phi_2(z) = \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2} (1-t)^{\beta_1-1} dt \times \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1} dt - \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1-1} dt \times \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2} (1-t)^{\beta_1} dt.
$$

Then, note that

$$
\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1} dt = \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1-1} dt - \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2} (1-t)^{\beta_1-1} dt
$$

$$
\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2} (1-t)^{\beta_1} dt = \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2} (1-t)^{\beta_1-1} dt - \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2-1} (1-t)^{\beta_1-1} dt.
$$

It follows that

$$
\Phi_2(z) = -\left( \int_0^1 f_a(t, z) g_a(t, z) dt \right)^2 + \left( \int_0^1 f_a^2(t, z) dt \right) \times \left( \int_0^1 g_a^2(t, z) dt \right) > 0,
$$

40
by the Cauchy Schwartz inequality with
\[ f_a(t, z) = e^{\frac{A}{2}(z-a) t - \beta_2 (1-t)^{\beta_2-1} - \beta_1 (1-t)^{\beta_1-1}}. \]
\[ g_a(t, z) = e^{\frac{A}{2}(z-a) t - \beta_2 (1-t)^{\beta_2-1} - \beta_1 (1-t)^{\beta_1-1}}. \]

Finally, we note that \(-1 + \int_0^1 e^{A(z-a) t - \beta_2 (1-t)^{\beta_2-1}(1-t)^{\beta_1-1}} dt > 0\). We conclude that \(\phi'_2 > 0\). It follows easily that \(\varphi'_2 < 0\). As \(\varphi_2(x) = 1 + i > 0\) and \(\lim_{x \to \infty} \varphi_2(x) = (k-i)\beta_2 + 1 + i < 0\) by assumption A1, we conclude that \(\varphi_2\) has a unique root \(x^* > \frac{k}{k-i} > 1\) and note that \((k-i)\beta_2 + 1 + i < 0\) is necessary and sufficient.

\[ \frac{\partial \pi^*}{\partial k} < 0 \text{ and } \frac{\partial \pi^*}{\partial i} > 0. \]  

Totally differentiating relationship (14) with respect to \(k\) and evaluating at \(x = x^*\) leads to
\[ (x^* - 1) \frac{H'_2(x^*)}{H_2(x^*)} + \varphi'_2(x^*) \frac{\partial x^*}{\partial k} = 0. \]
Since \(\varphi'_2(x^*) < 0\) and \((x^* - 1) \frac{H'_2(x^*)}{H_2(x^*)} < 0\), we deduce that \(\frac{\partial x^*}{\partial k} < 0\). Similarly, totally differentiating \(\varphi_2(x^*)\) with respect to \(i\) leads to
\[ -\frac{x^*H'_2(x^*)}{H_2(x^*)} + 1 + \varphi'_2(x^*) \frac{\partial x^*}{\partial i} = 0. \]
Since \(\varphi'_2(x^*) < 0\) and \(-\frac{x^*H'_2(x^*)}{H_2(x^*)} + 1 > 0\), we deduce that \(\frac{\partial x^*}{\partial i} > 0\).

Appendix C


P.1. \(\frac{\partial \pi^*}{\partial k} < 0 \text{ and } \frac{\partial \pi^*}{\partial i} > 0. \) As \(H_2\) is independent of \((k,i)\) and \(u = \frac{H'_2(x)}{H_2(x^*)}\), fixing \(u\), we have
\[ \frac{\partial \pi^*}{\partial k} = \frac{\partial \pi^*}{\partial x} \frac{\partial x}{\partial k} = \frac{\partial \pi^*}{\partial x} uH''_2(x^*) \frac{\partial x^*}{\partial k} < 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial k} < 0. \]

Similarly
\[ \frac{\partial \pi^*}{\partial i} = \frac{\partial \pi^*}{\partial x} uH''_2(x^*) \frac{\partial x^*}{\partial i} > 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial i} > 0. \]

P.2. For all \(u \in (0,1), \frac{\partial \pi^*}{\partial u} < 0. \) Recall that
\[ \pi^* = -\frac{\mu - r}{\sigma^2} xH''_2(x). \]
As \( H_2(x) \sim B(-\beta_2, \beta_1 + 1)x^{\beta_2} \), we find that \( \frac{xH''_2(x)}{H_2(x)} \sim \beta_2 - 1 \). Then, we have

\[
-\frac{xH''_2(x)}{H_2(x)} = (\beta_1 - 1) \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} = (\beta_1 - 1)(-1 + \varpi(x)),
\]

with \( \varpi(x) = \frac{\varphi(x)}{\varphi'(x)} \), where auxiliary functions \( \varphi \) and \( \psi \) are defined as \( \varphi(x) = \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} \) and \( \psi(x) = \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-1} dt} \). It follows that

\[
\frac{d\varpi(x)}{dx} = \frac{\varphi'(x)\psi(x) - \psi'(x)\varphi(x)}{\psi^2(x)},
\]

with \( \varphi'(x) = \frac{1}{x^2} \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} \) and \( \psi'(x) = \frac{1}{x^2} \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-1} dt} \). Next note that

\[
\psi(x) = \frac{\Lambda}{x^2} \left( \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} - \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} \right)
\]

\[
\psi'(x) = \frac{\Lambda}{x^2} \left( \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} - \int_0^1 e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-2} dt} \right).
\]

Thus \( \frac{d\varpi(x)}{dx} \) has the same sign as

\[
\varphi'(x)\psi(x) - \psi'(x)\varphi(x) = - \left( \int_0^1 f(t,x)g(t,x)dt \right)^2 + \left( \int_0^1 f^2(t,x)dt \right) \times \left( \int_0^1 g^2(t,x)dt \right) > 0
\]

by the Cauchy-Schwartz inequality with

\[
f(t,x) = e^{-\frac{At}{x} t^{1-\beta_2}(1-t)^{\beta_1-1}}
\]

\[
g(t,x) = e^{-\frac{At}{x} t^{\beta_2}(1-t)^{\beta_1-1}}.
\]

We conclude that \( \frac{d\varpi(x)}{dx} > 0 \) for all \( x \). It follows that

\[
\frac{d\pi^*}{du} = \frac{\partial \pi^*}{\partial x} \times \frac{\partial x}{\partial u} < 0 \text{ as } \frac{\partial u}{\partial x} = -J''(x) < 0.
\]

We deduce that for all \( u \in [0,1] \), \( -\frac{f'(u)}{J''(u)} \leq 1 - \beta_2 \). Then, integrating this relationship and using the fact that \( f'(u) \sim K_0 u^{\frac{1}{1-\beta_2}} \), we find that for all \( u \in [0,1] \), \( f'(u) \leq K_0 u^{\frac{1}{1-\beta_2}} \), which implies that

\[
f(u) \leq \frac{\beta_2 - 1}{\beta_2} K_0 u^{\frac{\beta_2}{\beta_2 - 1}}. \]

