Extracting Latent States from High Frequency Option Prices

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Abstract

We propose the realized option variance as a new observable variable to integrate high frequency option prices in the inference of option pricing models. Using simulation and empirical studies, this paper documents the incremental information offered by this realized measure. Our empirical results show that the information contained in the realized option variance improves the inference of model variables such as the instantaneous variance and variance jumps of the S&P 500 index. Parameter estimates indicate that the risk premium breakdown between jump and diffusive risks is affected by the omission of this information.

Keywords: High frequency data, realized option variance, options, jump-diffusions, particle filter.
1 Introduction

The increasing availability of high frequency data has paved the way for a better understanding of asset prices and their underlying risks. The fine granularity of intraday prices and recent advances in econometrics provide new evidence regarding the importance of stochastic variance and jumps as sources of risk (Todorov and Tauchen, 2011; Andersen et al., 2015a, among others). In addition, mounting nonparametric evidence suggests that these risk factors constitute an important proportion of the average equity and variance risk premiums (see Bollerslev and Todorov, 2011; Andersen et al., 2015b). In the context of option pricing, these findings seem to indicate that more precise estimates could be obtained by using the full set of observed intraday prices. Nonetheless, estimating models directly from high frequency prices is a daunting task since it would require not only the computation and storage of large option panels, but also explicit assumptions about the microstructure effects that govern these prices.

This paper proposes a parsimonious framework to incorporate intraday prices in the estimation of jump-diffusion option pricing models. Rather than using directly the time series of these prices, we summarize this information with realized measures of variance that can be used as observable variables at lower frequencies. In particular, we propose the realized option variance (ROV) as a new variable to synthesize the information from high frequency option prices. The paper employs simulated and empirical evidence to document how the inclusion of intraday information from option prices provides more precise estimates of the instantaneous variance and its jumps. These reported gains ultimately lead to important economic implications in terms of option pricing errors and risk premium identification.

Our strategy to identify model components – namely, stochastic variance, return jumps and variance jumps – relies on exploiting the complementary information of the rich set of observable variables. The first source of information corresponds to daily time series of the underlying asset price, which provides the primary source of information about price dynamics under the physical measure. Aside from these time series, we employ daily measures of intraday price variation using the realized variance and the bipower variation (see Andersen et al., 2001; Barndorff-Nielsen and Shephard, 2004, among others). These measures capture jump and diffusive variation of the underlying process by aggregating intraday prices in different ways. Studies such Christoffersen et al. (2014) and Christoffersen et al. (2015) have shown the economic value of including these measures in the specification of option pricing models.

The second source of information comes from the cross-section of option prices. These prices have
been extensively used in the literature to extract information about price dynamics under the risk-neutral measure, risk premiums related to jump and variance risk, and parameters that govern the conditional return distribution under the physical measure (see Bates, 2000; Chernov and Ghysels, 2000; Pan, 2002a; Eraker, 2004, among others).\(^1\) We enrich the cross-section of option prices with information from their intraday prices. For each option in the inference set, we employ squared intraday price variations across the day to obtain the realized option variance. The idea behind this new measure is to capture the intraday dynamics under the physical measure associated with the model components. As argued in Andersen et al. (2015a), functionals of instantaneous variance and jump intensities – such as option prices – inherit the behaviour of these latent variables at small scales. Thus, we compile intraday activity of these latent components to facilitate their identification at a lower frequency.

To understand the information content of realized option variances, we employ a model of asset returns that exhibits stochastic variance, jumps in returns with stochastic intensity, as well as jumps in variance using the general affine framework of Duffie et al. (2000). These characteristics are similar to those studied in Pan (2002b), Eraker et al. (2003), Brodie et al. (2007) and Johannes et al. (2009), among others. We favour this specification because it allow us to study the role of intraday information on model identification within a framework that has been extensively used in financial studies. The existing literature on option pricing has almost exclusively focused on model specification\(^2\) and estimation\(^3\), but has not given much regard to the question of incremental information from observable variables. This paper addresses this essential question and provides evidence of the economic gains associated with the addition of high-frequency information to the estimation set.

We use this model to characterize the realized option variance in closed-form and show how this measure is impacted by the model components. A distinctive characteristic of this measure is that, depending on the option’s moneyness, specific features of the underlying processes can be isolated. For instance, deep out-of-the-money options have very low deltas and variance vegas, so most of the variability in these option prices comes from discontinuous sample paths generated by return and variance

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\(^1\)More recently, the emergence of derivative contracts on the VIX index has allowed studies such as Cont and Kokholm (2013) and Bardgett et al. (2015) to extract information about the expected future realized variance. Given that our objective is to work with realized measures from option prices and their underlying, we leave for further research this alternative source of information.

\(^2\)Model features include jumps in volatility (Eraker et al., 2003; Pan, 2002b; Todorov and Tauchen, 2011), co-jumps (Chernov et al., 2003; Eraker, 2004; Brodie et al., 2007), jump arrival intensities (Bates, 2000), and multifactor stochastic volatility with jumps (Bardgett et al., 2015), among others.

\(^3\)A non-exhaustive list of methods for the estimation of these models includes Markov chain Monte Carlo methods (Jones, 1998; Eraker, 2001), nonparametric approaches (Aït-Sahalia and Lo, 1998), the simulated method of moments (Duffie and Singleton, 1983; Batt and Tauchen, 2010), the generalized method of moments (Pan, 2002b), and approximate maximum likelihood (Bates, 2006), among others.
jumps. Likewise, at-the-money options are highly sensitive to changes in the underlying asset’s variance, giving additional information to the one provided in the realized variance. Since contributions of diffusive and jump components are difficult to disentangle from a single realized measure, including realized option variances of different moneyness is appealing to identify the latent components empirically. Employing multiple sources that load differently on latent variables constitutes an alternative approach to those using different sample schemes to untangle variance and jumps (e.g., Barndorff-Nielsen and Shephard, 2004).

The intuition obtained from the parametric model about the behaviour of realized option variances is complemented empirically by documenting several properties of these realized measures. We employ intraday data from options on the S&P 500 index and from futures prices on the E-mini S&P 500 to construct a sample of daily observations that extends from July 2004 to December 2012. We analyze the large cross-section of option data by constructing realized option variance surfaces across moneyness and maturities. These surfaces show an important level of commonality between ROVs and the realized variance of the index returns, especially when large, sporadic shocks happen. Nonetheless, a principal component analysis reveals additional information in the realized option variances that is unavailable in the index’s realized variance. We extend these analyses by looking at the economic relationship between realized option variances and index return variations. Predictive regressions reveal that lagged values of selected realized option variances help forecast the one-day ahead realized variance and the jump variation activity of the underlying.

The paper investigates several empirical implications of adding the ROVs as additional observable variables in the estimation of option pricing models. Our results show that the addition of the ROVs produces lower posterior standard deviations of the latent quantities, especially for variance jumps. We note that including ROVs produces less frequent, larger variance jumps and more frequent, smaller negative return jumps. These differences have nontrivial implications on the identification of short-term risk premiums. First, the average equity risk premium decreases from 4.51% to 3.39% with the addition of this information. Decomposing this risk between diffusive and jump risk shows that when ROVs are included, the diffusive risk premium is less compensated (decreases from 4.45% to 1.76%).

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4 Events such as the financial crisis of 2008, the flash crash episode of 2010, or the downgrade of the U.S. debt in 2011 have common spikes in all time series. Not surprisingly, this degree of commonality is expected as a result of intense trading activity in both markets (Stephan and Whaley, 1990).

5 The jump variation activity is measured as the positive difference between realized variance and bipower variation.

6 Recently, we have been made aware that Audrino and Fengler (2015) have developed, independently, a competing measure of the realized option variance similar to ours. Whereas they use intraday option prices to analyze the consistency of option pricing models, we study the economic implications of aggregating intraday option prices for model identification.
the contrary, the average jump risk premium increases from 0.06% to 1.63%, but this increment is not sufficient to compensate for the overall average decrease. Second, the compensation for variance risk increases in absolute value from −0.90 bps to −3.41 bps by adding ROVs. This increase is driven by a higher compensation for variance jump risk (−3.24 bps with ROV, −0.12 bps without) and a lower compensation for variance diffusive risk (−0.17 bps with ROV, −0.78 bps without). Thus, not only the total average risk premiums are affected by omitting intraday option price information, but also the risk premium breakdown between diffusive and jump risks. The differences in the jump activities and the risk premiums have important repercussions on the accuracy of model option prices. We conduct in- and out-of-sample analyses and find that adding information from intraday option prices produces smaller forecasting errors of implied volatilities.

Several studies in the literature of asset pricing use nonparametric methods to infer the underlying structure of asset returns and their volatilities from high-frequency prices. Unlike these studies, the proposed method employs variables from intraday prices to study their information content within a parametric framework. The strategy of using time-series from several sources offers at least two advantages over previous approaches. First, the approach provides a robust way to identify the model components of the underlying data generating process since multiple signals are employed jointly. Second, the integration of different observable sources linking common fundamental variables allows the method to mitigate the effects of microstructure noise associated with high-frequency estimators.

This paper is also related to studies that deal with the filtering of continuous-time jump-diffusion models. Our paper differs from this literature in several ways. First, we adapt the sequential importance resampling (SIR) algorithm (Gordon et al., 1993) in such a manner that, in addition to index returns and option prices, the filter employs realized variance, bipower variation, and realized option variances as observable sources. Second, we complement Johannes et al. (2009) and estimate the proposed model

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7Todorov and Tauchen (2011) find evidence of discontinuous co-movements between the instantaneous volatilities and returns using high frequency data of VIX and the S&P 500 index. Andersen et al. (2015a) employ a nonparametric framework to infer latent instantaneous volatilities and jump intensities with intraday Black-Scholes implied volatilities. Using option prices, the authors find that the dynamics for IV of deep OTM puts behave as a pure jump process, whereas those of near-the-money are better characterized by a diffusive process. Bandi and Reno (2016) use high-frequency data for the S&P 500 index to estimate its realized variance over intraday intervals. Using a novel moment-based procedure, the authors find support for the existence of independent jumps in the instantaneous variances and co-jumps between the price process and that of the variance.

8In addition to aggregating information from different sources, we also use the sub-sampling methodology of Zhang et al. (2005) to compute individual variance estimators. To reduce bid-ask bounce effects, mid-point option prices are employed in the computation of ROV. An advantage of the filtering approach used in the model’s estimation is that measurement errors are easily accommodated into the estimation of the model, which adds an additional mechanism to control for microstructure noise in high-frequency estimates.

9Johannes et al. (2009) and Bardgett et al. (2015) use an optimal filtering methodology that combines time-discretization schemes with Monte Carlo methods to obtain latent states. Eraker (2004) employs Markov chain Monte Carlo simulation to estimate the posterior distribution of parameters, as well as volatility and jump processes.
using the filter. We employ a weighted likelihood to ensure that the joint estimation is not dominated by one particular source. This approach results in a procedure that is less computationally intensive than the one employed in Eraker (2004).

The rest of this paper is organized as follows. Section 2 describes the model and provides insights about the information content of observable variables. Section 3 presents the estimation approach. Simulated-based results are given in Section 4. Section 5 conducts a nonparametric study of realized option variances. Section 6 presents empirical option pricing implications of adding ROVs in the estimation set. Finally, Section 7 concludes.

2 Framework

This section presents a parametric model that captures different properties of asset prices. We focus our attention on a model with the following latent factors: diffusive stochastic volatility, return jumps, and instantaneous variance jumps. These three risk factors induce different behaviour in the asset price and hence can be identified from an econometric perspective, given the right amount of data (Eraker et al., 2003). This insight is explored in the second part of this section, in which we discuss the list of available data sources and link these quantities to different features of the data generating process.

2.1 Model

The model belongs to the family of stochastic volatility models with jumps (SVJ). This affine jump-diffusion model yields semi-closed form solutions for option pricing as shown in Duffie et al. (2000). Under the objective measure \( \mathbb{P} \), the process governing the dynamics of the log-equity price, \( Y \), and its instantaneous stochastic variance, \( V \), are

\[
\begin{align*}
\text{d}Y_t &= \alpha_t \, \text{d}t + \sqrt{V_t} \, \text{d}W_{Y,t} + \text{d}J_{Y,t}, \\
\text{d}V_t &= \kappa (\theta - V_t) \, \text{d}t + \sigma \sqrt{V_t} \, \text{d}W_{V,t} + \text{d}J_{V,t}, \tag{2}
\end{align*}
\]

\[
\begin{align*}
W_{Y,t} &= \rho W_{V,t} + \sqrt{1 - \rho^2} W_{\perp,t}, \\
J_{Y,t} &= \sum_{n=1}^{N_{Y,t}} Z_{Y,n}, \quad Z_{Y,n} \sim \mathcal{N}(\mu_Y; \sigma^2_Y) \\
J_{V,t} &= \sum_{n=1}^{N_{V,t}} Z_{V,n}, \quad Z_{V,n} \sim \text{Exp}(\mu_V)
\end{align*}
\]
where \(\{W_{t}\}_{t \geq 0}\) and \(\{W_{u,t}\}_{t \geq 0}\) are two independent standard \(\mathbb{P}\)-Brownian motions and \(\{\alpha_r^P\}_{t \geq 0}\) is a predictable process such that
\[
\alpha_r^P = r - q + \left(\eta_Y - \frac{1}{2}\right)V_r + \left(\gamma_Y - \left(\varphi_{Z_Y}(1) - 1\right)\right)\lambda_{Y,r}.
\] (3)

The risk-free interest rate is given by \(r\) and the instantaneous dividend yield by \(q\). The parameters \(\eta_Y\) and \(\gamma_Y\) capture the diffusive and jump risk premiums, respectively.\(^{10}\) The function \(\varphi_{Z_Y}(1)\) is the cumulant generating function of \(Z_Y\) evaluated at 1. Log-equity jumps are generated by a Poisson process \(\{N_{Y,t}\}_{t \geq 0}\) with stochastic intensity that depends on the instantaneous variance: \(\lambda_{Y,r} = \lambda_{Y,0} + \lambda_{Y,1}V_r\). The size of these jumps are given by Gaussian random variables with mean \(\mu_Y\) and standard deviation \(\sigma_Y\).\(^{11,12}\)

Regarding jumps in volatility, these are governed by a Poisson process \(\{N_{V,t}\}_{t \geq 0}\) that has a constant intensity \(\lambda_{V,r} = \lambda_{V,0}\). Variance jump sizes \(\{Z_{V,n}\}_{n=1}^\infty\) are given by independent exponentially distributed random variables with mean \(\mu_V\).\(^{13,14}\)

The drift process \(\alpha_r^P\) is a by-product of the Radon-Nikodym derivative used in this study. The change of measure is specified so that model dynamics under the both measures keep the same structure. This implies that each risk factor is priced, from log-equity and variance Brownian motions to the jump processes \(J_{Y,t}\) and \(J_{V,t}\). Similar methodologies are used in Bates (1991, 2006), Liu et al. (2005), Eraker (2008), Christoffersen et al. (2012), Ornthanalai (2014), among others. We refer the interested reader to Section B.1 of the Appendix for more details.

### 2.2 Links Between Theoretical and Empirical Quantities

We now analyze the variables used to conduct inference of models displaying latent characteristics such as those embedded in Equations (1) and (2).

\(^{10}\)Our Radon-Nikodym derivative explained in Appendix A.1 includes four equivalent martingale measure processes: \(\Lambda_{\alpha,r}\), \(\Lambda_{\alpha,u,r}\), \(\Gamma_r\) and \(\Gamma_Y\). The process \(\Lambda_{\alpha,u,r}\) is a \(\mathbb{P}\)-martingale, as in Heston (1993) among others. \(\Lambda_{\alpha,u,r}\) is defined analogously. Moreover, \(\gamma_Y\) is a nontrivial function of \(\Gamma_r\). Even though \(\eta_Y\) and \(\Gamma_r\) are not involved directly in our \(\mathbb{P}\)-measure modelling, these two parameters deal with the change of measure of the variance diffusive and jump components.

\(^{11}\)This kind of jump process is used by Bates (1996), Bakshi et al. (1997), Duffie et al. (2000), Pan (2002b), Eraker et al. (2003), Johannes et al. (2009) to name a few.

\(^{12}\)For this model, the jump convexity correction is given by \(\varphi_{Z_Y}(1) - 1 = \exp(\mu_Y + \sigma_Y^2/2) - 1\).

\(^{13}\)As argued in Bandi and Reno (2016), among others, jumps in log-equity and variance processes tend to happen at the same time – as in the so-called stochastic volatility with correlated jumps (SVCJ) specification. Although we do not include this feature directly in the proposed framework, both Poisson processes could jump during a given interval, which reconciles with the co-jump empirical evidence. Moreover, our specification makes the inference of latent states more difficult as the noise to signal ratio is higher.

2.2.1 Index Prices

The first source comes from log-prices, which provide information about the price dynamics under the physical measure $\mathbb{P}$:

Observable #1: $Y_t$. \(\text{(4)}\)

As pointed out in Merton (1980), these prices provide information about volatility by arbitrarily increasing the sample frequency. The fundamental variable that links intraday log-prices with price volatility is the quadratic variation ($QV$). This variable employs intraday log-prices over the interval $[0, t]$ in the following way:

\[
QV_t = [Y_t, Y_t]_t = \lim_{N \to \infty} \sum_{j=1}^N \left( Y_{t(j/N)} - Y_{t((j-1)/N)} \right)^2
\]

where $N$ is the number of elements in an equidistant grid dividing the interval $[0, t]$. Under the proposed jump-diffusion model, the quadratic variation is composed of the integrated variance and the log-price jump induced variation:

\[
QV_t = \int_0^t V_s - ds + \sum_{n=1}^{N_{Y_t}} (Z_{Y_n})^2.
\]

Over a time interval of length $\tau$ (e.g., one day), the increments of the quadratic variation are given by:

\[
\Delta QV_{t-\tau, t} = QV_t - QV_{t-\tau} = \int_{t-\tau}^t V_s - ds + \sum_{n=N_{Y_{t-\tau}}+1}^{N_{Y_t}} (Z_{Y_n})^2. \tag{5}\]

As argued by Andersen et al. (2001), estimates of these increments can be obtained with the realized variance ($RV$), which is computed from intraday log-prices in the following way:

Observable #2: $RV_{t-\tau, t} = \sum_{j=1}^N \left( Y_{t-\tau + j\tau/N} - Y_{t-\tau + (j-1)\tau/N} \right)^2$. \(\text{(6)}\)

In general, if $N$ is large enough, $\Delta QV_{t-\tau, t}$ and $RV_{t-\tau, t}$ should be similar.\(^{15}\)

Whereas the quadratic variation provides an overall measure of price volatility, the integrated vari-

\(^{15}\)Protter (2004) shows that $RV_{t-\tau, t}$ converges uniformly in probability to $\Delta QV_{t-\tau, t}$ as the number of intraday observations increases.
ance, defined by

\[ I_t = \int_0^t V_s \, ds \]

depends only on the diffusive component of the variance process and its jumps. Increments of the integrated variance over an interval \( \tau \) are given by:

\[ \Delta I_{t-\tau} = I_t - I_{t-\tau} = \int_{t-\tau}^t V_s \, ds, \]  

which can be estimated using the realized bipower variation:

\[ \text{Observable #3: } BV_{t-\tau} = \frac{\pi N}{2} \sum_{j=2}^N \left| Y_{t-\tau+j\tau/N} - Y_{t-\tau+(j-1)\tau/N} \right| \left| Y_{t-\tau+(j-1)\tau/N} - Y_{t-\tau+(j-2)\tau/N} \right|. \]  

In general, if \( N \) is large enough, \( \Delta I_{t-\tau} \) and \( BV_{t-\tau} \) should be similar. Thus, in order to identify log-price jump activity, it is necessary to combine the information contained in \( RV \) and \( BV \).

### 2.2.2 Option-Based Information

Option prices provide an additional source of information because they are sensitive to changes in price and volatility. Under usual conditions, the price of an option is given by

\[ O_t = \mathbb{E}^Q \left[ e^{-r(T-t)} F(Y_T, K) \right] Y_t, V_t, \]

where \( F(Y_T, K) \) is the payoff at time \( T \), \( K \) the strike price of the option, and \( Q \) a risk-neutral probability measure.\(^{17,18}\) Rather than using option prices directly, we employ the implied volatility (IV) resulting

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\(^{16}\)In a similar class of models, Barndorff-Nielsen and Shephard (2004) prove that \( BV_{t-\tau} \) converges to \( \Delta I_{t-\tau} \) as the number of intraday observations increases.

\(^{17}\)Several authors highlight different advantages of adding these prices to the information set. First, these prices are conditional functions of stock returns, which allows researchers to estimate parameters governing the shape of these distributions (Jackwerth and Rubinstein, 1996; Dumas et al., 1998). Second, option prices help capture the wedge between the measures \( \mathbb{P} \) and \( Q \), thus providing information about volatility and jump risk premiums (Chernov and Ghyssels, 2000; Pan, 2002b; Eraker, 2004; Santa-Clara and Yan, 2010; Christoffersen et al., 2012). Third, option prices are highly informative about the instantaneous variance level (Broadie et al., 2007).

\(^{18}\)Within the proposed framework, semi closed-forms for option prices exist and are provided in Appendix A.3.
from the Black-Scholes formula:\textsuperscript{19}

Observable 4: \( \sigma_{t,i}^{BS}, t \in \{1, 2, \ldots, n_t\} \) (9)

The literature on option pricing has relied almost exclusively on end-of-day option prices. With the availability of high frequency option prices, we now explore how to construct variables that capture information from fine grids. To this end, we extend the well-established concept of realized variance of log-prices to option prices and compute what we call the realized option variance (ROV).

To understand ROV, we need to first look at the quadratic variation of the option price, as the former quantity corresponds to an approximation of the latter. In the proposed framework, we can characterize this variation for a European option as

\[ [O, O]_t = \int_0^t \left( \left( \frac{\partial O_u}{\partial Y} (Y_{u^-}, V_{u^-}) \right)^2 + 2 \sigma \rho \frac{\partial O_u}{\partial Y} (Y_{u^-}, V_{u^-}) \frac{\partial O_u}{\partial V} (Y_{u^-}, V_{u^-}) + \sigma^2 \left( \frac{\partial O_u}{\partial V} (Y_{u^-}, V_{u^-}) \right)^2 \right) V_{u^-} du \]

\[ + \sum_{0 < u \leq t} \left( O_u (Y_u, V_u) - O_u (Y_{u^-}, V_{u^-}) \right)^2. \] (10)

The last term of Equation (10) is associated with the jump variation contribution and can be rewritten as:

\[ \sum_{0 < u \leq t} \left( O_u (Y_u, V_u) - O_u (Y_{u^-}, V_{u^-}) \right)^2 = \sum_{0 < u \leq t} \left( O_u (Y_u, V_u) - O_u (Y_{u^-}, V_u) \right)^2 + \sum_{0 < u \leq t} \left( O_u (Y_u, V_u) - O_u (Y_{u^-}, V_{u^-}) \right)^2, \]

where the first term relates to log-equity jumps and the second one to variance jumps.

The realized equivalent of the change in option quadratic variation

\[ \Delta OQV_{t-\tau, t} = [O, O]_t - [O, O]_{t-\tau} \]

\textsuperscript{19}Implied volatilities do not only provide measures that are invariant to price levels, but also ones that can be characterized with call and put options of different maturities (the so-called IV surface). There is indeed a bijection between option prices and implied volatilities. See Renault (1997) for the benefits of using implied volatilities over option prices. We digress from the pure common calibration approach and use a time series of cross-sections of options to capture the links between the physical and the risk-neutral parameters.

\textsuperscript{20}Since the option price is a smooth function of \( Y \) and \( V \), Itô’s lemma can be applied to determine its quadratic variation. Details are available in the Internet Appendix B.2. Internet Appendix C.2.1 provides a description on how to compute the derivatives used in Equation (10).
is the realized option variance. This variable is computed from intraday option prices according to:

\[
\text{Observable } #5: \quad \text{ROV}_{t} = \sum_{j=1}^{N} \left( O_{t-t+j/N} - O_{t-t+(j-1)/N} \right)^2, \quad t \in \{1, \ldots, n_t\}. \tag{11}
\]

Comparing Equations (10) and (11), we conclude that the realized option variance depends on the delta of the option through \( \frac{\partial O_t}{\partial y} = \frac{\partial O_t}{\partial S} S \) and the variance vega \( \frac{\partial O_t}{\partial \sigma} \). The option moneyness determines the model components that drive the realized option variance. Indeed, a deep in-the-money (ITM) call option has a delta close to one and variance vega close to zero, implying that its quadratic variation is approximately equal to:

\[
\left[ O, O \right]_{\text{ITM}} = \int_0^T \exp(2Y_u) V_u \, du + \sum_{0 < u \leq t} (\Delta O_u)^2.
\]

In this case, we would expect the ITM option’s quadratic variance to be “perfectly elastic” to the integrated variance and the jump induced variation (in a day without jumps, we expect that the option quadratic variance for such a derivative to be proportional to the integrated variance of the underlying).\(^{21}\) Therefore, information obtained using ITM option quadratic variance is somewhat redundant when one has already included the realized variance as a source of information. For ATM options, the variance vega is at its highest value and delta should be close to \( \frac{1}{2} \), so Equation (10) is approximated to:

\[
\left[ O, O \right]_{\text{ATM}} \approx \int_0^T \left( \frac{\exp(2Y_u)}{4} + \rho \sigma \frac{\partial O_u}{\partial v} + \sigma^2 \left( \frac{\partial O_u}{\partial v} \right)^2 \right) V_u \, du + \sum_{0 < u \leq t} (\Delta O_u)^2.
\]

For this level of moneyness, we expect that the option quadratic variation responds more than proportionally to changes in the integrated variance. Finally, for deep OTM options, delta and variance vega should be close to zero, so most of the option quadratic variation comes from jump-induced variations:

\[
\left[ O, O \right]_{\text{OTM}} \approx \sum_{0 < u \leq t} (\Delta O_u)^2.
\]

This is consistent with nonparametric evidence of Andersen et al. (2015a) showing that OTM put options behave like a pure-jump process.

We end this section by summarizing how realized moments and other observed quantities can be combined to identify different sources of risk. First, the difference between the realized variance and the

\(^{21}\) We look at this relation by running log-regressions of ROV over RV for different moneyness (controlling with a variable related to jumps, or excluding days with detected jumps). Section 4.2 shows examples of these regressions using simulated data. We observe that coefficients for ATM are higher than those of ITM, and that the latter should be higher than those of OTM.
bipower variation,
\[ RV_{t-\tau, t} - BV_{t-\tau, t} \equiv \sum_{n=N_{t-\tau+1}}^{N_t} (Z_{Y,n})^2, \]
provides information about the presence and the importance of log-equity price jumps; however, this difference does not give enough information to identify the sign of the jump. To some extent, daily log-equity price should capture the sign of large return jumps.

Second, OTM option realized variations, combined with realized measures of log-equity prices, help to identify jumps in the volatility process. Indeed,
\[
ROV_{t-\tau, t} \equiv \sum_{t-\tau < u \leq t} \left[ O_u(Y_u, V_u) - O_u(Y_u, V_u - \delta_u) \right]^2 + \sum_{t-\tau < u \leq t} \left[ O_u(Y_u, V_u) - O_u(Y_u - \delta_u, V_u) \right]^2.
\]
In the presence of log-equity price jumps, \( RV_{t-\tau, t} - BV_{t-\tau, t} \) already informs us on the (absolute) size of the log-equity jump, meaning that \( ROV_{t-\tau, t} \) contains non-redundant information about the variance jumps. Notice that in the absence of log-equity price jumps, \( RV_{t-\tau, t} - BV_{t-\tau, t} \) is close to zero, so \( ROV_{t-\tau, t} \) becomes
\[
ROV_{t-\tau, t} \equiv \sum_{t-\tau < u \leq t} \left[ O_u(Y_u, V_u) - O_u(Y_u, V_u) \right]^2.
\]

3 Filtering and Estimation

This section provides details about the implementation of a SIR-type filter following Gordon et al. (1993). First, the SIR filter is adapted to account for the realized variance, the bipower variation and the realized option variances as observables. Then, the calculation of the likelihood function with this filter is discussed.

3.1 Filtering Algorithm

The filter is constructed over samples of daily observations, so we define the elapsed time between two consecutive time steps by \( \tau = 1/252 \) and denote the observed sample by \( \{z_{k\tau}\}_{k=1}^T \), where
\[
z_{k\tau} = \left[ Y_{k\tau}, RV_{(k-1)\tau, k\tau}, BV_{(k-1)\tau, k\tau}, \sigma_{k\tau}^{BS}, ROV_{(k-1)\tau, k\tau} \right].
\]
We define \( \sigma_{k\tau}^{BS} = [\sigma_{k\tau,1}^{BS}, \ldots, \sigma_{k\tau, n_{BS}}^{BS}] \) as the implied volatility vector of the \( n_{BS} \) options available on day \( k \) and \( ROV_{(k-1)\tau, k\tau} = [ROV_{(k-1)\tau, k\tau, 1}, \ldots, ROV_{(k-1)\tau, k\tau, n_{ROV}}] \) the corresponding vector of realized option vari-
The first step is to simulate particles (intraday paths) for the latent variables. The interconnection between the instantaneous variance and the log-price process requires the generation of the complete model. More precisely, assuming that the log-price $Y_{(k-1)\tau}$ and its instantaneous variance $V_{(k-1)\tau}$ are known at the end of day $k-1$, the variables are simulated based on a time discretization of Equations (1) and (2). The $M$ intraday steps are required to capture the possibility of having multiple jumps in a single day and to ensure that the discretization scheme is not too far away from the original model. Appendix B.2.1 provides a detailed description of the simulation step.