Properties of Value Function \( F \).
\textbf{P1:} \( F_2 > 0, F_{22} < 0. \) \( F_2(W, M) = f(u) - uf'(u) = J(x) > 0. \) Then \( F_{22}(W, M) = \frac{u^2}{M}f''(u) < 0. \)

\textbf{P2:} \( \frac{\partial f(u)}{\partial c_F} < 0. \) Let \( c_{F_2} > c_{F_1} \) given. Let \( F^i \) denote the value function for parameter \( c_{F_i}, i = 1, 2. \)

Using the HJB satisfied by \( F^1, \) it is easy to check that \( F^1 \) satisfies:

\[
F^1(W_0, M_0) = \max_{\pi} E_0 \left[ \int_0^{\bar{t}} e^{-\theta t} \left[ (c_{F_2} - c_{F_1})W_t \left[ (f^1)'(u_t) - 1 \right] dt + c_{F_2}W_t dt + kdM_t \right] \right]
\]

s.t. \( dW_t = (r - c_{F_2})W_t dt + (\mu - r)\pi_t W_t dt + \sigma\pi_t W_t dw_t - (k - i) dM_t. \)

Recall we established that \( (f^1)'(u_t) = x_t > 1. \) As \( (c_{F_2} - c_{F_1})W_t \left[ (f^1)'(u_t) - 1 \right] > 0, \) we deduce that \( F^1 > F^2. \) Finally, since \( (1 + i)f(1) = (1 + k)x^* - 1, \) we deduce that \( \frac{\partial f^*}{\partial c_F} = \frac{1 + i}{1 + k} \frac{\partial f(1)}{\partial c_F} < 0. \)

\textbf{P3:} \( \frac{\partial f(u)}{\partial k} < 0 \) and \( \frac{\partial f(u)}{\partial t} > 0. \) Recall that

\[
f(u) = -\frac{H_2(x) - xH'_2(x)}{H'_2(x^*)} \quad \text{and} \quad u = -\frac{H'_2(x)}{H'_2(x^*)}.
\]

Fixing \( u \geq 0, \) we have

\[
\frac{\partial f(u)}{\partial k} = \frac{H_2(x) - xH'_2(x)}{[H'_2(x^*)]^2} \frac{\partial x^*}{\partial k} + \frac{xH_2''(x) - xH'_2(x)}{H'_2(x^*)} \frac{\partial x}{\partial k}.
\]

Rearranging terms and simplifying, we find that \( \frac{\partial f(u)}{\partial k} = \frac{H_2(x)H'_2(x^*)}{[H'_2(x^*)]^2} \frac{\partial x^*}{\partial k} < 0. \) Similarly, \( \frac{\partial f(u)}{\partial t} = \frac{H_2(x)H'_2(x^*)}{[H'_2(x^*)]^2} \frac{\partial x^*}{\partial t} > 0. \)

\textbf{Process } x. \quad \text{Recall that for } u_t < 1,

\[
du_t = \left[ (r - c_F)u_t - \frac{(\mu - r)^2}{\sigma^2} \frac{f'(u_t)}{f''(u_t)} \right] dt - \frac{\mu - r}{\sigma} \frac{f'(u_t)}{f''(u_t)} du_t.
\]

Formally write \( dx_t = \mu_x dt + \sigma_x dw_t; \) as \( u_t = -J'(x_t), \) applying Ito’s lemma for \( x_t > x^* \) leads to

\[
du_t = -\left( J''(x_t)\mu_x + \frac{1}{2} J'''(x_t)\sigma_x^2 \right) dt - J''(x_t)\sigma_x dw_t.
\]

Then, as \( x_t = f'(u_t), u_t = -J'(x_t) \) and \( f''(u_t) = -\frac{1}{J''(x_t)} \), identifying the drift and the diffusion terms from relationship (1), we find that

\[
\sigma_x = -\frac{\mu - r}{\sigma} x_t \quad \quad -(r - c_F)J'(x_t) + \frac{(\mu - r)^2}{\sigma^2} x_t J''(x_t) = -J''(x_t)\mu_x - \frac{\sigma_x^2}{2} J''(x_t).
\]
Then, recall that function $J$ is solution of (6), so differentiating both sides of (6) and rearranging terms yields

$$(r - c_F)J'(x) = [(\theta + \delta - r + c_F)x - c_F]J''(x) + \frac{(\mu - r)^2}{\sigma^2} x J'''(x) + \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} x^2 J''''(x).$$

Given what precedes, we must have

$$\mu x_t = (\theta + \delta - r + c_F)x - c_F.$$

$$\frac{\partial \pi^*}{\partial k} < 0 \text{ and } \frac{\partial \pi^*}{\partial i} > 0.$$  As $H_2$ is independent of $(k, i)$ and $u = \frac{H_2'(x)}{H_2'(x^*)}$, fixing $u$, we have

$$\frac{\partial \pi^*}{\partial i} = \frac{\partial \pi^* u H_2''(x^*)}{\partial x} \frac{\partial x^*}{\partial i} > 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial i} > 0.$$

Similarly

$$\frac{\partial \pi^*}{\partial k} = \frac{\partial \pi^* u H_2''(x^*)}{\partial x} \frac{\partial x^*}{\partial k} < 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial k} < 0.$$  ■

Appendix D

Appendix D1

**Revenue Decomposition.** Recall that for $x > x^*$ functions $g_k$ satisfies the following ODE

$$(\theta + \delta)g_k(x) = [(\theta + \delta - r + c_F)x - c_F]g'_k(x) + \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} x^2 g''_k(x),$$

with $\lim_{x \to \infty} g_k = 0$. The solution of (15) that vanishes when $x$ goes to $\infty$ is given by

$$g_k(x) = \frac{-A_k H_2(x)}{H_2'(x^*)},$$

where $A_k > 0$ is a constant and recall that $-\frac{H_2(x)}{H_2'(x^*)} = f(u) - u f'(u)$. Furthermore, we have $f_k(1) = g_k(x^*) = \frac{-A_k H_2(x^*)}{H_2'(x^*)}$ and $f'_k(1) = -\frac{g'_k(x^*)}{g''_k(x^*)} = \frac{-A_k H_2'(x^*)}{H_2'(x^*)}$. As $f_k$ satisfies boundary condition (3) at $u = 1$, solving for constant $A_k$ leads to

$$A_k = \frac{k}{-(k+1) \frac{H_2'(x^*)}{H_2'(x^*)} + (1 + i) \frac{H_2'(x^*)}{H_2'(x^*)}}.$$
To show that indeed \( A_k > 0 \), recall that

\[
\varphi_2(x) = k(ax - 1)\frac{H'_2(x)}{H_2(x)} + 1 + i,
\]

with \( \varphi_2(x^*) = 0 \) and \( \varphi'_2(x^*) < 0 \). We have

\[
\varphi'_2(x) = ka\frac{H'_2(x)}{H_2(x)} + k(ax - 1)\frac{H''_2(x)}{H_2(x)} - k(ax - 1)\frac{H'_2(x)H'_2(x)}{H_2(x)H_2(x)}.
\]