To obtain end-of-day quantities, we aggregate the $M$ simulated values as explained in Appendix B.2.2, which produces daily simulated particles:

$$x_{kr} = \left[ Y_{kr}, V_{kr}, \Delta I_{(k-1)\tau,kr}, \Delta QV_{(k-1)\tau,kr}, IV_{kr} (Y_{kr}, V_{kr}), AOQV_{(k-1)\tau,kr} \right]$$

where $\Delta I_{(k-1)\tau,kr}$ is the integrated variance generated as a by-product of the simulation stage, $\Delta QV_{(k-1)\tau,kr}$, is the simulated quadratic variation derived from the integrated variance and the simulated jumps, $IV_{kr} (Y_{kr}, V_{kr})$ represents the vector of the $n_{kr}$ model implied volatilities based on the simulated log-price and variance values, and $AOQV_{(k-1)\tau,kr}$ is the vector of the corresponding option quadratic variation calculated from Equation (10).

Further distributional assumptions about the measurement errors are required to connect the observed variables to the state variables. Since the realized variance is based on a finite sample of returns, it has not converged to its limit. Hence, the relative error in the realized variance follows

Assumption #1: \( RE_{k}^{RV} = \frac{\Delta QV_{(k-1)\tau,kr} - RV_{(k-1)\tau,kr}}{RV_{(k-1)\tau,kr}} \sim N\left(0, \eta_1^2\right) \) \hspace{1cm} (12)

Similarly, the relative error between bipower variation and integrated variance is assumed to be normally distributed:

Assumption #2: \( RE_{k}^{BV} = \frac{\Delta I_{(k-1)\tau,kr} - BV_{(k-1)\tau,kr}}{BV_{(k-1)\tau,kr}} \sim N\left(0, \eta_2^2\right) \) \hspace{1cm} (13)

---

22The calculation of realized option variances requires the generation of intraday latent quantities. In this regard, an aggregation step is proposed. Specifically, we use the simulation method of Appendix B.2.1 to generate paths on a fine grid (i.e., $M$ intraday values per day).

23We are aware of Barndorff-Nielsen and Shephard’s (2002) asymptotic results as the number $M$ of intraday observations tends to infinity. However, the time discretization of our empirical implementation is too coarse to pretend that the asymptotic distribution has been reached. In fact, the goal here is to get an estimation procedure as efficient as possible and a large $M$ makes the simulation step very time consuming. The Monte Carlo study shows that good precision is attained with quite small $M$. 
The relative implied volatility error of the $i$th option follows:

\[ \text{Assumption #3: } RE^{IV}_{k,i} = \frac{IV_{k,t,i}(Y_{k,t}, V_{k,t}) - \sigma_{BS}^{BS}_{k,t,i}}{\sigma_{BS}^{BS}_{k,t,i}} \sim N\left(0, \eta_3^2\right) \] (14)

where $IV_{k,t,i}(Y_{k,t}, V_{k,t})$ is the model $IV$ and $\sigma_{BS}^{BS}_{k,t,i}$ is the market implied volatility. Finally, $ROV_{(k-1)r,k,t,i}$ converges to $\Delta OQV_{(k-1)r,k,t,i}$. Again the relative error is assumed Gaussian:

\[ \text{Assumption #4: } RE^{ROV}_{k,i} = \frac{\Delta OQV_{(k-1)r,k,t,i} - ROV_{(k-1)r,k,t,i}}{ROV_{(k-1)r,k,t,i}} \sim N\left(0, \eta_4^2\right). \] (15)

All these relative errors are assumed to be independent.

### 3.2 Estimation

In the SIR method, the proposal distribution depends on the most recent values of state variables.\(^{24}\) To ensure that the joint estimation is not dominated by one particular source, the likelihood associated with each observation receives a weight inversely proportional to the number of sources of a given type. That is, log-equity price, $RV$, and $BV$ receive a weight of 1, and $\sigma_{BS}$ and $ROV$ a weight of one over $n_{k,r}$.\(^{25}\)

Based on Assumptions (12), (13), (14) and (15), the weighted contribution to the likelihood function at time $k\tau$ for particle $x_{k,t}$ is

\[
f(z_{k\tau}|x_{k,t}) = \Phi\left(Y_{k,t}; \mu_{k,t}, \sigma_{k,t}^2\right) \Phi\left(RE^{RV}_{k,i}; 0, \eta_1^2\right) \Phi\left(RE^{BV}_{k,i}; 0, \eta_2^2\right) \\
\quad \times \left( \prod_{i=1}^{n_{k,t}} \Phi\left(RE^{IV}_{k,i}; 0, \eta_3^2\right) \right)^{1/n_{k,r}} \left( \prod_{i=1}^{n_{k,t}} \Phi\left(RE^{ROV}_{k,i}; 0, \eta_4^2\right) \right)^{1/n_{k,r}},
\]

where $\Phi(\cdot; m, s^2)$ is the density function of a Gaussian variable with mean $m$ and variance $s^2$. The conditional expectation $\mu_{k,t}$ and standard deviation $\sigma_{k,t}$ are defined in Appendix B.1. Let

\[
f(z_{k\tau}) = \frac{1}{N_{x}} \sum_{x_{k\tau}} f(z_{k\tau}|x_{k\tau})
\]

\(^{24}\)The SIR method might suffer from “sample impoverishment” issues. To circumvent this issue, the auxiliary particle filter (APR) of Pitt and Shephard (1999) could be used. Yet, we did not consider this avenue as the observable vector contains much more than just the log-equity price, making the use of the APR particularly difficult.

\(^{25}\)This idea is similar to the weighted likelihood estimator. Hu and Zidek (2002) study the properties of the weighted likelihood estimator and show that the key asymptotic results continue to hold.
be the average likelihood at time $k\tau$ across particles and the log-likelihood function up to time $k\tau$ to be defined recursively:

$$\mathcal{L}(z_{k\tau}) \propto \mathcal{L}(z_{(k-1)\tau}) + \log f(z_{k\tau}) .$$

As usual in the SIR filter, particles are resampled according to their weights before handling the information of the following day. More precisely, the path $x_{r(k+1)\tau}$ is resampled proportionally to its likelihood, that is,$^{26}$

$$\omega(x_{k\tau}) \propto \omega(x_{(k-1)\tau}) f(z_{k\tau} | x_{k\tau}) .$$

Although the particle filter approximation of the likelihood function is asymptotically consistent at any point as the number of particles increases, the log-likelihood function is not a continuous function of the parameters and this could cause problems for gradient-based optimizers (e.g., Hürzeler and Künsch, 2001). Indeed, particle filters are known for being ill-suited for maximum likelihood estimation when using naive resampling methodologies. To circumvent the issue, we use the continuous resampling of Malik and Pitt (2011) which generates a continuous likelihood as a function of the unknown parameters.

### 4 Simulation-Based Results

This section provides two sets of simulated-based results. The first one presented in Section 4.1 studies the performance of the filter introduced in Section 3, conditional on the information provided by different sources. The second one is conducted in Section 4.2 and analyzes the information conveyed in ROV for estimation purposes.

Table 1 shows the parameters employed in this section and are qualitatively consistent with the ones estimated in the current literature, e.g., Eraker et al. (2003) and Eraker (2004), among others.$^{27,28}$

The long-term expected variance $\theta = 0.02$ corresponds to about $\sqrt{0.02} = 14\%$ annualized volatility. The parameter $\rho$ is constrained to zero to reduce the signal to noise ratio, making it more difficult to estimate the volatility. The average log-price jump intensity is 15.8%, meaning that about 16 jumps per year are expected. The log-equity price jumps are on average negative, with an average magnitude of $-2\%$ and standard deviation of 2%. The average variance jump is about 3%. The theoretical average

$^{26}$Because of our particular resampling strategy, $\omega(x_{(k-1)\tau}) = 1/N$, $N$, being the number of particles, and the weight reduces to $\omega(x_{k\tau}) \propto f(z_{k\tau} | x_{k\tau})$.

$^{27}$They are also consistent with the ones estimated in Subsection 6.1.

$^{28}$A few other specifications have been tested and results were qualitatively robust to these other parameters.
Table 1: Model parameters used in the simulation study.

<table>
<thead>
<tr>
<th>Log-price process</th>
<th>Variance process</th>
<th>Standard deviations of error terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_Y$</td>
<td>0.500</td>
<td>$\eta_V$</td>
</tr>
<tr>
<td>$\gamma_Y$</td>
<td>0.005</td>
<td>$\Gamma_V$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.010</td>
<td>$\kappa$</td>
</tr>
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<td>$\theta$</td>
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<td>40.000</td>
<td>$\mu_Y$</td>
</tr>
<tr>
<td>$\mu_Y$</td>
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<td>$V_0$</td>
</tr>
<tr>
<td>$\sigma_Y$</td>
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<td></td>
</tr>
<tr>
<td>$Y_0$</td>
<td>log(1000)</td>
<td></td>
</tr>
</tbody>
</table>

diffusive risk premium is given by $\eta_Y \theta = 1\%$. The jump risk premium is about $\gamma_Y (\lambda_{Y,0} + \lambda_{Y,1} \theta) = 7.9\%$.\textsuperscript{29} Consistent with most econometric studies, $\eta_V$ is negative: the $\mathbb{Q}$-measure variance persistence is thus lower than its $\mathbb{P}$ counterpart (3.75 vs. 4, respectively) and the long-term expected variance is higher under the risk-neutral measure (0.0213 vs. 0.02, respectively). Finally, since $\Gamma_V$ is higher than zero, the average variance jump size under the $\mathbb{Q}$-measure is slightly higher (0.0309) than the one under the physical measure (0.03). Throughout the simulation experiments, we use 50,000 particles in the filter.

4.1 Filter’s Performance as a Function of Information

We simulate 100 sample paths of the data generating process (DGP) using the Broadie and Kaya (2006) drift-interpolation scheme of van Haastrecht and Pelsser (2010). The sampling frequency is set at 1/1,560 of a day (every 15 seconds during a 6.5 hour trading day) and the length of a path is set to one year. Quantities such as integrated variance and quadratic variation are computed for each path over a daily frequency. Additional to a path of the DGP, we also compute intraday option prices for short-term options (30 days to maturity) with call-equivalent deltas of 0.20, 0.35, 0.50, 0.65, and 0.80, and a for long-term options (90 days to maturity) with same call-equivalent deltas.\textsuperscript{30} Intraday option prices are used to compute realized option variances for each day, while implied volatilities are deduced from end-of-the-day option prices. To take into account measurement errors, all end-of-day quantities include error terms as defined in Assumptions (12) to (15). These observations constitute our set of observables, from which the filter is run for different data aggregation periods of $M = 1, 2, 3, 5, \text{ and } 10$.

We start with a graphical illustration of how different sources of information impact filter estimates. Figure 1 compares five different densities of the true instantaneous variance for a randomly selected day

\textsuperscript{29}An unreported study shows that our results are robust to other choices of $\eta_Y$ and $\gamma_Y$.

\textsuperscript{30}The call-equivalent delta is simply the delta of the option assuming it is a call (regardless of its real type).
Figure 1: Example of instantaneous variance densities using different data sources.

$Y$ means daily log-equity value, $RV$ means realized variance, $BV$ means bipower variation, $IV$ means implied volatility, and $ROV$ means realized option variance. The parameters of Table 1 are used. The vertical line represents the true instantaneous variance value. The density are constructed by using Gaussian kernels along with simulated particles obtained using the SIR-based methodology.

during which a variance jump occurred. Each density corresponds to the posterior instantaneous variance density function obtained from running the filter with a specific set of variables (i.e., the information set). When only returns are employed in the filter, we observe that the filter lacks precision as values are highly dispersed on the left of the true value. When $RV$ and $BV$ are included, the distribution exhibits a mode closer to the true value with a heavy tail on the right. However, the main mode still exhibits a downward bias. An interesting impact in the distribution is observed when option prices are included in the data set, since the data points cluster closer to the true variance value producing a distribution that is noticeably more peaked, that is, the variability of the estimator is reduced, but the downward bias is still present. Finally, the inclusion of realized option variances centers the distribution around the true value.

We now turn to a more detailed analysis of the filter performance regarding the use of different information sources. This time, we employ the root mean square error (RMSE) to compare the filtered mean of a given variable with the one computed from simulated paths. We regard the mean obtained from simulations as the true value for that variable. To assess the impact of discretization errors in the filtering process, we use different intraday steps (data augmentation) in the simulation of the particles. Performance results are summarized in the first five columns of Table 2 for each latent variable. Consider first the case of the instantaneous variance in Panel A. Notice that no matter which information set is employed, increasing $M$ has little effect on the performance of the filter for this variable. This effect
comes as no surprise due to the low discretization bias present at daily frequencies. On the other hand,
we do observe large differences depending on the source of information. The most important case corre-
sponds to a fourfold decrease when \( RV \) is combined with log-returns. We still observe improvements of
about 10% each time other observable variables are included in the information set, which goes in line
with the patterns observed in Figure 1. This result complements the findings of Johannes et al. (2009), in
which the authors report similar gains when daily option prices are included in addition to asset returns.
The large improvement offered by \( RV \) is also observed when filtering other measures of variance such
as the quadratic variation (Panel B) and the integrated variance (Panel C).

Concerning the performance of jump estimation, Panel D shows results regarding log-equity jump
dsizes and Panel E does so for volatility jumps. The results indicate that data augmentation benefits the
identification of log-equity jump sizes when \( RV \) is added to the information set, but it has little effect
when other sources are employed. Most of the gains in RMSE come from the inclusion of \( RV \), and to
some extent from \( BV \). There is little or no benefit when option prices and \( ROVs \) are added. These results
contrast with those of volatility jumps, where there is little benefit from including \( RV \) in the filter and the
largest RMSE gains are observed when \( ROV \) is included, yielding an almost threefold decrease in RMSE
for \( M = 10 \). Notice also how data augmentation helps reduce RMSE for volatility jump size estimation,
which shows how difficult it is to estimate jumps in volatility at the daily level even if parameters are
known. If we look at the sum of the RMSE for both types of jumps, we observe that the information
content of \( ROV \) is very useful for volatility jump estimation, even with no data aggregation (\( M = 1 \)).

To analyze the effect of discretization bias in the filtering algorithm’s performance, we compare the
filtered mean of a variable with the one computed from the filter using an intraday step of \( M = 15 \).
Results in the rightmost columns of Table 2 show that the previous analysis still hold when considering
filtered values instead of the simulated ones.

The above results about jump size estimation are complemented with jump detection results from
Table 3. The focus of this set of results is on the ability of a given source to identify the occurrence
of jumps. A jump is detected when the filtered jump probability is greater than 50%. This statistic,
when compared with a true simulated jump, provides an idea of the general performance of the filter.
On the other hand, when the statistic is compared with the filtered jump (a jump detected with the filter
using \( M = 15 \)), it quantifies the ability of a given source to identify the occurrence of a jump despite
the discretization bias. According to Panel A, the log-equity price jump times are filtered adequately
when log-equity values and realized variance are included, with a level of concordance of about 96.9%
Table 2: RMSE for instantaneous variance, integrated variance, quadratic variance, log-equity jump, and instantaneous variance jump across 100 simulated paths.