Then, using the fact that \( \varphi_2(x^*) = 0 \) leads to

\[
\varphi'_2(x^*) = \left(1 + i\right)\frac{H''_2(x^*)}{H_2(x^*)} + \left(1 + i\right)\frac{H'_2(x^*)}{H_2(x^*)} - \frac{H''_2(x^*)}{H_2(x^*)} - \frac{H'_2(x^*)}{H_2(x^*)} - \frac{H'_2(x^*)}{H_2(x^*)} = \left(-k + 1\right)\frac{H''_2(x^*)}{H_2(x^*)} + \left(1 + i\right)\frac{H'_2(x^*)}{H_2(x^*)},
\]

as \( ka + 1 + i = k + 1 \), which indeed implies that \( A_k > 0 \) as \( \varphi_2(x^*) < 0 \) and \( -\frac{H''_2(x^*)}{H_2(x^*)} < 0 \). Next, observe that

\[
f'_k(u) = -\frac{A_kH'_2(x)}{H'_2(x)} = -A_kuf''(u) > 0.
\]

As \( \pi^* \) is decreasing in \( u \), we have \( 1 < \frac{f'(u)}{u(f''(u))}(f''(u) + uf'''(u)) \), which implies that \( f''_k < 0 \). Finally

\[
f_k(u) = A_k\left(1 - \frac{uf'(u)}{f(u)}\right).
\]

Using relationship (13), we find that \( \lim_{\theta \to 0^-} f_k(u) = A_k \frac{\theta}{1 - \beta_2} \), so in particular \( A_k < 1 - \beta_2 \). Furthermore, observe that

\[
\frac{f(u)}{uf''(u)} = -\frac{H_2(x)}{xH'_2(x)} + 1.
\]

Following the same steps as for \( \pi^* \) as for P.2. in Appendix C, one can show that function \( \lambda \) with \( \lambda(x) = -\frac{xH'_2(x)}{H_2(x)} \) is increasing in \( x \). Since, \( f_k(u) = A_k\frac{\theta}{1 + \lambda(x)} \), we deduce that \( \frac{\partial}{\partial \mu} \left[ \frac{f_k(u)}{f(u)} \right] = \frac{\partial}{\partial \mu} \left[ \frac{f_k(u)}{f(u)} \right] \times \frac{\partial \mu}{\partial \mu} > 0 \) as \( \frac{\partial}{\partial \mu} \left[ \frac{f_k(u)}{f(u)} \right] < 0 \) and \( \frac{\partial \mu}{\partial \mu} < 0 \). Appendix D2

**Expected Time until the next Hit.** Let \( f(x, a) = E[e^{-a\tau}] \) be the Laplace transform of the hitting time \( \tau \), for \( x_0 = x > x^* \) and \( a \geq 0 \). Set \( a_0 = \theta + \delta - r + c_F \), \( a_1 = c_F \) and \( a_2 = \frac{1}{2}(\mu - \sigma^2) \), so that \( \frac{a_0}{a_2} = 1 - \beta_1 - \beta_2 \) and \( \frac{a_1}{a_2} = \Lambda \). For \( x > x^* \), function \( f \) is a smooth function that satisfies the following ODE for

\[
af(x, a) = (a_0x - a_1)f_1(x, a) + a_2x^2f_{11}(x, a),
\]
with \( f(x^*, a) = 1 \) and \( \lim_{x \to \infty} f(x, a) = 0 \). The solution is given by

\[
f(x, a) = \left( \frac{x}{x^*} \right)^{\beta_2,a} \frac{\int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt}{\int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt},
\]

where \( \beta_{1,a} \) and \( \beta_{2,a} \) are respectively the positive and negative roots of the quadratic \( Q_a \) with

\[
Q_a(y) = a_2 y^2 + (a_0 - a_2) y - a.
\]

Then, we have

\[
\frac{\partial f(x, a)}{\partial \Lambda} = -\frac{1}{x} \left( \frac{x}{x^*} \right)^{\beta_2,a} D^2 \frac{1}{\varphi(x/x^*)} \left[ \varphi(L) - \varphi(L^2) \right]
\]

with \( D = \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt \) and \( \varphi(L) = \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt \). Using the same approach as in Appendix C for property P2, one can show that function \( \varphi \) is decreasing, so that \( \frac{\partial f(x, a)}{\partial \Lambda} > 0 \), since \( x > x^* \). Next, we make use of the identity \( E[t] = -\frac{\partial f(x, a)}{\partial a} \). Note that as \( a \) goes to 0, we have \( \beta_{1,a} \sim \beta_1 + \beta_2 - \frac{a}{a_0 - a_2} \) and \( \beta_{2,a} \sim \frac{a}{a_0 - a_2} \) and

\[
\beta_{2,a} \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt = \frac{L}{x} \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a+1} dt - (\beta_{1,a} - \beta_{2,a} + 1) \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt.
\]

Taking a Taylor expansion of order 1 in variable \( a \) around \( a = 0 \) yields

\[
-\frac{L}{x} \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a+1} dt - (\beta_{1,a} - \beta_{2,a} + 1) \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt = \frac{L}{x} \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a+1} dt - (\beta_{1,a} - \beta_{2,a} + 1) \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1,a} dt
\]

The term of order 0 is given by

\[
- \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1+\beta_2} \left[ \frac{L}{x}(1-t) + 1 + \beta_1 + \beta_2 \right] dt = -1,
\]

whereas the term of order 1 is given by

\[
\frac{a}{a_0 - a_2} \times \int_0^1 e^{- \frac{\Lambda}{x} t^{\beta_2,a-1}} (1-t)^{\beta_1+\beta_2} \left[ \left( \frac{L}{x}(1-t) + 1 + \beta_1 + \beta_2 \right) \ln[t(1-t)] + 2 \right] dt.
\]
Let denote

\[ A(x) = \int_0^1 e^{-\frac{\Lambda t}{x}} (1-t)^{\beta_1 + \beta_2} \left[ 1 + \left( \frac{\Lambda (1-t)}{x} + \beta_1 + \beta_2 + 1 \right) \ln t \right] dt. \]

It follows that

\[ f(x,a) = 1 + \frac{a}{a_0 - a_2} \left( \ln \frac{x}{x^*} + A(x^*) - A(x) \right) + o(a). \]

Since \( E[\tau] = -\frac{\partial f(x,a)}{\partial a} \bigg|_{a=0} \), the desired result follows. Finally, as \( \frac{\partial f(x,a)}{\partial \Lambda} > 0 \), we have \( \frac{\partial E[\tau]}{\partial \Lambda} < 0 \), which implies that \( A' > 0 \).

**Appendix E: Verification Theorem**

We follow Dybvig (1996) and Panageas and Westerfield (2009). For any feasible strategy \( \pi \), define the process

\[ Q^\pi_t = \int_0^t e^{-(\theta + \delta)s} (c_F W^\pi_s ds + kdM^\pi_s) + e^{-(\theta + \delta)t} F(W^\pi_t, M^\pi_t), \]

where \( F \) is the proposed (optimal) value function. Let \( \bar{M} > 0 \) and denote \( \tau = \inf_{t \geq 0} \{ M_t \geq \bar{M} \} \).

**Step 1:** We look for a function \( F \) that satisfies the following ODE

\[(\theta + \delta)F = c_F W + (r - c_F)W \bar{F}_1 - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{F^2}{\bar{F}_1^2},\]

with boundary conditions

\[ k \bar{F}_1(M, M) = k + (1 + i) \bar{F}_2(M, M) \]
\[ \bar{F}(W, M) = 0 \text{ for all } 0 < W \leq \bar{M} \]
\[ F(0, M) = 0 \text{ for all } 0 < M \leq \bar{M}. \]

Let us consider the following Legendre transform \( W = -J_1(x, M) \) and \( x = F_1(W, M) \). The solution we are looking for is of the form \( J(x, M) = K(M)H_2(x) \), for some smooth function \( K \). It follows that

\[ F(W, M) = K(M) \left[ H_2(x) - xH'_2(x) \right], \]

As \( H''_2 > 0 \), by the implicit function theorem, define function \( X \) such that

\[ M = -J_1(X(M), M) = -K(M)H'_2(X(M)). \]
The boundary condition at \( W = M \) leads to

\[
(k - i)X(M) - k = (1 + i)K'(M)H_2(X(M))
\]

\[
M = -K(M)H_2'(X(M)),
\]

with \( K(M) = 0 \) so that we must have \( X(M) = 0 \). Eliminating function \( K \), we find that \( X \) satisfies the following ODE

\[
1 + i + k \left[ \frac{k - i}{k}X(M) - 1 \right] \frac{H_2'(X(M))}{H_2(X(M))} = (1 + i)M X'(M) \frac{H_2''(X(M))}{H_2'(X(M))}.
\]

Set \( M = e^m \) and define \( x(m) = X(e^m) \). Function \( x \) is solution of the autonomous ODE

\[
(1 + i)x'(m) \frac{H_2''(x(m))}{H_2'(x(m))} = \varphi_2(x(m)).
\]

(16)

with \( x(m) = 0 \), where \( \varphi_2(x) = 1 + i + k \left[ \frac{k - i}{k}x - 1 \right] \frac{H_2'(x)}{H_2(x)} \) and \( \overline{m} = \ln \overline{M} \). Recall that function \( \varphi_2 \) is decreasing on \((\frac{k - i}{k}, \infty)\) and \( \varphi_2 \) is positive on \((0, x^*)\) and negative on \((x^*, \infty)\). If for \( m_0 < \overline{m} \) we have \( \varphi_2(x(m_0)) < 0 \), then as \( \frac{H_2''}{H_2'} < 0 \), function \( x \) will be increasing on \([m_0, \overline{m}]\) and we shall have \( x(m) > x^* \), which leads to a contradiction. Hence, for all \( m \in [0, \overline{m}] \), \( x'(m) < 0 \) and \( x(m) < x^* \). Furthermore, integrating ODE (16) and using the fact \( x(m) = 0 \) leads to

\[
\int_{0}^{x(m)} - \frac{H_2''(y)}{H_2'(y)} \frac{dy}{\varphi_2(y)} = \frac{\overline{m} - m}{1 + i},
\]

(17)

for all \( m < \overline{m} \). It remains to show that as \( \overline{m} \) goes to infinity, function \( x \) converges to \( x^* \). Given \( \overline{m} \), function \( x \) takes values in the bounded interval \([0, x^*]\). For all \( m > 0 \), as the right hand side of relationship (17) goes to infinity and \( x(m) \) is bounded, the integral on the left hand side must be unbounded. This implies that, for all \( m \), we must have \( \lim_{\overline{m} \to \infty} x(m) = x^* \) as \( \varphi_2(x^*) = 0 \), otherwise we get a contradiction. Once \( x \) is known, we can recover function \( K \) and verify that \( \lim_{\overline{M} \to \infty} K(M) = -\frac{M}{H_2'(x^*)} \). We conclude that as \( \overline{M} \) goes to infinity, \( \overline{F} \) converges to \( F \), our candidate function. Finally, note that \( \overline{F}_1 > 0, \overline{F}_{11} = -\frac{1}{\overline{J}_1} < 0 \). Then, by proceeding in the same way as for function \( F \) (see Appendix C) one can show that for all \( W \leq M \), \( F_1(W, M) \leq K_0(K(M))^{\frac{1}{1-\beta_2}} W^{\frac{1}{\beta_2-1}} \), with \( K_0 = (-\beta_2 B(-\beta_2, 1 + \beta_1))^{\frac{1}{1-\beta_2}} \) and \( \pi^* \leq \frac{\mu - \nu}{\sigma^2}(1 - \beta_2) \). This implies that \( W \pi^* \) and \( W^{\pi^*} F_1 \) are bounded on \([0, \overline{M}]^2 \).