<table>
<thead>
<tr>
<th>Panel A: Instantaneous variance</th>
<th>RMSE (true)</th>
<th>RMSE (filtered)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>M</td>
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</tr>
<tr>
<td></td>
<td>1 2 3 5 10</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>Y, RV and BV</td>
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<tr>
<td>Y, RV, BV, IV and ROV</td>
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<table>
<thead>
<tr>
<th>Panel B: Quadratic variation increment</th>
<th>RMSE (true)</th>
<th>RMSE (filtered)</th>
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<td>Y, RV, BV, IV and ROV</td>
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<td>0.0259</td>
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<table>
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<tr>
<th>Panel C: Integrated variance increment</th>
<th>RMSE (true)</th>
<th>RMSE (filtered)</th>
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</tr>
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<td>Y</td>
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<table>
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<th>Panel D: Log-equity jump</th>
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<th>RMSE (filtered)</th>
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<td>Y and RV</td>
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<th>Panel E: Instantaneous variance jump</th>
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<th>RMSE (filtered)</th>
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<td>Y</td>
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<td>Y and RV</td>
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<tr>
<td>Y, RV, BV, IV and ROV</td>
<td>4.5677</td>
<td>4.2681</td>
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</table>

Y means daily log-equity, RV means realized variance, BV means bipower variation, IV means implied volatility, and ROV means realized option variance. Quantities were multiplied by 1,000. Filtered values are computed as the mean of resampled particles obtained via a SIR particle filter with \( M = 15 \) and using \( Y, RV, BV, IV \) and ROV.
Table 3: Average jump times in percentages.

Panel A: Average log-equity jump times

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<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
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Panel B: Average log-equity jump times

<table>
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<td>94.08</td>
<td>94.08</td>
<td>94.08</td>
<td>94.08</td>
<td>94.08</td>
<td>94.08</td>
<td>94.08</td>
</tr>
<tr>
<td>Y and RV</td>
<td>94.08</td>
<td>94.15</td>
<td>94.29</td>
<td>94.41</td>
<td>94.48</td>
<td>94.48</td>
<td>94.48</td>
<td>94.48</td>
<td>94.48</td>
<td>94.48</td>
</tr>
<tr>
<td>Y, RV and BV</td>
<td>94.08</td>
<td>94.47</td>
<td>94.79</td>
<td>95.03</td>
<td>95.12</td>
<td>95.12</td>
<td>95.12</td>
<td>95.12</td>
<td>95.12</td>
<td>95.12</td>
</tr>
<tr>
<td>Y, RV, BV and IV</td>
<td>94.31</td>
<td>95.21</td>
<td>95.33</td>
<td>95.37</td>
<td>95.38</td>
<td>95.38</td>
<td>95.38</td>
<td>95.38</td>
<td>95.38</td>
<td>95.38</td>
</tr>
<tr>
<td>Y, RV, BV, IV and ROV</td>
<td>97.60</td>
<td>97.33</td>
<td>97.29</td>
<td>97.31</td>
<td>97.31</td>
<td>97.31</td>
<td>97.31</td>
<td>97.31</td>
<td>97.31</td>
<td>97.31</td>
</tr>
</tbody>
</table>

In this table, we show the average number of times a jump has been adequately filtered (i.e., whether the probability of having a jump on a given day is higher than 0.5). Hit reveals the proportion of time that jumps are adequately filtered. Y means daily log-equity, RV means realized variance, BV means bipower variation, IV means implied volatility, and ROV means realized option variance. Filtered values are computed as the mean of resampled particles obtained via a SIR particle filter with $M = 15$ and using Y, RV, BV, IV and ROV.

(98.7% with the discretization bias). This level is slightly improved when more sources are employed.

As observed with jump sizes, the detection of volatility jumps is improved when ROV are introduced, with concordance levels of 97.3% (99.8% with the discretization bias).

4.2 Information Embedded in ROV

The previous tests highlight the benefit of using ROVs as a source for disentangling jumps and volatility in the filtering process. The tests conducted in this section study more closely the link between ROV and information contained in RV and jumps.

The first experiment consists of simulating 1,000 paths of one day for which there are no log-equity and variance jumps in the DGP. Along the paths, a 30-day European call option is priced so that ROV can be computed at the end of each day.

The simulated data produces daily values of RV and ROV for a random sample of options, which allows us to measure the degree of redundancy between these two variables with the following regression:

$$ROV_i = \gamma_1 \left( \frac{\partial O_i}{\partial y} \right)^2 RV_i + \gamma_2 \left( 2 \frac{\partial O_i}{\partial y} \frac{\partial O_i}{\partial v} \right) RV_i + \gamma_3 \left( \frac{\partial O_i}{\partial v} \right)^2 RV_i + e_i.$$  \hspace{1cm} (16)

\( ^{31}\)Each path is generated at a frequency of 1/1,560 of a day using the Broadie and Kaya (2006) drift-interpolation scheme of van Haastrecht and Pelsser (2010).

\( ^{32}\)Each path has its own option for which the option’s delta is uniformly simulated between 0.1 and 0.9 and the initial spot variance between 0.01 and 0.10.
This regression model comes from fixing the option derivatives in Equation (10) to their beginning-of-the-day values, so that the impacts of changes in $RV$ are only modulated by option parameters from sample to sample.

### Table 4: Information content of $ROV$ when jumps are absent.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>1.0136</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>-0.0234</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>0.4356</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.9891</td>
</tr>
</tbody>
</table>

The following regression is applied to $ROV$:

$$
ROV_i = \gamma_1 \left( \frac{\partial O_i}{\partial y} \right)^2 RV_i + \gamma_2 \left( 2 \frac{\partial O_i}{\partial y} \frac{\partial O_i}{\partial v} \right) RV_i + \gamma_3 \left( \frac{\partial O_i}{\partial v} \right)^2 RV_i + \varepsilon_i
$$

The derivatives are computed based on beginning-of-the-day information. Values in bold are statistically different (at a significance level of 5%) from their theoretical values (i.e., 1, 0, and 0.25 respectively). We use a simulated sample of 1000 observations.

As reported in Table 4, the $R$-squared close to one reveals that, in the absence of jumps, $ROV$ is redundant with respect to the information embedded in $RV$. Regarding regression estimates, the individual hypothesis that $\gamma_1 = 1$, $\gamma_2 = 2\sigma \rho = 0$, and $\gamma_3 = \sigma^2 = 0.25$ cannot be rejected with a confidence level of 95%, suggesting that this specification follows Equation (10) closely with the assumption of constant option derivatives. This supports the sampling technique used in the filter to compute particles of $ROV$ with few intraday steps, as option derivatives vary little during a day.

To analyze the impact that jumps in the underlying process might have on the information content of $ROV$, the next experiment accounts for the specificity of the option contract. For each type of contract (call or put), and moneyness (call-equivalent delta of 0.20, 0.35, 0.50, 0.65, or 0.8), the regression

$$
\log(ROV_i) = \beta_0 + \beta_1 \log(RV_i) + \beta_2 \Delta N_{Y,i} + \beta_3 \Delta N_{V,i} + \varepsilon_i
$$

are run separately. In the latter, $\Delta N_{Y,i}$ and $\Delta N_{V,i}$ are variables that capture the number of log-equity price jumps and variance jumps during day $i$, respectively.

Table 5 presents two specifications of the previous regression model. The first one includes information about log-return jumps and the second one adds to this specification information about volatility jumps. Taking $R$-squared as a measure of redundancy, we observe that $RV$ and $ROV$ are more redundant when the option is in-the-money (delta higher than 0.5 for calls and lower than 0.5 for puts). As the option becomes out-of-the-money, redundancy decreases. A closer look at the coefficients in this regression show the sensitiveness of $ROV$ to different types of information. Whereas the coefficient associated
to RV increases with the moneyness of the option, those associated with jump activity do so (in absolute value) when the moneyness decreases. These two pieces of evidence show that the ROV of OTM options provides complementary information not contained in RV.

Table 5: Information content of ROV when jumps are present.

<table>
<thead>
<tr>
<th>Δ^c</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>SE</td>
</tr>
<tr>
<td>0.20</td>
<td>$\beta_0$</td>
<td>13.945 (0.383)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>1.333 (0.041)</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>-1.064 (0.086)</td>
</tr>
<tr>
<td></td>
<td>$\beta_3$</td>
<td>1.067 (0.045)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.627</td>
<td>0.763</td>
</tr>
<tr>
<td>0.35</td>
<td>$\beta_0$</td>
<td>13.796 (0.285)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>1.199 (0.030)</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>-0.726 (0.064)</td>
</tr>
<tr>
<td></td>
<td>$\beta_3$</td>
<td>0.763 (0.034)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.742</td>
<td>0.829</td>
</tr>
<tr>
<td>0.50</td>
<td>$\beta_0$</td>
<td>13.681 (0.215)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>1.119 (0.023)</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>-0.497 (0.048)</td>
</tr>
<tr>
<td></td>
<td>$\beta_3$</td>
<td>0.538 (0.026)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.833</td>
<td>0.882</td>
</tr>
<tr>
<td>0.65</td>
<td>$\beta_0$</td>
<td>13.548 (0.141)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>1.048 (0.015)</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>-0.282 (0.031)</td>
</tr>
<tr>
<td></td>
<td>$\beta_3$</td>
<td>0.310 (0.018)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.922</td>
<td>0.940</td>
</tr>
<tr>
<td>0.80</td>
<td>$\beta_0$</td>
<td>13.470 (0.083)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>1.004 (0.009)</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>-0.131 (0.019)</td>
</tr>
<tr>
<td></td>
<td>$\beta_3$</td>
<td>0.150 (0.011)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.972</td>
<td>0.976</td>
</tr>
</tbody>
</table>

The following regression is applied to ROV:

$$\log(ROV_i) = \beta_0 + \beta_1 \log(RV_i) + \beta_2 \Delta N_Y, i + \beta_3 \Delta N_V, i + \epsilon_i$$

where $\Delta N_Y, i$ is the number of log-equity price jumps jump during day $i$ and $\Delta N_V, i$ is the number of variance jumps jump during day $i$. We estimate two versions of Equation (17): in Regression (1) we force $\beta_3 = 0$, and in Regression (2) we let $\beta_3$ be different than zero. $R^2$ and Newey and West (1987) standard errors (SE, in parentheses) are reported. All coefficients estimated are statistically significant at a significance level of 5%.

5 Exploring Realized Option Variances Empirically

This section starts by presenting our datasets. Next, we analyze nonparametrically the information contained in realized option variances by constructing surfaces of these variations. We complement this investigation with a principal component analysis (PCA) of realized option variances. Finally, we pro-
vide evidence of the economic relationship between realized option variances and index return variability using predictive regressions.

5.1 Data

The sample period employed in this study is from July 2004 to December 2012. This period offers several empirical features such as periods of high economic uncertainty.

We start by constructing a time series measure of the realized option variance at the daily level. We employ tick-by-tick Level I quote data provided by Tick Data for European options written on the S&P 500 index. Tick Data prices come from the Options Price Reporting Authority (OPRA), the national market system that provides information about last sale reports and quotation information. We employ midquote prices instead of trade prices to mitigate the effect of bid-ask spread bounces in the total variation of the option price. For each trading day, we start with 390 one-minute prices from 9:30 AM to 4:00 PM. Next, we construct five different grids with five-minute prices and compute the daily variation according to Equation (11) over each grid. The average of these five values provides an estimate of the option daily variation. This procedure helps to mitigate the presence of microstructure noise, as suggested by Zhang et al. (2005) for the case of realized volatility estimation. For a quote to be included in the dataset, we require its bid price to be higher than zero and the quote not to have any condition code or be eligible for automatic execution. Each day, we restrict our attention to a representative sample set composed of OTM and ATM options. As was argued in Subsection 2.2, ROVs for ITM options yield equivalent information to that of RV, so we exclude these options from our sample. Only options with positive volume and bid prices are included in the sample, as well as those satisfying the no-arbitrage conditions of Bakshi et al. (1997).

Additional to the option’s price variation, we compute several variables that capture different aspects of the market activity for each day in the sample. The first is the daily Black-Scholes implied volatility (IV), which is computed for the option with a forward-to-strike ratio closest to 1 (ATM option) and a maturity closest to 30 business days. The second is the realized variance RV of index returns, constructed from one-minute returns of the E-mini S&P futures contract prices. Finally, the third variable is the bipower variation BV, which accounts for the variability of the diffusive component governing the return process. These last two variables are constructed with five sub-grids, following Zhang et al. (2005) sub-

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33Our final ROV dataset contains 682,380 data points, for an average of 286 realized option variance per day.
34Errors in the option quote dataset could artificially increase the computed ROV. To discard such outliers, the last permille (0.1%) of the ROV sample (i.e., largest values) is removed.
sampling methodology. Descriptive statistics about these variables and the panel of realized option variances are provided in the Internet Appendix E.1.

5.2 Understanding Realized Option Variances

The surfaces (indexed by time) constitute a parsimonious way to study and understand the behaviour of option prices, as shown in Andersen et al. (2015b) for the case of implied volatilities. To interpret the large cross-section of option data that spans distinct maturities and moneyness, we construct a sequence of realized option variance surfaces across these two dimensions. For a given day, we collect the ROVs for all the ATM and OTM options in our dataset and perform a locally weighted scatter plot smoothing across moneyness and maturities.35

Following Andersen et al. (2015b), Figure 2 presents specific ROV surface characteristics such as its level, term structure, skew, and skew term structure across time. The ROV level comes from the ATM option \((\Delta^e = 0.5)\) with 30 days to maturity. The ROV term structure (TS) is defined as the difference between the ROV of the ATM with 90 days to maturity minus the ROV of the ATM option with 30 days to maturity. The ROV skew represents the difference between shorter dated OTM put options (i.e., \(\Delta^e = 0.9\)) and OTM call options (i.e., \(\Delta^e = 0.1\)), both with 30 days to maturity. Finally, the ROV skew term structure (Skew TS) is the difference between longer and shorter dated skew, with the longer dated (i.e., 90 business days) skew defined analogously to the shorter one.

Figure 2 also presents realized volatility and realized jump variation of index returns. Realized jump variation captures the variability induced by discontinuous activity in the index return, which explains its erratic spikes and less persistent behaviour. Sparks in different characteristics of the ROV surface are associated with shocks to realized variance and jump activity of the option’s underlying asset (Figure 2 and Table 6).