**Step 2:** Let \( T > 0 \) and denote \( \hat{T} = \pi \land T \). For \( t \leq \hat{T} \), applying Itô’s lemma for semi-martingales (see
Grossman and Zhou (1993)), we have
\begin{align*}
Q_t^\pi &= Q_0^\pi + \int_0^t e^{-(\theta + \delta)s} \mathcal{A}(W_s^\pi, M_s^\pi) ds + \int_0^t \sigma \pi_s W_s^\pi F_1(W_s^\pi, M_s^\pi) dw_s \\
&\quad + \int_0^t \left[ k - (k - i) F_1(W_s^\pi, M_s^\pi) + (1 + i) F_2(W_s^\pi, M_s^\pi) \right] dM_s^\pi,
\end{align*}

where
\[ \mathcal{A}(W, M) = c_F W + \frac{\sigma^2}{2} \pi^2(W)^2 F_{11} + \pi W(\mu - r) F_1 - (\theta + \delta) F \leq 0, \]

for all strategy \( \pi \) and equal to 0 for \( \pi^* = -\frac{\mu - r}{\sigma^2} \). It follows that
\[ \int_0^{\hat{\tau}} \sigma \pi_s W_s^\pi F_1(W_s^\pi, M_s^\pi) dw_s \geq Q_{\hat{\tau}}^\pi - Q_0^\pi \geq -Q_0^\pi \quad \text{as} \quad Q_{\hat{\tau}}^\pi \geq 0. \quad (18) \]

The left hand side of the inequality is a local martingale that is bounded from below and hence a supermartingale. Thus
\[ 0 \geq E_0 \left[ \int_0^{\hat{\tau}} \sigma \pi_s W_s^\pi F_1(W_s^\pi, M_s^\pi) dw_s \right] \geq E_0[Q_{\hat{\tau}}^\pi] - Q_0^\pi. \]

It follows that \( Q_0^\pi \geq E_0[Q_{\hat{\tau}}^\pi] \). For optimal strategy \( \pi^* \), since \( W^{\pi^*} \) and \( W^{\pi^*} F_1^{\pi^*} \) are bounded, the inequality is an equality as the left hand side in relationship (18) is actually a martingale. Then, by Lebesgue Monotone Convergence Theorem, we have
\[ \lim_{T \to \infty} E_0[Q_T^{\pi^*}] = Q_0^{\pi^*} \geq Q_0^\pi \geq \lim_{T \to \infty} E_0[Q_T^{\pi^*}]. \]

The left and right hand sides of the inequality converge to \( \overline{F}(W_0, M_0) \) and \( E_0 \left[ \int_0^\tau (c_F W_s^\pi ds + kdM_s^\pi) e^{-(\theta + \delta)s} \right] \) respectively, as for any admissible investment strategy, the corresponding value function satisfies the transversality condition (2). Finally, letting \( \overline{M} \) goes to infinity and using once again Lebesgue Monotone Convergence Theorem combined with the fact that \( \overline{F} \) converges to \( F \), we obtain that
\[ F(W, M) \geq E_0 \left[ \int_0^\infty (c_F W_s^\pi ds + kdM_s^\pi) e^{-(\theta + \delta)s} \right], \]

for every feasible investment strategy \( \pi \). This concludes the proof. \( \blacksquare \)

Appendix F: Large Drawdown Prohibited

Appendix F1: No Management Fee \( c_F = 0 \)

**Proof of proposition 3.** Evaluating relationship (11) at \( u = \alpha \) and \( u = 1 \), using \( \pi^*(\alpha) = 0 \) and
the fact that \( f_\alpha \) satisfies the boundary condition (3) leads to the following system \( S_0 \)

\[
\begin{align*}
\alpha & = -\beta_1 K_1(x_\alpha^{**})^{\beta_1-1} - \beta_2 K_2(x_\alpha^{**})^{\beta_2-1} \\
0 & = \beta_1(\beta_1 - 1) K_1(x_\alpha^{**})^{\beta_1-1} + \beta_2(\beta_2 - 1) K_2(x_\alpha^{**})^{\beta_2-1} \\
1 & = -\beta_1 K_1(x_\alpha^*)^{\beta_1-1} - \beta_2 K_2(x_\alpha^*)^{\beta_2-1} \\
\frac{k - i}{1 + i} x_\alpha^* & = \frac{k}{1 + i} + K_1(x_\alpha^*)^{\beta_1} + K_2(x_\alpha^*)^{\beta_2},
\end{align*}
\]  

with \( 0 < x_\alpha^* < x_\alpha^{**} \). Combining relationships (19) and (20) leads to

\[
K_1(x_\alpha^{**})^{\beta_1-1} = -\frac{1 - \beta_2}{\beta_1(\beta_1 - \beta_2)} \alpha < 0 \tag{19}
\]

\[
K_2(x_\alpha^{**})^{\beta_2-1} = -\frac{\beta_1 - 1}{\beta_2(\beta_1 - \beta_2)} \alpha > 0. \tag{20}
\]

Then, combining relationships (21) and (22) leads to

\[
\begin{align*}
K_1(x_\alpha^*)^{\beta_1} & = \frac{1}{\beta_1 - \beta_2} \left[ -(\beta_2 \frac{k - i}{1 + i} + 1)x_\alpha^* + \frac{\beta_2 k}{1 + i} \right] \\
K_2(x_\alpha^*)^{\beta_2} & = \frac{1}{\beta_1 - \beta_2} \left[ (\beta_1 \frac{k - i}{1 + i} + 1)x_\alpha^* - \frac{\beta_1 k}{1 + i} \right].
\end{align*}
\]

Then set \( \varpi = \frac{x_\alpha^*}{x_\alpha^{**}} < 1 \). Eliminating constants \( K_1 \) and \( K_2 \), we find that

\[
\begin{align*}
-\frac{1 - \beta_2}{\beta_1} \alpha x_\alpha^* \varpi^{\beta_1-1} & = -(\beta_2 \frac{k - i}{1 + i} + 1)x_\alpha^* + \frac{\beta_2 k}{1 + i} \\
-\frac{\beta_1 - 1}{\beta_2} \alpha x_\alpha^* \varpi^{\beta_2-1} & = (\beta_1 \frac{k - i}{1 + i} + 1)x_\alpha^* - \frac{\beta_1 k}{1 + i}.
\end{align*}
\]

Eliminating \( x_\alpha^* \) yields

\[
\alpha[(1 - \beta_2)\varpi^{\beta_1-1} + (\beta_1 - 1)\varpi^{\beta_2-1}] = \beta_1 - \beta_2. \tag{23}
\]

For \( z \in (0, 1] \), define auxiliary function \( \Phi \) with

\[
\Phi(z) = \alpha \left( (1 - \beta_2)z^{\beta_1-1} + (\beta_1 - 1)z^{\beta_2-1} \right) - (\beta_1 - \beta_2).
\]