The surface level presents sporadic spikes that generate some persistence after their occurrence (top-left panel of Figure 2). ATM options exhibit realized variations that mimic the behaviour of the underlying realized variance (top-left and bottom-middle panels of Figure 2, as well as Table 6 showing that the correlation coefficient between the level and RV is 92%). Consequently, ATM realized option variance brings little new information when RV or BV are already part of the sample. The sporadic spikes are also

\footnote{The locally smoothing quadratic regression is performed using the Matlab procedure Lowess on log(ROV), and then transformed back to ROV. The Matlab procedure performs a local regression using weighted linear least squares with a 2nd degree polynomial model.}

23
Figure 2: **Realized option variance surface characteristics and realized measures.**
The realized option variance level is the realized option volatility for ATM options (i.e., $\Delta^e = 0.5$ and 30 business days). The realized option volatility term structure is the difference between long- and short-dated ATM options (i.e., 90 and 30 business days in our case). The realized option variance skew is the difference between short-dated OTM put options (i.e., $\Delta^e = 0.9$, 30 business days) and OTM call options (i.e., $\Delta^e = 0.1$, 30 business days). The realized option volatility skew term structure is the difference between long- and short-dated skew, with the long-dated (i.e., 90 business days) skew defined analogously to the short one. The realized volatility and the realized jump variation $JV = \max(RV - BV, 0)$ are computed using Zhang et al.’s (2005) microstructure-noise robust estimate.

observed at the same times in the other characteristics of the surface (top-middle, top-right and bottom-left panels of Figure 2), showing a level of commonality associated with large shocks. The other surface characteristics fluctuate around zero and are particularly elevated during turbulent market periods. More precisely, $ROV$ term structure (top-middle panel of Figure 2) shows that shocks are more important for short-dated options. Its low correlation with $RV$ and the realized jump variation, $JV = \max(RV - BV, 0)$, indicates that the $ROV$ term structure is a non-redundant information source (see Table 6). The $ROV$ skew (top-right panel of Figure 2) generally takes positive values, which means that OTM puts are more responsive to shocks than OTM calls. This type of responsiveness is more observed for short dated options than for longer dated ones, as evidenced from the negative sign associated with values of the skew term structure (bottom-left panel of Figure 2).

We now employ the PCA of the $ROV$ surface to look more closely at different aspects of the commonality among $ROV$ surface characteristics, realized variance, and jump activity of the S&P index. We extract from the $ROV$ surface 18 values per day and conduct a PCA over these values for the full sample. More precisely, we take nine equally spaced points over the call-equivalent delta dimension between 0.1 and 0.9 for maturities of 30 and 90 business days. Similar to what is observed for the implied volatility
Table 6: Correlation matrix for realized option variance characteristics and realized measures.

<table>
<thead>
<tr>
<th></th>
<th>Level</th>
<th>TS</th>
<th>Skew</th>
<th>Skew TS</th>
<th>RV</th>
<th>JV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level</td>
<td>1.000</td>
<td>-0.260</td>
<td>0.630</td>
<td>-0.384</td>
<td>0.920</td>
<td>0.626</td>
</tr>
<tr>
<td>TS</td>
<td></td>
<td>1.000</td>
<td>0.049</td>
<td>-0.098</td>
<td>-0.763</td>
<td>-0.008</td>
</tr>
<tr>
<td>Skew</td>
<td></td>
<td></td>
<td>1.000</td>
<td></td>
<td></td>
<td>0.423</td>
</tr>
<tr>
<td>Skew TS</td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
<td>0.703</td>
</tr>
<tr>
<td>JV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
</tr>
</tbody>
</table>

The realized option variance level is the realized option variance for ATM options (i.e., $\Delta e = 0.5$). The realized option variance term structure (TS) is the difference between longer and shorter dated ATM options (i.e., 90 and 30 business days in our case). The realized option variance skew is the difference between shorter dated OTM put options (i.e., $\Delta e = 0.9, 30$ business days) and OTM call options (i.e., $\Delta e = 0.1, 30$ business days). The realized option variance skew term structure (Skew TS) is the difference between longer and shorter dated skew, with the longer dated (i.e., 90 business days) skew defined analogously to the shorter one. The realized variance and the realized jump variation are computed using Zhang et al.’s (2005) microstructure-noise robust estimate and are multiplied by 1000.

The realized option variance level is the realized option variance for ATM options (i.e., $\Delta e = 0.5$). The realized option variance term structure (TS) is the difference between longer and shorter dated ATM options (i.e., 90 and 30 business days in our case). The realized option variance skew is the difference between shorter dated OTM put options (i.e., $\Delta e = 0.9, 30$ business days) and OTM call options (i.e., $\Delta e = 0.1, 30$ business days). The realized option variance skew term structure (Skew TS) is the difference between longer and shorter dated skew, with the longer dated (i.e., 90 business days) skew defined analogously to the shorter one. The realized variance and the realized jump variation are computed using Zhang et al.’s (2005) microstructure-noise robust estimate and are multiplied by 1000.

surface in Andersen et al. (2015b), the $ROV$ surface displays a dominant level type effect, as the first PC accounts for 94.44% of the total variation and displays a high degree of commonality with the surface level. The second PC captures 4.73% of the total variation, while the following ones account for 0.32%, 0.18%, 0.11%, and 0.05%, respectively.

We run in-sample regressions of surface characteristics on PCs to determine whether the $ROV$ surface can be summarized in one simple specification (e.g., the first component of a PCA) or whether different $ROV$ measures are needed to appropriately capture the nonlinear changes in the surface. Thus, our model is given by:

$$
Char_t = \beta_0 + \beta_1 PC_{1,t} + \beta_2 PC_{2,t} + \beta_3 PC_{3,t} + \beta_4 PC_{4,t} + \beta_5 PC_{5,t} + \beta_6 PC_{6,t} + \varepsilon_t,
$$

where $Char_t$ is the characteristic of interest at time $t$ (i.e., Level, TS, Skew, Skew TS) and $PC_{n,t}$ represents the $n^{th}$ principal component at time $t$. In addition, we also regress $RV$ and $JV$ on PCs to further understand the relation between $ORV$ and realized variations of the Index. To account for heteroskedasticity and autocorrelation, standard errors are computed following Newey and West (1987).

As shown in Table 7, $ROV$ characteristics, $RV$, and $JV$ all share commonality effects driven by the first PC. Nonetheless, the loadings on PCs vary across characteristics and return variations. Notice how successful PCs are at capturing the behaviour of index return variation, as evidenced from $R$-squared values of 90% for realized variance and 45% for realized jump variation. However, PCs impact these variables differently, suggesting that the information contained in the surface is impounded differently in these variations. This difference is also evidenced by looking at the persistence of regression residuals, which have different autocorrelation patterns. Persistent residuals in the realized variance regression
Table 7: Realized option volatility characteristics, realized measures, and principal component regressions.

<table>
<thead>
<tr>
<th></th>
<th>Level</th>
<th>TS</th>
<th>Skew</th>
<th>Skew TS</th>
<th>RV</th>
<th>JV</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC_1</td>
<td>0.395</td>
<td>-0.009</td>
<td>0.029</td>
<td>-0.011</td>
<td>0.515</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.002)</td>
<td>(0.002)</td>
<td>(0.002)</td>
<td>(0.036)</td>
<td>(0.003)</td>
</tr>
<tr>
<td>PC_2</td>
<td>0.268</td>
<td>-0.072</td>
<td>-0.107</td>
<td>0.051</td>
<td>-0.060</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.006)</td>
<td>(0.007)</td>
<td>(0.007)</td>
<td>(0.074)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>PC_3</td>
<td>-0.434</td>
<td>0.938</td>
<td>-0.014</td>
<td>-0.005</td>
<td>0.697</td>
<td>0.070</td>
</tr>
<tr>
<td></td>
<td>(0.016)</td>
<td>(0.034)</td>
<td>(0.046)</td>
<td>(0.043)</td>
<td>(0.358)</td>
<td>(0.041)</td>
</tr>
<tr>
<td>PC_4</td>
<td>-0.360</td>
<td>0.230</td>
<td>0.048</td>
<td>-0.066</td>
<td>-0.536</td>
<td>-0.088</td>
</tr>
<tr>
<td></td>
<td>(0.016)</td>
<td>(0.028)</td>
<td>(0.031)</td>
<td>(0.046)</td>
<td>(0.445)</td>
<td>(0.057)</td>
</tr>
<tr>
<td>PC_5</td>
<td>0.027</td>
<td>-0.126</td>
<td>-0.028</td>
<td>0.170</td>
<td>1.203</td>
<td>0.079</td>
</tr>
<tr>
<td></td>
<td>(0.029)</td>
<td>(0.045)</td>
<td>(0.032)</td>
<td>(0.044)</td>
<td>(0.575)</td>
<td>(0.086)</td>
</tr>
<tr>
<td>PC_6</td>
<td>-0.427</td>
<td>0.375</td>
<td>0.101</td>
<td>-0.050</td>
<td>0.988</td>
<td>0.207</td>
</tr>
<tr>
<td></td>
<td>(0.043)</td>
<td>(0.057)</td>
<td>(0.050)</td>
<td>(0.056)</td>
<td>(0.642)</td>
<td>(0.111)</td>
</tr>
</tbody>
</table>

$R^2$ and Newey and West (1987) standard errors (in parentheses) are reported. Values in bold are statistically significant at a significance level of 5%.

The following linear regressions are performed:

$$\text{Char}_t = \beta_0 + \beta_1 \text{PC}_{1,t} + \beta_2 \text{PC}_{2,t} + \beta_3 \text{PC}_{3,t} + \beta_4 \text{PC}_{4,t} + \beta_5 \text{PC}_{5,t} + \beta_6 \text{PC}_{6,t} + \epsilon_t,$$

where Char_t is the characteristic of interest at time t (i.e., Level, TS, Skew, Skew TS, RV and JV) and PC_{n,t} represents the n-th principal component at time t. The first six principal components extracted from the S&P 500 realized option volatility surface from July 2004 to December 2012. The first sample autocorrelation coefficient and the average sample autocorrelation over two to ten and eleven to twenty lags of the regression residuals are exhibited. The values of RV and JV = max(RV - BV, 0) are multiplied by 1000 in the regressions.

suggest that either the relation is nonlinear, or that there are missing factors in the model (e.g., lag values of RV). On the other hand, the jump realized variation regression exhibits low persistent residuals.

What can we learn from these results? First, realized option variances are intertwined with measures of index return variation, revealing an important degree of commonality that is useful for analyzing index return dynamics such as variances and sporadic shocks. Notwithstanding, realized option variations also exhibit information that is not shared with index return variations, suggesting their pertinence as an alternative source of information about the dynamics of the underlying generating process. Second, the fact that different characteristics of the ROV surface cannot be summarized in one simple specification of PCs suggests that various realized option variances should be used in parametric studies; given the large number of options available at a given point in time, a well-selected subset of realized option variance should be employed for empirical analyses.

5.3 Predicting Index Return Variations

The economic relationship between realized option variations and index return variations is now studied with predictive regressions. The motivation behind this exercise is to look at the role of realized
Table 8: Correlation matrix of realized values used in the predictive regressions.

<table>
<thead>
<tr>
<th></th>
<th>log(RV)</th>
<th>JV</th>
<th>log(IV_ATM)</th>
<th>log(ROV_{\Delta e=0.1})</th>
<th>log(ROV_{\Delta e=0.9})</th>
</tr>
</thead>
<tbody>
<tr>
<td>log(RV)</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JV</td>
<td>0.4876</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(IV_ATM)</td>
<td>0.8511</td>
<td>0.4121</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(ROV_{\Delta e=0.1})</td>
<td>0.8470</td>
<td>0.3961</td>
<td>0.7116</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>log(ROV_{\Delta e=0.9})</td>
<td>0.8691</td>
<td>0.3993</td>
<td>0.7202</td>
<td>0.8194</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

option variations as economic variables that predict different types of index return variability.

The first model consists of explaining one-day ahead index return realized variances. We employ the logarithm of realized variance as a dependent variable and estimate the following model:

$$\log(RV_{t+1}) = \beta_0 + \beta_1 \log(RV_t) + \beta_2 JV_t + \beta_3 IV_{\text{ATM}}^{log_{+,-},t} + \beta_4 ROV_{\Delta e=0.1}^{log_{+,-},t} + \beta_5 ROV_{\Delta e=0.9}^{log_{+,-},t} + \epsilon_t, \quad (18)$$

The variables of interest in this model are lagged values of realized jump variation (JV), ATM implied volatility (IV_ATM), and realized option variances of OTM options with call-equivalent deltas (\(\Delta e\)) of 0.1 and 0.9. The maturity of all options is 30 days. To remove potential collinearity issues across the regressors (see Table 8), we employ the orthogonal component that results from the projection of a given measure on RV in our regressions. That is, we define IV_ATM^{log_{+,-},t} as the residual of the following regression:

$$\log(IV_{t}^{\text{ATM}}) = \alpha_1 \log(RV_t) + IV_{\text{ATM}}^{log_{+,-},t},$$

where IV_{t}^{\text{ATM}} is the ATM implied volatility (with maturity of 30 days). The variables ROV_{\Delta e=0.1}^{log_{+,-},t} and ROV_{\Delta e=0.9}^{log_{+,-},t} are computed similarly from the realized option variance for OTM calls with call-equivalent delta of 0.1 and the realized option variance for OTM calls with call-equivalent delta of 0.9.

Panel A of Table 9 presents estimates and t-statistics of six specifications of the regression in Equation (18). The first four specifications show that only the orthogonal components of IV_ATM and ROV_{\Delta e=0.1}^{log_{+,-},t} are statistically significant explaining the one-day ahead realized variation of index returns after controlling for the lagged value of the dependent variable. They enter in the regression model with positive signs, which confirms the forward-looking nature of IV_ATM. Interestingly, it is the residual component of ROV associated with OTM calls that predicts the subsequent realized variance, even after controlling for IV_ATM. As discussed in Section 2.2, the ROV for an OTM option is related to the jump activity in the log-equity price and variance processes. Thus, the explanation power of this
Table 9: Predictive regressions.

Panel A: Logarithm of realized variance

<table>
<thead>
<tr>
<th>Regression</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>−0.5493</td>
<td>0.8733</td>
<td>−3.2276</td>
<td></td>
<td></td>
<td></td>
<td>0.7490</td>
</tr>
<tr>
<td>(2)</td>
<td>−0.2279</td>
<td>0.9425</td>
<td>−2.9616</td>
<td>0.9432</td>
<td></td>
<td></td>
<td>0.7839</td>
</tr>
<tr>
<td>(3)</td>
<td>−0.8316</td>
<td>0.8128</td>
<td>−2.8130</td>
<td></td>
<td></td>
<td>0.1197</td>
<td>0.7515</td>
</tr>
<tr>
<td>(4)</td>
<td>−0.3872</td>
<td>0.9081</td>
<td>−3.4894</td>
<td></td>
<td></td>
<td>−0.0438</td>
<td>0.7494</td>
</tr>
<tr>
<td>(5)</td>
<td>−0.5424</td>
<td>0.8750</td>
<td>−2.4922</td>
<td>0.9529</td>
<td>0.1347</td>
<td></td>
<td>0.7871</td>
</tr>
<tr>
<td>(6)</td>
<td>−0.1785</td>
<td>0.9531</td>
<td>−3.0435</td>
<td>0.9408</td>
<td>−0.0136</td>
<td></td>
<td>0.7839</td>
</tr>
</tbody>
</table>

Variations of following linear regression are performed:

$$X_{t+1} = \beta_0 + \beta_1 \log(RV_t) + \beta_2 JV_t + \beta_3 IV_{\text{ATM}}^{\log, \perp, t} + \beta_4 ROV_{\log, \perp, t}^{0.1} + \beta_5 ROV_{\log, \perp, t}^{0.9} + \epsilon_t,$$

where $X_{t+1} \in \{RV_{t+1}, JV_{t+1}\}$. $RV_t$ is the realized variance at time $t$, $JV_t$ is the realized jump variation at time $t$, $IV_{\text{ATM}}^{\log, \perp, t}$ is the residual of the following regression

$$\log(IV_{\text{ATM}}^{\log, \perp, t}) = \alpha_1 \log(RV_t) + IV_{\text{ATM}}^{\log, \perp, t},$$

where $IV_{\text{ATM}}^{\log, \perp, t}$ is the ATM implied volatility (with maturity of 30 days). The variables $ROV_{\log, \perp, t}^{0.1}$ and $ROV_{\log, \perp, t}^{0.9}$ are computed similarly from the realized option variance for OTM calls with call-equivalent delta of 0.1 and the realized option variance for OTM calls with call-equivalent delta of 0.9. We compute Newey-West standard errors. These are in parentheses in the above table. Values in bold are statistically significant at a significance level of 5%. Results for bipower variation are both qualitatively and quantitatively similar to those of realized variance.
variable might come from two sources. The first one is the high persistence of the variance process, so that volatility jumps lead to higher activity in subsequent periods. The second possibility is that jumps induce future variability by arriving in clusters or by increasing the volatility directly. Fulop et al. (2014) provide evidence of self-exciting jump clustering during turbulent market periods.