\( \Phi \) is a continuously differentiable function with

\[
\Phi'(z) = \alpha(1 - \beta_2)(\beta_1 - 1)z^{\beta_2-2} \left( z^{\beta_1-\beta_2} - 1 \right) < 0 \text{ for all } z \in (0, 1). \]
Hence, \( \Phi \) is strictly decreasing with \( \lim_{\partial \varpi} = \infty \) and \( \Phi(1) = (\alpha - 1)(\beta_1 - \beta_2) < 0 \). We conclude that \( \Phi \) has a unique root \( \varpi \) in \((0, 1)\) that is independent of \( k \). Totally differentiating relationship (23) with respect to \( \alpha \) leads to

\[
(1 - \beta_2)\varpi^{\beta_1 - 1} + (\beta_1 - 1)\varpi^{\beta_2 - 1} + \Phi'(\varpi) \frac{\partial \varpi}{\partial \alpha} = 0,
\]

which leads to \( \frac{\partial \varpi}{\partial \alpha} > 0 \). It remains to check the condition \( x^*_\alpha > 0 \) or equivalently

\[
\beta_1((k-i)\beta_2 + 1 + i) - \alpha(1 - \beta_2)(1 + i)\varpi^{\beta_1 - 1} < 0.
\]

Set \( \varpi^* = \left[ \frac{\beta_1((k-i)\beta_2 + 1 + i)}{\alpha(1 + i)(1 - \beta_2)} \right]^{\frac{1}{\beta_1 - 1}} \geq 0 \). It is easy to verify that the condition is met whenever \( \varpi > \varpi^* \).

We find that

\[
x^*_\alpha = \frac{\beta_2\beta_1 k}{\beta_1((k-i)\beta_2 + 1 + i) - \alpha(1 - \beta_2)(1 + i)\varpi^{\beta_1 - 1}}.
\]

Recall that \((1 + k)f'(1) = k + (1 + i)f(1)\) and \( x^*_\alpha = f'(1) \). As clearly \( \frac{\partial f(1)}{\partial \alpha} < 0 \), we obtain that \( \frac{\partial x^*_\alpha}{\partial \alpha} < 0 \). Then, as \( x^{**} = \frac{x^*}{\varpi} \), we get \( \frac{\partial x^{**}}{\varpi} < 0 \) and, we can recover constants \( K_1 \) and \( K_2 \). Since \( K_1 = -\frac{1-\beta_1}{\beta_1(1-\beta_2)} \alpha(x^{**})^{1-\beta_1} \), we deduce that \( \frac{\partial K_1}{\partial \alpha} < 0 \). Finally observe that

\[
\pi^* = -\frac{1}{\sigma^2} \frac{-\mu - r}{\beta_1(1 - \beta_2)K_1 x^{\beta_1 - 1} + \beta_2 x^{\beta_2 - 1}} K_2 x^{\beta_2 - 1} \frac{\beta_2 K_2}{\beta_1 K_1 f'(u)(\beta_1 - \beta_2)},
\]

which is increasing in \( u \) as \( f' > 0 \), \( K_1 < 0 \) and \( K_2 > 0 \).

\[
\frac{\partial x^*_\alpha}{\partial \alpha} < 0. \quad \text{Recall that} \ u = -\beta_1 K_1 x^{\beta_1 - 1} - \beta_2 K_2 x^{\beta_2 - 1}, \quad \text{so, fixing} \ u, \ \text{we have}
\]

\[
\frac{1}{x} \frac{\partial x}{\partial \alpha} = -\frac{\beta_1 x^{\beta_1 - 1} \frac{\partial K_1}{\partial \alpha} + \beta_2 x^{\beta_2 - 1} \frac{\partial K_2}{\partial \alpha}}{\beta_1(1 - \beta_2)K_1 x^{\beta_1 - 1} + \beta_2 x^{\beta_2 - 1}}.
\]

Since \( \pi^* = \frac{-\mu - r}{\sigma^2} f'(u) = \frac{-\mu - r}{\sigma^2 u} (\beta_1(1 - \beta_2)K_1 x^{\beta_1 - 1} + \beta_2(1 - \beta_2)K_2 x^{\beta_2 - 1}) \), it follows that

\[
\frac{\partial \pi^*}{\partial \alpha} = -\frac{1}{\sigma^2 u} \frac{\beta_1(1 - \beta_2) \frac{\partial K_1}{\partial \alpha} x^{\beta_1 - 1} + \beta_2(1 - \beta_2) \frac{\partial K_2}{\partial \alpha} x^{\beta_2 - 1}}{\beta_1(1 - \beta_2)K_1 x^{\beta_1 - 1} + \beta_2 x^{\beta_2 - 1}}
\]

\[
+ (\beta_1(1 - \beta_2)K_1 x^{\beta_1 - 1} + \beta_2(1 - \beta_2)K_2 x^{\beta_2 - 1})(\beta_1(1 - \beta_2)K_1 \frac{\partial K_1}{\partial \alpha} - (\beta_1 - 1)K_1 \frac{\partial K_2}{\partial \alpha})
\]

Since \( -\beta_1(1 - \beta_2)K_1 x^{\beta_1 - 1} - \beta_2(1 - \beta_2)K_2 x^{\beta_2 - 1} = -x^2 J''(x) < 0 \), we conclude that \( \frac{\partial \pi^*}{\partial \alpha} \) has a constant
sign, independently of \( u \). Then

\[
\pi_1^* = \frac{\mu - r}{\sigma^2} \left( \beta_1(\beta_1 - 1)K_1(x_\alpha^*)^{\beta_1-1} + \beta_2(\beta_2 - 1)K_2(x_\alpha^*)^{\beta_2-1} \right)
\]

\[
= \frac{\mu - r}{\sigma^2} \beta_1 \beta_2 \left( \frac{k}{1+i} x_\alpha^* - \frac{k - i}{1+i} - \frac{\beta_1 + \beta_2 - 1}{\beta_1 \beta_2} \right).
\]

It follows that \( \frac{\partial \pi_1^*}{\partial \alpha} = \frac{\mu - r}{\sigma^2} \beta_1 \beta_2 K_1 x_\alpha^* \frac{\beta_1 - 1}{\beta_1 - \beta_2} \alpha < 0 \), as \( \frac{\partial x_\alpha^*}{\partial \alpha} < 0 \). The desired results follows.