The second model we consider consists of explaining one-day ahead jump activity on the index return realized variances. This time we run the regression:

\[ JV_{t+1} = \beta_0 + \beta_1 \log(RV_t) + \beta_2 JV_t + \beta_3 IV_{\text{ATM}}^{\Delta_0} + \beta_4 ROV_{\text{ATM}}^{\Delta_0=0.1} + \beta_5 ROV_{\text{ATM}}^{\Delta_0=0.9} + \epsilon_t. \]  

(19)

Panel B of Table 9 shows that lagged values of \( RV \) and realized jump variance have explanatory power over future jump activity. This relationship provides evidence in favour of the parametric model we are considering in Section 2, since the intensity of the counting process governing jump returns is a function of volatility. Regarding \( IV \), its residual component also has a positive effect on jump realized variance. Contrary to what was observed in the \( RV \) regression, it is the \( ROV \) of the OTM put that now has a significant relation with the dependent variable. However, this time, the coefficient of the relationship is negative. Given that jump activity is measured as the difference between total and diffusive variance, the negative sign might indicate that the jump activity driving the \( ROV \) of OTM puts exerts more influence over future diffusive variance than return jumps. Notwithstanding, if we compare the magnitude of this coefficient with the one of OTM calls in Equation (18), we note that the significance of the relationship is more important in the latter case.

Lastly, we emphasize that the regression models of Equation (18) and (19) are of predictive nature, so our analysis focuses on the identification of variables that are important in the determination of future realized variances. The fact that \( R \)-squareds in Panel A are higher than those reported in Panel B just shows how much easier it is to predict future total variance than to predict variance induced by jump activity as variance is generally persistent.

6 Option Pricing Implications

This section provides empirical results using the model introduced in Section 2. We start by analyzing parameter estimates of the model with different sets of information. We then use these parameters to disentangle latent variables and analyze the information content of realized option variances. Finally, we look at the fitting performance using in- and out-of-sample analyses.
The index parameters are estimated using daily index returns, realized variances, bipower variations, daily option prices, and realized option variances (in the first case), from July 2004 to December 2012. Parameters are estimated using multiple simplex search method optimizations (fminsearch in Matlab) combined with the filter of Section 3. Robust standard errors are computed from the outer product of the gradient at optimal parameter values. Each day, we restrict our attention to OTM or ATM options with maturities closest to 30 and 90 days, and call-equivalent deltas closest to 0.2, 0.35, 0.5, 0.65 and 0.8.

### 6.1 Parameter Estimates

We obtain model parameters using the estimation procedure described in Section 3.2. As explained in Section 2.2, this procedure combines several observables into a likelihood function that is computed using a Monte Carlo based filtering approximation. The first set of observable variables are related to S&P 500 returns. We use daily log returns, realized variance, and bipower variation. The second set is composed of daily option prices and their realized variances (ROVs). Option prices come from OptionMetrics and correspond to European S&P 500 index option contracts. As argued in Bates (2000), the daily overabundance of option data becomes a hurdle for estimation routines, so we use a representative sample of options by restricting our attention to OTM or ATM options with maturities closest to 30 and 90 days, and call-equivalent deltas closest to 0.2, 0.35, 0.5, 0.65 and 0.8. Options with positive volume and bid price are included in the sample, as well as those satisfying the no-arbitrage conditions of Bakshi et al. (1997). Thus, the final sample of options is composed of a panel of 21,400 contracts.

To assess the information contained in ROVs, we first estimate the model excluding these variables.

---

**Table 10: S&P 500 parameter estimates.**

<table>
<thead>
<tr>
<th>Panel A: With ROV</th>
<th>Log-price process</th>
<th>Variance process</th>
<th>Standard deviations of error terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>η_Y</td>
<td>0.7635 (0.000230)</td>
<td>η_Y</td>
<td>-0.2199 (0.000013)</td>
</tr>
<tr>
<td>γ_Y</td>
<td>0.0058 (0.000010)</td>
<td>Γ_Y</td>
<td>0.5811 (0.000019)</td>
</tr>
<tr>
<td>ρ</td>
<td>-0.4336 (0.000011)</td>
<td>κ</td>
<td>5.7808 (0.000061)</td>
</tr>
<tr>
<td>λ_{3,0}</td>
<td>2.1530 (0.000498)</td>
<td>θ</td>
<td>0.0085 (0.000177)</td>
</tr>
<tr>
<td>λ_{3,1}</td>
<td>29.0744 (0.000682)</td>
<td>σ</td>
<td>0.8935 (0.000070)</td>
</tr>
<tr>
<td>µ_Y</td>
<td>-0.0050 (0.000018)</td>
<td>λ_{3,0}</td>
<td>7.6498 (0.000090)</td>
</tr>
<tr>
<td>σ_Y</td>
<td>0.0163 (0.000129)</td>
<td>µ_{3,0}</td>
<td>0.0214 (0.000101)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Without ROV</th>
<th>Log-price process</th>
<th>Variance process</th>
<th>Standard deviations of error terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>η_Y</td>
<td>1.8734 (0.003111)</td>
<td>η_γ</td>
<td>-1.5870 (0.00033)</td>
</tr>
<tr>
<td>γ_Y</td>
<td>0.0009 (0.000004)</td>
<td>Γ_γ</td>
<td>0.7061 (0.000016)</td>
</tr>
<tr>
<td>ρ</td>
<td>-0.3912 (0.000015)</td>
<td>κ</td>
<td>4.7614 (0.000107)</td>
</tr>
<tr>
<td>λ_{3,0}</td>
<td>0.0045 (0.025987)</td>
<td>θ</td>
<td>0.0030 (0.000825)</td>
</tr>
<tr>
<td>λ_{3,1}</td>
<td>30.0106 (0.000992)</td>
<td>σ</td>
<td>0.9121 (0.000010)</td>
</tr>
<tr>
<td>µ_Y</td>
<td>-0.0062 (0.000049)</td>
<td>λ_{3,0}</td>
<td>11.4438 (0.000065)</td>
</tr>
<tr>
<td>σ_Y</td>
<td>0.0189 (0.000464)</td>
<td>µ_{3,0}</td>
<td>0.0174 (0.000041)</td>
</tr>
<tr>
<td>V_0</td>
<td>0.0118 (0.000174)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
from the set of observables and then re-estimate it using the complete set. Table 10 reports parameter estimates and standard errors (in parentheses) obtained with and without the information from realized option variances. We first consider the parameters governing the jump process intensities. We observe a decrease in the intensity of the jump process governing volatility jumps, $\lambda_{V0}$, when ROVs are introduced, but an increase in the parameter governing the size of the jumps, $\mu_Y$. This suggests that ROV is detecting less frequent jumps with higher magnitudes in the volatility process. Regarding jumps in the log-equity price process, we observe an increase in the base intensity, $\lambda_{Y0}$, as well as in the average size of jumps, $\mu_Y$, in absolute value. These values show that ROV is favouring more frequent negative jumps with lower magnitudes, which is also observed from the average intensity – 0.0112 for the estimation with ROV and 0.0028 without.\(^{39}\)

We next provide visual evidence of the informativeness of ROVs for disentangling jumps. Figure 3 displays filtered price jumps and variance jumps excluding ROVs (first row) and with all data sources (second row). The dynamics of both jump processes consistently differ across sources, showing that jumps for log-prices are sporadic and largely negative on average and that jumps in volatility are more frequent and tend to be small on average. Note the striking decrease in the number and sizes of volatility jumps when ROV is employed in the filtering process.

### 6.2 Informativeness of Data Sources

Next, we focus on the informativeness of ROV about different latent variables of the model. Using the set of parameters estimated in Table 10, we filter different latent quantities using all data sources and compare the average standard deviations of these quantities with those obtained when filtering without ROV. Table 11 shows the posterior standard deviation (PSD) of these latent quantities under the two filtering procedures.\(^{40}\) The PSD of variance jumps decreases approximately five times when ROVs are added. This dramatic decrease confirms our simulated results that show how ROV embeds vital information about the presence of variance jumps. Note also, as it is the case in the simulation results, the PSDs of the instantaneous variances, quadratic variations, and integrated variances also fall when ROVs are included. Regarding the slight increase in the PSD for log-return jumps, this could be associated with the fact that these jumps are less frequent than jumps in volatility, which increases the estimation

\(^{39}\)For the estimation results including ROVs in the information set, 51.7% of variance jumps happen on days where a log-equity jump is filtered. This percentage of common jumps at a daily frequency shows that our specification is coherent with previous models such as the SVCJ, in which this feature is directly incorporated into the data process.

\(^{40}\)The posterior standard deviation is the standard deviation of the posterior density (as given by the particle filter). It allows us to assess the uncertainty around estimated latent variables.
Figure 3: Filtered log-equity price jumps (left panels) and variance jumps (right panels).
The figure shows the log-equity price jumps (left panels) and variance jumps (right panels). The top panels show the filtered
jumps based on returns, option implied volatilities, realized variances and bipower variations. The bottom panels also include
realized option variances as a new source of information.

Table 11: Posterior standard deviation of filtered quantities with and without ROV.

<table>
<thead>
<tr>
<th></th>
<th>With ROV</th>
<th>Without ROV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instantaneous variance</td>
<td>4.821×10⁻３</td>
<td>5.961×10⁻３</td>
</tr>
<tr>
<td>Quadratic variation</td>
<td>1.252×10⁻０５</td>
<td>1.410×10⁻０５</td>
</tr>
<tr>
<td>Integrated variance</td>
<td>1.088×10⁻０５</td>
<td>1.255×10⁻０５</td>
</tr>
<tr>
<td>Log-equity price jumps</td>
<td>2.886×10⁻０４</td>
<td>1.804×10⁻０４</td>
</tr>
<tr>
<td>Variance jumps</td>
<td>7.038×10⁻０４</td>
<td>3.297×10⁻０３</td>
</tr>
</tbody>
</table>

The posterior standard deviation is calculated from the particles generated in the particle filtering scheme. Parameters from
Table 10 are used to infer the filtered values and their posterior standard deviation.

uncertainty about this value.

6.3 Risk Premiums

We turn next to analyzing the impact of jump identification on equity and variance risk premiums. Since jumps play a
determinant role on the shorter end of the risk-premium term structure (see Bardgett et al., 2015), we focus our analysis on this part. We follow Bollerslev and Todorov (2011) and define
the integrated equity risk premium as:

\[
IERP(t, T) = \frac{1}{T - t} \left( \mathbb{E}_t^P [Y_T - Y_t] - \mathbb{E}_t^Q [Y_T - Y_t] \right)
\]
which can be decomposed into two parts: $IERP_{\text{Diffusion}}(t, T)$ and $IERP_{\text{Jump}}(t, T)$. Similarly, the integrated variance risk premium is defined as

$$IVRP(t, T) = \frac{1}{T-t} \left( E^P_t [\Delta QV_{t,T}] - E^Q_t [\Delta QV_{t,T}] \right)$$

and, again, could be decomposed into a diffusive and a jump component. These two integrated risk premiums are computed in closed-form solutions; derivations of these quantities are given in Internet Appendix B.

Figure 4 exhibits the evolution of one-day integrated equity ($IERP$) and variance risk premiums ($IVRP$) for the two information sets under consideration. In line with previous studies, we observe that both premiums vary significantly across time and have sporadic spikes during turbulent times. Regarding the $IERP$, we find that the average level of compensation decreases with the addition of $ROV$ to the information set – it passes from 4.51% to 3.39%. This decrease can be explained by the fact that $ROV$ is favouring more frequent negative price jumps, so compensation for bearing this type of risk increases (i.e., 0.06% without $ROV$ vs. 1.63% with $ROV$, on average) at the expense of a decrease in the compensation of equity diffusive risk (4.45% when $ROV$ is excluded and 1.76% when $ROV$ is used). However, given the infrequent character of these jumps, the combined compensation decreases in average with the addition of this new information.\footnote{Estimates for equity jump risk are close but somewhat lower than those reported in Broadie et al. (2007), Christoffersen et al. (2012), and Ornthanalai (2014). Variations in the sampling period, the datasets, and the methodologies could also explain these small differences to some extent.}

When we look at the effect of $ROV$ on the compensation for variance risk (rightmost panels of Figure 4), a different pattern emerges: the inclusion of $ROV$ increases the average premium for bearing this risk. Adding $ROV$ produces less frequent volatility jumps but increases their sizes, which impacts positively the average compensation for variance jump risk ($-3.24$ bps with $ROV$ vs. $-0.12$ bps without). In addition, variance diffusive risk premium decreases from $-0.78$ bps to $-0.17$ bps. In opposite to what was observed for $IERP$ case, the combined compensation increases with the incorporation of $ROV$ as jumps in volatility are more frequent than those in prices.

In conclusion, the previous results suggest that discontinuous risks’ premiums are underestimated when information about intraday option prices is excluded from the estimation step. The effect of this misspecification on the total compensation for equity and variance risk is not trivial, since the contribution of discontinuous risk play an important role at short horizons.
Figure 4: Integrated equity and variance risk premiums over a time horizon of 1 day (IERP and IVRP, respectively).

The figure shows both the integrated diffusive and jump risk premiums as given by the model. Both areas are stacked. The top panels show the filtered premiums based on returns, option implied volatilities, realized variances and bipower variations. The bottom panels also include realized option variances as a new source of information.