Imposing \( f_\alpha(\alpha) = 0 \) instead of \( \pi^*(\alpha) = 0 \). Condition (20) is now replaced by \( J_\alpha(x_\alpha^{**}) + \alpha x_\alpha^{**} = 0 \). We find that

\[
K_1(x_\alpha^{**})^{\beta_1-1} = -\frac{1-\beta_2}{\beta_1 - \beta_2} \alpha < 0
\]

\[
K_2(x_\alpha^{**})^{\beta_2-1} = -\frac{\beta_1 - 1}{\beta_1 - \beta_2} \alpha < 0,
\]

and using condition (21) leads to

\[
\alpha [(1 - \beta_2) \beta_1 \varpi^{\beta_1-1} + (\beta_1 - 1) \beta_2 \varpi^{\beta_2-1}] = \beta_1 - \beta_2,
\]

where \( \varpi = \frac{x_\alpha^*}{x_\alpha^{**}} \). For \( z > 0 \), define auxiliary function \( \Psi \) with

\[
\Psi(z) = \alpha \left( (1 - \beta_2) \beta_1 z^{\beta_1-1} + (\beta_1 - 1) \beta_2 z^{\beta_2-1} \right) - (\beta_1 - \beta_2).
\]

\( \Psi \) is a smooth function with

\[
\Psi'(z) = \alpha(1 - \beta_2)(\beta_1 - 1)z^{\beta_2-2} \left( \beta_1 z^{\beta_1-1} - \beta_2 \right) > 0.
\]

Since \( \lim_{0} \Psi = -\infty \), \( \Psi(1) = (\alpha - 1)(\beta_1 - \beta_2) < 0 \) and \( \lim_{\infty} \Psi = \infty \), we deduce that \( \Psi \) admits a unique root strictly greater than 1, so we must have \( \varpi > 1 \), i.e., \( x_\alpha^* > x_\alpha^{**} \). Furthermore, since \( K_1 < 0 \) and \( K_2 < 0 \), this implies that for all \( x \) in \( (x_\alpha^{**}, x_\alpha^*) \), we have \( J''_\alpha(x) > 0 \), i.e., for all \( u > \alpha \), \( f''_\alpha(u) > 0 \). Value function \( f_\alpha \) is globally convex and thus cannot be solution of equation (10). To have a well-defined problem (4), we need to impose \( \pi \leq \pi_{\text{max}} \), with \( \pi_{\text{max}} > 0 \), which optimally shall always binds.

Appendix F2: General Case. Recall that \( a = \frac{k-i}{k} \). We want to show that system (S) has a unique
solution with $0 \leq x_\alpha^* \leq x_\alpha^{**}$. Solving for $K_1$ and $K_2$ we find that

$$K_1 = \frac{(1 + i)H_2(x_\alpha^*) + k(ax_\alpha^* - 1)H_2'(x_\alpha^*)}{(1 + i)W(H_1, H_2)(x_\alpha^*)}$$

$$K_2 = \frac{(1 + i)H_1(x_\alpha^*) + k(ax_\alpha^* - 1)H_1'(x_\alpha^*)}{(1 + i)W(H_2, H_1)(x_\alpha^*)}$$

Eliminating $(K_1, K_2)$ and using properties of the Wronskien leads to

$$\frac{(1 + i)H_2(x_\alpha^*) + k(ax_\alpha^* - 1)H_2'(x_\alpha^*)}{(1 + i)W(H_1, H_2)(x_\alpha^*)} = \frac{\alpha H_2''(x_\alpha^{**})}{W(H_1', H_2')(x_\alpha^{**})}$$

$$\frac{(1 + i)H_1(x_\alpha^*) + k(ax_\alpha^* - 1)H_1'(x_\alpha^*)}{(1 + i)W(H_2, H_1)(x_\alpha^*)} = \frac{\alpha H_1''(x_\alpha^{**})}{W(H_1', H_2')(x_\alpha^{**})}.$$  \hfill (24)

Note that function $\Phi_2$ where $\Phi_2(x) = (1 + i)H_2(x) + k(ax - 1)H_2(x) = H_2(x)\varphi_2(x)$ is the product of two (strictly) decreasing functions that are positive on $(0, x^*)$. Then set $\Phi_1$ where $\Phi_1(x) = (1 + i)H_1(x) + k(ax - 1)H_1'(x) = H_1(x)\varphi_1(x)$ with

$$\varphi_1(x) = (1 + i) + k(ax - 1)\frac{H_1'(x)}{H_1(x)}.$$

Following the exact same steps as in Appendix B, one can show that function $\varphi_1$ is (strictly) increasing with $\lim_{x \to \infty} \varphi_1(x) = (k - i)\beta + 1 + i > 0$ and $\varphi_1(x) \sim -\frac{kA}{\sqrt{x}}$. We deduce that $\varphi_1$ has a unique root $x_{\alpha_{\min}}$ and note that $x_{\alpha_{\min}} < 1$. We conclude that $\Phi_1$ is increasing and positive on $(x_{\alpha_{\min}}, \infty)$. It follows that $\Phi = \frac{\Phi_2}{\Phi_1}$ is decreasing and positive on $(x_{\alpha_{\min}}, x^*)$. Then, note that function $\frac{H_2''}{H_1'}$ is a decreasing and positive function. By the Implicit Function Theorem, we can write $x_\alpha^* = \Psi(x_\alpha^{**})$, where $\Psi$ is a positive increasing function, independent of $\alpha$, that takes value in $(x_{\alpha_{\min}}, x^*)$ with $\lim_{x \to 0} \Psi(x) = x_{\alpha_{\min}}$ and $\lim_{x \to \infty} \Psi(x) = x^*$. We look for a fixed point $\bar{x}$ of $\Psi$. Using relationship (26) and rearranging terms, we find that $\bar{x}$ must satisfy

$$(1 + i) \left[ H_2(\bar{x})H_1''(\bar{x}) - H_1(\bar{x})H_2''(\bar{x}) \right] = k(a\bar{x} - 1)W(H_1', H_2')(\bar{x}).$$
As $H_1$ and $H_2$ are solutions of (6) and given the property of the Wronskian, we have

$$(1 + i) [(1 - \beta_1 - \beta_2)x - \Lambda] = k(a\bar{x} - 1)\beta_1\beta_2,$$

with $a = \frac{k - i}{k}$, which leads to

$$\bar{x} = \frac{\Lambda(1 + i) - k\beta_1\beta_2}{(1 + i)(1 - \beta_1 - \beta_2) - (k - i)\beta_1\beta_2}. \quad (27)$$

The condition $(k - i)\beta_2 + 1 + i < 0$ implies that $(1 + i)(1 - \beta_1 - \beta_2) - (k - i)\beta_1\beta_2 > 0$, so indeed $\bar{x} > 0$. Therefore $\Psi$ has a unique fixed point. Then, as $\Phi$ is decreasing and $\Phi(\bar{x}) > 0$, $\Phi(x^*) = 0$, we must have $x < x^*$. Since we are looking for $x_\alpha^* \leq x_\alpha^{**}$, we shall restrict our attention to $x_\alpha^* \leq x^* \leq x_\alpha^{**}$.

For $\alpha = 0$, the solution is $(x_\alpha^*, x_\alpha^{**}) = (x^*, \infty)$. Next, we show that for $\alpha = 1$, the solution is $x_\alpha^* = x_\alpha^{**} = \bar{x}$. It is enough to that check $x_\alpha^* = x_\alpha^{**} = \bar{x}$ satisfy relationship (24) when $\alpha = 1$, which is indeed the case. Then, combining relationships (25) and (24) to eliminate the term $k(ax_\alpha^* - 1)$ leads to

$$W(H'_1, H'_2)(x_\alpha^{**}) = \alpha \left[ H''_2(x_\alpha^{**})H'_1(x_\alpha^*) - H''_1(x_\alpha^{**})H'_2(x_\alpha^*) \right].$$

Fix $x \geq \bar{x}$, and for $\bar{x} \leq y \leq x$, consider auxiliary function $G(y; x) = H''_2(x)H'_1(y) - H''_1(x)H'_2(y)$. $G$ is a smooth function of $y$ with

$$G'(y; x) = H''_2(x)H''_1(y) - H''_1(x)H''_2(y).$$

Notice that $G(x; x) = W(H'_1, H'_2)(x)$. We want to show that $G' < 0$, or equivalently that function $R$ with $R(z) = \frac{H'_2(z)}{H''_1(z)}$ is decreasing. We have

$$R'(z) = \frac{W(H''_1, H''_2)(z)}{(H''_1(z))^2}$$

$$\quad = \frac{\beta_1\beta_2(\beta_1 - 1)(\beta_2 - 1)x^{-4}W(H_1, H_2)(z)}{(H''_1(z))^2} < 0.$$
function $\Delta$ with for $x \in [\overline{x}, \infty)$, $\Delta(x) = \Gamma(\Psi(x); x)$. We have $\Delta(\overline{x}) = 1$ and

$$\lim_{x \to \infty} \Delta(x) = \lim_{x \to \infty} \frac{-\beta_1 \beta_2 \Lambda^{\beta_2 - \beta_1} \Gamma(-\beta_2) \Gamma(\beta_1 + 1) e^{-\frac{\Delta}{x}} x^{\beta_1 + \beta_2 - 3} - \overline{x}^{\beta_1 + \beta_2 - 3}}{H''_2(x) \overline{H}'_1(x) - H''_1(x) \overline{H}'_2(x)} = \lim_{x \to \infty} \frac{-\beta_1 \beta_2 \Lambda^{\beta_2 - \beta_1} \Gamma(-\beta_2) \Gamma(\beta_1 + 1) e^{-\frac{\Delta}{x}} x^{\beta_1 + \beta_2 - 3}}{-H''_2(x) \overline{H}'_1(x) \beta_1 (\beta_1 - 1) \Lambda^{\beta_2 - \beta_1} \beta_1 x^{\beta_1 - 2}} = 0,$$

as $\beta_2 - 1 < 0$ and where we have used the fact that $\Psi(\overline{x}) = \overline{x}$ and $\lim_{x \to \infty} \Psi = x^*$. By the Intermediate Value Theorem, we deduce that the equation $\Delta(x) = \alpha$, with $\alpha < 1$ has (at least) one root. Given what precedes, $\Delta$ has at most one root, so the root is indeed unique. Finally, to show that $J$ is strictly convex, it is enough to show that for all $x_\alpha^* < x < x_\alpha^{**}$, we have $K_1 H''_1(x) + K_2 H''_2(x) > 0$, or equivalently $K_1 + K_2 \frac{H''_1(x)}{H''_1(x_\alpha^*)} > 0$, which is true as $\frac{H''_1(x)}{H''_1(x_\alpha^*)}$ is a decreasing and $K_1 + K_2 \frac{H''_1(x_\alpha^{**})}{H''_1(x_\alpha^{**})} = 0$. ■

**Fund Manager Compensation Decomposition.** Recall that $g'_{\alpha,k}(x) = -J''_{\alpha}(x) J'_{\alpha,k}(u)$ and $f_{\alpha,k}(u) = g_{\alpha,k}(x)$ so the boundary conditions at $x = x_\alpha^*$ and $x = x_\alpha^{**}$ are

$$A_{k_1} H'_1(x_\alpha^{**}) + A_{k_2} H'_2(x_\alpha^{**}) = 0$$

$$-(1 + k) [A_{k_1} H'_1(x_\alpha^*) + A_{k_2} H'_2(x_\alpha^*)] = J''_{\alpha}(x_\alpha^*) (k + (1 + i) [A_{k_1} H_1(x_\alpha^*) + A_{k_2} H_2(x_\alpha^*)]).$$

Solving for $(A_{k_1}, A_{k_2})$ yields

$$A_{k_1} = -\frac{k J''_{\alpha}(x_\alpha^*)}{(1 + k) H'_2(x_\alpha^*) \left[ \frac{H'_1(x_\alpha^*)}{H''_1(x_\alpha^*)} - \frac{H'_1(x_\alpha^{**})}{H''_1(x_\alpha^{**})} \right] + (1 + i) J''_{\alpha}(x_\alpha^*) H'_2(x_\alpha^*) \left[ \frac{H'_2(x_\alpha^*)}{H''_2(x_\alpha^*)} - \frac{H'_2(x_\alpha^{**})}{H''_2(x_\alpha^{**})} \right]}$$

$$A_{k_2} = -\frac{k J''_{\alpha}(x_\alpha^*)}{(1 + k) H'_1(x_\alpha^*) \left[ \frac{H'_1(x_\alpha^*)}{H''_1(x_\alpha^*)} - \frac{H'_1(x_\alpha^{**})}{H''_1(x_\alpha^{**})} \right] + (1 + i) J''_{\alpha}(x_\alpha^*) H'_1(x_\alpha^*) \left[ \frac{H'_2(x_\alpha^*)}{H''_2(x_\alpha^*)} - \frac{H'_2(x_\alpha^{**})}{H''_2(x_\alpha^{**})} \right].$$

**Appendix G**

The dependent variable in columns 1-6 is the change in standard deviation of monthly returns during the 6 months that follow the anniversary of each fund's inception date relative to the previous 6 months. The first three columns only consider funds with inception in January, columns four through six includes all. The dependent variable in column 7 is the standard deviation of monthly returns during the 12 months that follow the anniversary of each fund's inception date. The sample in column 7 includes only the funds with neither incentive fee nor HWM provision. Value relative to HWM is the value of the fund divided by the HWM, that is, it corresponds to 1 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

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| N | 8133 | 8133 | 8133 | 35552 | 35552 | 35552 | 4652 |
| R-sqr | 0.334 | 0.337 | 0.336 | 0.228 | 0.23 | 0.23 | 0.742 |

| Strategy fixed effects | YES | YES | YES | YES | YES | YES | YES |
| Fund fixed effects | NO | NO | NO | NO | NO | NO | YES |
| Year fixed effects | YES | YES | YES | YES | YES | YES | YES |
7 References