6.4 Out-of-Sample Assessment

To investigate whether the previous documented differences have significant implications on the model’s performance, we assess the goodness of fit of the two parameter sets reported in Section 3.2. We employ the parameters obtained over the sample period between July 2004 and December 2012, and use daily information between January 2013 and December 2013 to compute one-day ahead forecast errors.\(^{42}\) We first assess the ability of both parameter sets to fit historical implied volatilities. To perform this exercise, we use the relative implied volatility root mean square error (RIVRMSE), defined as follows:

\[
\text{RIVRMSE} = \sqrt{\left( \sum_k \sum_i O_k \left( \frac{IV_{k,t,i}(Y_{tk}, \hat{V}_{tk}) - \sigma_{BS}^{k,t,i}}{\sigma_{BS}^{k,t,i}} \right)^2 \right)},
\]

where \(IV\) is the one-day ahead model implied volatility, \(\hat{V}_{tk}\) is the one-day ahead instantaneous variance on day \(k\), and \(O_k\) represents the number of options in a subset of all the options available on day \(k\).\(^{43}\) Here, \(\sigma_{BS}\) denotes the Black-Scholes implied volatility associated with the observed option price. Regarding the sample, we employ all ATM and OTM options available in OptionMetrics for 2013 to compute

\(^{42}\text{We also perform an in-sample analysis that employs data over the same period the parameters were estimated. These results are presented in the Internet Appendix.}\)

\(^{43}\text{To compute model-predicted implied volatilities on day } t, \text{ we calculate one-day ahead expectations of model variables for day } t \text{ using the filter’s predictive distribution resulting from day } t - 1.\)
The leftmost columns of Table 12 show the RIVRMSE for the panel of 2013 options. We observe that the average fit provided by the parameter set with ROVs is better than the one that excludes these variances – an RIVRMSE of 28.94 with ROV vs. 31.95 without. When we analyze these results by moneyness and maturity, we also observe gains for all cases expect when the equivalent delta is greater than 0.80.

We use the Diebold and Mariano (1995, DM henceforth) test to see if the apparent predictive superiority of ROV based forecasts is not particular to this sample. Using both RIVRMSE time series, we compare their forecasting accuracy and test for:

\[ H_0 : \mathbb{E}[d_t] = 0, \forall t \]
\[ H_1 : \mathbb{E}[d_t] > 0, \forall t \]

where \( d_t = \text{RIVRMSE}_{\text{without}, t} - \text{RIVRMSE}_{\text{with}, t} \) is the time-\( t \) loss differential between the forecast produced without ROV and the one including it. The DM test statistic is 8.09 and is significant at a 1% level, confirming that there exists a differential between the two forecasts and that the one based on ROV information produces more accurate results on average.

In order to identify which contracts would benefit more from the inclusion of ROVs, we assess the goodness of fit of both parameter sets for these variances. To this end, we use a criterion similar to the RIVRMSE in Equation (20) and compute the relative realized option variance root mean square error (RROVRMSE), defined as:

\[
\text{RROVRMSE} = \left( \frac{1}{\sum_k \hat{O}_k} \sum_k \sum_{i=1}^{\hat{O}_k} \left( \frac{\Delta OQV(k-1)_{r,kr,i} - \text{ROV}(k-1)_{r,kr,i}}{\text{ROV}(k-1)_{r,kr,i}} \right)^2 \right)^{\frac{1}{2}},
\]

where \( \Delta OQV \) is the option quadratic variation increment computed from the model and ROV is the observed realized option variance. To compute RROVRMSEs, we employ a dataset of 68,857 options for which ROVs were available.

The rightmost columns of Table 12 shows RROVRMSE by moneyness and maturity. The parameter set based on ROVs produces the lowest forecast errors, as shown by the average means for the whole sample. As expected, the average fit of these quantities is lower since ROVs are included in the informa-

---

44 We restrict our analysis to maturities of at least one week and at most one year. As before, observations violating no-arbitrage restrictions are excluded.

45 The lag in the Diebold and Mariano (1995) is selected as the first partial autocorrelation that is within confidence bounds.
Nonetheless, it is remarkable to observe that the overall RROVRMSE is about 1.35 times lower when \( ROV \) is included and that there are significant differences across contracts. Note that the largest gains in RROVRMSE correspond to the biggest gains in RROVRMSE. Similar to the RIVRMSE case, we apply the DM test to both RROVRMSE series and obtain a value of 16.17, confirming statistically the important differences between the two sets of parameters.

The above evidence is consistent with the view of \( ROV \) as a new source of information to disentangle jumps in volatility and prices. The parameter set obtained with the addition of \( ROV \) supports a variance process that has less frequent jumps with higher magnitudes and a price process that has more frequent negative jumps with lower magnitudes. This set also has a different attribution of risk premiums between diffusive and discontinuous innovations. It seems most likely that these features are important in the pricing of options, as they consistently produce lower forecasting errors of implied volatilities in-sample and out-of-sample.

Table 12: One-day ahead out-of-sample performances, in terms of RIVRMSE and RROVRMSE (2013).

<table>
<thead>
<tr>
<th></th>
<th>RIVRMSE</th>
<th>RROVRMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>With ( ROV )</td>
<td>Without ( ROV )</td>
</tr>
<tr>
<td>DTM &lt; 60</td>
<td>18.85</td>
<td>19.09</td>
</tr>
<tr>
<td>60 ≤ DTM &lt; 120</td>
<td>18.62</td>
<td>20.75</td>
</tr>
<tr>
<td>120 ≤ DTM &lt; 180</td>
<td>22.50</td>
<td>27.59</td>
</tr>
<tr>
<td>180 ≤ DTM</td>
<td>50.14</td>
<td>55.54</td>
</tr>
<tr>
<td>( \Delta \varepsilon &lt; 0.2 )</td>
<td>34.97</td>
<td>45.40</td>
</tr>
<tr>
<td>0.20 ≤ ( \Delta \varepsilon &lt; 0.35 )</td>
<td>33.16</td>
<td>39.70</td>
</tr>
<tr>
<td>0.35 ≤ ( \Delta \varepsilon &lt; 0.5 )</td>
<td>36.38</td>
<td>41.39</td>
</tr>
<tr>
<td>0.50 ≤ ( \Delta \varepsilon &lt; 0.65 )</td>
<td>37.38</td>
<td>38.37</td>
</tr>
<tr>
<td>0.65 ≤ ( \Delta \varepsilon &lt; 0.80 )</td>
<td>30.85</td>
<td>31.08</td>
</tr>
<tr>
<td>0.80 ≤ ( \Delta \varepsilon )</td>
<td>21.18</td>
<td>19.61</td>
</tr>
<tr>
<td>All</td>
<td>28.94</td>
<td>31.95</td>
</tr>
</tbody>
</table>

The sample of S&P 500 options is acquired via OptionMetrics. Options violating arbitrage conditions were discarded. The IV sample size is 77,310. The sample of S&P 500 options is acquired via Tick Data. Options violating arbitrage conditions were discarded. The ROV sample size is 68,857. Model prices and ROVs are calculated by using the parameters of Table 10. RIVRMSEs and RROVRMSEs are given in percentages.

7 Concluding Remarks

This paper proposes the realized option variance as a new observable quantity to summarize information from intraday option prices. Recent nonparametric studies have shown that these prices can provide important information about latent characteristics of the data generating process. We show that including the information from the option prices has important implications when conducting parametric inference of option pricing models in terms of parameter identification and pricing errors. The observ-
able variables used in our analyses provide a portrait of their incremental information, helping with the identification of latent features governing the asset prices dynamics.

One of the reasons behind the information gains of this measure is its ability to capture different properties of the DGP depending on the moneyness of the option employed. Whereas ITM and ATM options provide ROVs that are responsive to discontinuous and diffusive innovations, the ROV of OTM options are sensitive to changes in the discontinuous part of the variance and the log-equity processes.

The paper empirically documents how realized option variance behaves for options on the S&P 500 and studies its relation with the realized variance of the index. For instance, using a principal component analysis for the surface of realized option variances, we find that ROV contains additional information when compared to the one embedded in realized variance. The paper further explores the economic implications of not using high frequency option prices for model estimation. Model parameters estimated without this information are not able to correctly disentangle diffusive and discontinuous innovations, which produces an incorrect attribution of risk premiums among these two competitive sources of risk.

References


A Pricing

A.1 Radon-Nikodym Derivative

Let $\mathcal{F}_t = \sigma \{ W_{V,t}, W_{\perp,t}, J_{Y,t}, J_{V,t}\}_{t \geq 0}$ be the $\sigma$–field generated by the past and actual noise terms. The market being incomplete, there are infinitely many equivalent martingale measures. We restrict the choice among those that have a Radon-Nikodym derivative of the form

$$\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = \exp \left( - \int_0^t \Lambda_{V,u} \, dW_{V,u} - \int_0^t \Lambda_{\perp,u} \, dW_{\perp,u} + \Gamma_Y J_{Y,t} + \Gamma_V J_{V,t} \right) \right),$$

the latter being an extended version of the Girsanov theorem. The predictable process $\{ \Lambda_{V,t} \}_{t \geq 0}$ and $\{ \Lambda_{\perp,t} \}_{t \geq 0}$ and the constants $\Gamma_Y$ and $\Gamma_V$ characterize the risk premiums embedded in this framework by linking the $P$– and $Q$–parameters together.

This approach is different from the one proposed in Duffie et al. (2000), Pan (2002b), and Broadie et al. (2007) in which the $P$– and $Q$–parameters are allow to vary independently. However, it shares...
similarities to Christoffersen et al. (2012) and Ornthanalai (2014) that consider GARCH models with jumps. It is also related to Bates (2000): in the latter, the author restricts the value of certain parameters to be consistent with the time series behaviour of returns.

Building on the properties of exponential martingales,\(^\text{46}\)

\[
E^P \left[ \exp \left( - \int_0^t \Lambda_{V_{t'}} dW_{V_{t'}} - \int_0^t \Lambda_{\perp_{t'}} dW_{\perp_{t'}} + \Gamma_{Y_{t'}} + \Gamma_{V_{t'}} \right) \right] \\
= \exp \left( \frac{1}{2} \int_0^t \left( \Lambda_{V_{t'}}^2 + \Lambda_{\perp_{t'}}^2 \right) dt + \left( \varphi^P_{Z_{t'}} (\Gamma_Y) - 1 \right) \int_0^t \Lambda_{Y_{t'}} dt + \left( \varphi^P_{Z_{t'}} (\Gamma_V) - 1 \right) \int_0^t \Lambda_{V_{t'}} dt \right)
\]

where \( \varphi^P_{Z_{t'}} (\Gamma_Y) \) and \( \varphi^P_{Z_{t'}} (\Gamma_V) \) represent the moment generating functions of the log-equity price and variance jump size,\(^\text{47}\)

\[ \varphi^P_{Z_{t'}} (\Gamma_Y) = \exp \left( \mu_Y \Gamma_Y + \frac{1}{2} \sigma_Y^2 \Gamma_Y^2 \right) \quad \text{and} \quad \varphi^P_{Z_{t'}} (\Gamma_V) = (1 - \Gamma_V \mu_V)^{-1}. \] (22)

A.2 Model under the Risk-Neutral Measure \( Q \)

For the diffusion components of the model, the risk-neutral Brownian motion are constructed in the usual way:

\[ W^Q_{V_{t'}} = W_{V_{t'}} + \int_0^t \Lambda_{V_{t'}} dt, \quad W^Q_{\perp_{t'}} = W_{\perp_{t'}} + \int_0^t \Lambda_{\perp_{t'}} dt \] (23)

and \( W^Q_{Y_{t'}} = \rho W_{V_{t'}} + \sqrt{1 - \rho^2} W_{\perp_{t'}} = W_{Y_{t'}} + \int_0^t \Lambda_{Y_{t'}} dt \) where \( \Lambda_{Y_{t'}} = \rho \Lambda_{V_{t'}} + \sqrt{1 - \rho^2} \Lambda_{\perp_{t'}} \).

The risk-neutral jump component are obtained from a direct comparison of the \( P \)- and \( Q \)-versions of the moment generating functions of the jump increments (\( J_{Y_{t'}} - J_{Y_{t}} \) and \( J_{V_{t'}} - J_{V_{t}} \)).\(^\text{48}\) Indeed, the

\(^{46}\)It is required that \( E^P \left[ \int_0^t \Lambda_{V_{t'}}^2 dt \right] < \infty \) and \( E^P \left[ \int_0^t \Lambda_{\perp_{t'}}^2 dt \right] < \infty \). Details are provided in Lemmas 4 and 6 of the Internet Appendix.

\(^{47}\)Provided that \( \Gamma_Y < \frac{1}{\rho^2} \).

\(^{48}\)Lemma 8 of the Internet Appendix shows that

\[
\varphi^P_{J_{Y_{t'}} - J_{Y_{t}}}(a) = E^P_{F_t} \left[ \exp \left( \mu_Y a + \frac{1}{2} \sigma_Y^2 a^2 \right) \right. \\
\varphi^P_{J_{Y_{t'}} - J_{Y_{t}}}(a) = E^P_{F_t} \left[ \exp \left( \frac{1}{2} \sigma_Y^2 \right) \right] \exp \left( \int_0^t \Lambda_{V_{t'}} dt \right) \\
\varphi^P_{J_{Y_{t'}} - J_{Y_{t}}}(a) = E^P_{F_t} \left[ \exp \left( \frac{1}{2} \sigma_Y^2 \right) \right] \exp \left( \varphi^P_{Z_{t'}} (\Gamma_Y) \int_0^t \Lambda_{Y_{t'}} dt \right) \\
\varphi^P_{J_{Y_{t'}} - J_{Y_{t}}}(a) = E^P_{F_t} \left[ \exp \left( (1 - a \mu_Y)^{-1} \right) \int_0^t \Lambda_{V_{t'}} dt \right] \\
\varphi^P_{J_{Y_{t'}} - J_{Y_{t}}}(a) = E^P_{F_t} \left[ \exp \left( (1 - a \varphi^P_{Z_{t'}} (\Gamma_Y) \mu_V)^{-1} \right) \varphi^P_{Z_{t'}} (\Gamma_Y) \int_0^t \Lambda_{V_{t'}} dt \right].
\]
change of measure affects the parameters:

\[ J_{Y,t}^Q = \sum_{n=1}^{N_{Y,t}^Q} Z_{Y,n}, \quad \text{and} \quad J_{V,t}^Q = \sum_{n=1}^{N_{V,t}^Q} Z_{V,n}, \]

where \( (N_{Y,t}^Q)_{t \geq 0} \) is a Cox process with predictable intensity \( (\lambda_{Y,t})_{t \geq 0} \), \( (N_{V,t}^Q)_{t \geq 0} \) is a Poisson process of intensity \( \lambda_{V,0}^Q \) and

\[
\begin{align*}
\lambda_{Y,t}^Q &= \varphi_{Z_t}^P (\Gamma_Y) \lambda_{Y,t}, \\
\mu_{Y}^Q &= \mu_Y + \Gamma_Y \sigma_Y^2, \\
\sigma_{Y}^Q &= \sigma_Y,
\end{align*}
\]

\[
\begin{align*}
\lambda_{V,t}^Q &= \varphi_{Z_t}^P (\Gamma_V) \lambda_{V,0}, \\
\mu_{V}^Q &= \varphi_{Z_t}^P (\Gamma_V) \mu_V.
\end{align*}
\]

The risk-neutral dynamics of the log-price process is established by imposing the discounted price process \( \{ \exp (\left((q - r) t \right) \exp (Y_t)) \}_{t \geq 0} \) to be a \( Q \)- martingale where \( r \) is the risk-free rate and \( q \) is the dividend rate.\(^{49}\) To get a semi-closed form for option prices, the risk-neutral stochastic differential equation (SDE) of the variance process is assumed to have a mean reverting behaviour, as in Heston (1993) among others. Implicitly, it constrains \(^{50}\)

\[
\Lambda_{V,t} = \eta_V \sqrt{V_t}
\] (24)

and the model under the risk-neutral measure is thus

\[
\begin{align*}
dY_t &= \alpha_{Y,t}^Q dt + \sqrt{\lambda_Y} dW_{Y,t} + dJ_{Y,t}^Q, \\
dV_t &= \kappa^Q \left( \theta^2 - V_t \right) dt + \sigma \sqrt{V_t} dW_{V,t} + dJ_{V,t}^Q,
\end{align*}
\] (25) (26)

where the correspondence between the \( \mathbb{P} \)- and \( \mathbb{Q} \)- parameters is

\[
\begin{align*}
\alpha_{Y,t}^Q &= r - q - \frac{1}{2} \lambda_Y - \left( \varphi_{Z_t}^P (1) - 1 \right) \lambda_{Y,t}, \\
\kappa^Q &= \kappa + \sigma \eta_V, \\
\lambda_{V,t}^Q &= \varphi_{Z_t}^P (\Gamma_Y) \lambda_{Y,t}, \\
\theta^Q &= \frac{\kappa \theta}{\kappa + \sigma \eta_V}.
\end{align*}
\]

\(^{49}\)Details are provided in Lemma 9 of the Internet Appendix.\(^{50}\) Indeed,

\[
\begin{align*}
dV_t &= \kappa^Q \left( \theta^Q - V_t \right) dt + \sigma \sqrt{V_t} dW_{V,t} + dJ_{V,t}^Q \\
&= \kappa^Q \left( \theta - V_t \right) - \sigma \sqrt{V_t} \Lambda_{V,0}^Q dt + \sigma \sqrt{V_t} dW_{V,t} + dJ_{V,t}^Q \\
&= \left( \kappa + \sigma \eta_V \right) \left( \frac{\kappa \theta}{\kappa + \sigma \eta_V} - V_t \right) dt + \sigma \sqrt{V_t} dW_{V,t} + dJ_{V,t}^Q.
\end{align*}
\]
A.3 Pricing

The price of a European call option with strike price $K$ and maturity $T$ is

$$C_t(Y_t, V_t) = \exp(Y_t) \exp(-q(T-t)) P_1(Y_t, V_t) - K \exp(-r(T-t)) P_2(Y_t, V_t)$$

(27)

where

$$P_1(y, v) = \frac{1}{2} \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{\exp(-iku) \varphi_{Y_t[Y_t, V_t]}(u, y, v)}{u} \right) du,$$

$$P_2(y, v) = \frac{1}{2} \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{\exp(-iku) \varphi_{Y_t[Y_t, V_t]}(u, y, v)}{u} \right) du.$$

and $\varphi_{Y_t[Y_t, V_t]}(u, y, v) = \mathbb{E}_t[\exp(uY_T)|Y_t = y, V_t = v]$ is the moment generating function of $Y_T$ conditional on time $t$ information. More precisely,

$$\varphi_{Y_t[Y_t, V_t]}(u, y, v) = \exp(\mathcal{A}(u, t, T) + uY_t + C(u, t, T) V_t)$$

where

$$C(u, t, T) = \frac{2C_0(\exp(-C_2(T-t)) - 1)}{C_2(\exp(-C_2(T-t)) + 1) - C_1(\exp(-C_2(T-t)) - 1)},$$

$$C_0 = \lambda_{Y_1}^Q(\varphi_{Z_0}^Q(1) - 1) u - \lambda_{Y_1}^Q(\varphi_{Z_0}^Q(u) - 1) + \frac{u - u^2}{2},$$

$$C_1 = \kappa^Q - \rho \sigma u,$$

$$C_2 = \sqrt{C_1^2 + 2\sigma^2 C_0},$$

and

$$\mathcal{A}(u, t, T) = D_0(T-t) + 6^2 \kappa^Q g_1(t, T) + \lambda_{Y_0}^Q g_2(t, T)$$

(28)

$$D_0 = -r + (r - q)u + \lambda_{Y_0}^Q(\varphi_{Z_0}^Q(1) - 1) u - \lambda_{Y_0}^Q(\varphi_{Z_0}^Q(u) - 1)$$

$$g_1(t, T) = \frac{2 \log(2C_2) - 2 \log\left(C_1 \left(e^{C_2(t-T)} - 1\right) + C_2 \left(e^{C_2(t-T)} + 1\right) + (C_1 + C_2)(T-t)\right)}{-\sigma^2}$$

$$g_2(t, T) = \frac{\mu^Q_V}{2} \frac{2 \log(2C_2) - 2 \log\left(C_1 \left(e^{C_2(t-T)} - 1\right) + C_2 \left(e^{C_2(t-T)} + 1\right) + 2C_0\mu^Q_V \left(e^{C_2(t-T)} - 1\right)\right)}{C_0(\mu^Q_V)^2 + C_1\mu^Q_V - \frac{\sigma^2}{2}(T-t)}$$

$$+ \frac{\mu^Q_V}{2} \frac{2C_0\mu^Q_V + C_1 - C_2}{C_0(\mu^Q_V)^2 + C_1\mu^Q_V - \frac{\sigma^2}{2}(T-t)}$$

43
The approach is inspired from Heston (1993) that relies on an inversion similar to the one of Gil-Pelaez (1951). The explicit form of the moment generating function is similar to the ones found by Filipovic and Mayerhofer (2009) and Duffie et al. (2000).\(^\text{51}\)

### B Estimation

#### B.1 Drift Term

The drift term process of Equation (1) is

\[
\begin{align*}
\alpha_{\text{u}_t}^p &= r - q - \frac{1}{2} V_{\text{u}_t} + \sqrt{V_{\text{u}_t}} \Lambda_{Y_{\text{u}_t}} + \left( \phi_{Z_{\text{u}_t}}^p (\Gamma_{Y}) - \phi_{Z_{\text{u}_t}}^p (1 + \Gamma_{Y}) \right) \lambda_{Y_{\text{u}_t}} \\
&= r - q - \frac{1}{2} V_{\text{u}_t} + \sqrt{V_{\text{u}_t}} \Lambda_{Y_{\text{u}_t}} + \left( \gamma_{Y} - \left( \phi_{Z_{\text{u}_t}}^p (1) - 1 \right) \right) \lambda_{Y_{\text{u}_t}},
\end{align*}
\]

where the moment generating functions, \(\phi_{Z_{\text{u}_t}}^p (\Gamma_{Y})\) and \(\phi_{Z_{\text{u}_t}}^p (\Gamma_{Y})\), of the log-equity price and variance jump size are defined at Equation (22). To obtain a drift term that depends on the instantaneous variance only, it is assumed that

\[
\Lambda_{\perp_{Y_{\text{u}_t}}} = \eta_{\perp} \sqrt{V_{\text{u}_t}}.
\]

Consequently, letting \(\eta_{Y} = \rho \eta_{V} + \sqrt{1 - \rho^2} \eta_{\perp}\) yields \(\gamma_{Y_{\text{u}_t}} = (\rho \eta_{V} + \sqrt{1 - \rho^2} \eta_{\perp}) \sqrt{V_{\text{u}_t}}\), and

\[
\alpha_{\text{u}_t}^p = r - q + \left( \eta_{Y} - \frac{1}{2} \right) V_{\text{u}_t} + \left( \gamma_{Y} - \left( \phi_{Z_{\text{u}_t}}^p (1) - 1 \right) \right) \lambda_{Y_{\text{u}_t}}.
\]

**Sketch of the proof.** Let

\[
Z_t = \left\{ - \int_0^t \Lambda_{V_{\text{u}_t}} dW_{V_{\text{u}_t}} - \int_0^t \Lambda_{\perp_{Y_{\text{u}_t}}} dW_{\perp_{Y_{\text{u}_t}}} - \frac{1}{2} \int_0^t \left( \Lambda_{V_{\text{u}_t}}^2 + \Lambda_{\perp_{Y_{\text{u}_t}}}^2 \right) du + \Gamma_{Y} J_{Y_{\text{u}_t}} - \left( \phi_{Z_{\text{u}_t}}^p (\Gamma_{Y}) - 1 \right) \int_0^t \lambda_{Y_{\text{u}_t}} du + \Gamma_{V} J_{V_{\text{u}_t}} - \left( \phi_{Z_{\text{u}_t}}^p (\Gamma_{V}) - 1 \right) \int_0^t \lambda_{V_{\text{u}_t}} du \right\}
\]

be associated to the Radon-Nikodym derivative of Equation (21) and \(D_t = \exp((-r - q)t)\) be the combination of the discount factor and the dividend yield. Since the discounted price have to be a \(Q\)-martingale, then for all \(0 < s < t\),

\[
1 = \mathbb{E}^Q_s \left[ \frac{D_t \exp(Y_t)}{D_s \exp(Y_s)} \right] = \mathbb{E}^p_s \left[ \frac{D_t \exp(Y_t + Z_t)}{D_s \exp(Y_s + Z_s)} \right],
\]

\(^{\text{51}}\)Details are available in the Internet Appendix C.
which means that \( \{D_t \exp(Y_t + Z_t)\}_{t \geq 0} \) is a \( \mathbb{P} \)-martingale. The computation of this last expectation leads to the final result.\(^{52}\)

\[ \square \]

### B.2 Filtering Procedure

#### B.2.1 Intraday Simulation

In what follows, \( h \) is set to \( \tau / M \) implying that the path simulation is performed with \( M \) steps per day. Assuming that \( Y_t \) and \( V_t \) are known, \( V_{t+h}, \Delta I_{t,t+h} \) and \( Y_{t+h} \) are generated as follows:

1. The jump indicator functions \( \mathbf{1}_{\{N_{V,t+h} - N_{V,t} = 1\}} \) and \( \mathbf{1}_{\{N_{V,t+h} - N_{V,t} = 1\}} \) are generated from independent Bernoulli random variables with a success probabilities of \( \lambda_{V,t} h \) and \( \lambda_{V,t} h \) respectively.\(^{53}\)

2. If needed, \( Z_{Y,t+h} \sim \mathcal{N}(\mu_Y; \sigma_Y^2) \) and \( Z_{V,t+h} \sim \exp(\mu_V) \) are simulated.

3. Time \( t + h \) variance \( V_{t+h} \) is simulated according to

\[
V_{t+h} = V_t + \kappa (\theta - V_t) h + \sigma \sqrt{V_t} (W_{V,t+h} - W_{V,t}) + Z_{V,t+h} \mathbf{1}_{\{N_{V,t+h} - N_{V,t} = 1\}}.
\]

where \( \mathbf{1}_{\{N_{V,t+h} - N_{V,t} = 1\}} \) is an indicator function that is worth 1 if there is a jump during the interval and 0 otherwise. Indeed, it corresponds to the Euler approximation of the variance process (2).

4. The integrated variance increment \( \Delta I_{t,t+h} = \int_t^{t+h} V_u \, du \) is simulated conditionally on \( V_t \) and \( V_{(t+h)^-} \). More precisely, the cumulant generating function of \( \Delta I_{t,t+h} \) is

\[
\log \left( \mathbb{E}^\mathbb{P} \left[ \exp(a \Delta I_{t,t+h}) | V_t, V_{(t+h)^-} \right] \right) = \log \left( 1 + \sum_{m=1}^\infty \mathbb{E}^\mathbb{P} \left[ (\Delta I_{t,t+h})^m | V_t, V_{(t+h)^-} \right] \frac{a^m}{m!} \right)
\]

\[
\approx \mathbb{E}^\mathbb{P} \left[ \Delta I_{t,t+h} | V_t, V_{(t+h)^-} \right] a + \operatorname{Var}^\mathbb{P} \left[ \Delta I_{t,t+h} | V_t, V_{(t+h)^-} \right] \frac{a^2}{2},
\]

which is the cumulant generating function of a Gaussian distribution. The conditional density function of \( \Delta I_{t,t+h} \) is approximated by

\[
\frac{1}{\sqrt{2\pi \operatorname{Var}^\mathbb{P} \left[ \Delta I_{t,t+h} | V_t, V_{(t+h)^-} \right]}} \exp \left( -\frac{1}{2} \frac{\left( x - \mathbb{E}^\mathbb{P} \left[ \Delta I_{t,t+h} | V_t, V_{(t+h)^-} \right] \right)^2}{\operatorname{Var}^\mathbb{P} \left[ \Delta I_{t,t+h} | V_t, V_{(t+h)^-} \right]} \right).
\]

\(^{52}\)Lemma 10 of the Internet Appendix provides all details.

\(^{53}\)Because of the Poisson process properties, the probability of observing a single jump is approximately \( \lambda_{V,t} h \) when \( h \) is small.
The first two centred moments of $\Delta I_{t,h}$ are available in closed form in Tse and Wan (2013).

5. The log-price is simulated according to $^{54,55,56}$

$$
Y_{t+h} = Y_t + c_1 h + c_2 \Delta I_{t,h} + \rho \sigma^{-1} (V_{(t+h)^-} - V_t) + \sqrt{1 - \rho^2} \int_t^{t+h} \sqrt{V_u} \, dW_{u,t} - \frac{\rho}{\sigma} \left( \sum_{n=N_y,t+1}^{N_y,t+h} Z_{V,n} \right) + \sum_{n=N_y,t+1}^{N_y,t+h} Z_{Y,n}.
$$

where

$$
c_1 = r - q + \left( \varphi_{Z_Y} (\Gamma_Y) - \varphi_{Z_Y} ((1 + \Gamma_Y)) \right) \lambda_{Y,t} - \frac{\rho \kappa \theta}{\sigma},
c_2 = \eta_Y - \frac{1}{2} + \left( \varphi_{Z_Y} (\Gamma_Y) - \varphi_{Z_Y} ((1 + \Gamma_Y)) \right) \lambda_{Y,1} + \frac{\rho}{\sigma} \kappa.
$$

In fact, from the integral version of Equation (2),

$$
\int_t^{t+h} \sqrt{V_u} \, dW_{V,u} = \sigma^{-1} \left( V_{(t+h)^-} - V_t - \kappa \theta h + \kappa \int_t^{t+h} V_u \, du - \sum_{n=N_y,t+1}^{N_y,t+h} Z_{V,n} \right).
$$

Replacing back in the integral form of the log-equity price (1), substituting the drift term of Equation (31) and noting that $\lambda_{Y,t-} = \lambda_{Y,0} + \lambda_{Y,1} V_{t-}$ completes the construction.

### B.2.2 Time Aggregation

Let $t = (k - 1) \tau$. Once Steps 1 to 5 of Appendix B.2.1 have been executed for each intraday periods $[t + jh]_{j=1}^M$, the integrated variance and quadratic variation increments are approximated by the aggregation of the simulated variables:

$$
\Delta I_{t,t+j} \equiv \sum_{j=1}^M \Delta I_{t+(j-1)h, t+jh} \text{ and } \Delta QV_{t,t+j} \equiv \Delta I_{t,t+j} + \sum_{j=1}^M Z_{Y,t+jh} I_{[N_y,t+jh,N_{y,t+jh+1}]}.
$$

$^{54}$Again, for small $h > 0$, the two last terms may be approximated with

$$
-\frac{\rho}{\sigma} Z_{(t+j)h} I_{[N_{y,t+jh},N_{y,t+jh+1}]} + Z_{Y,t+jh} I_{[N_{y,t+jh},N_{y,t+jh+1}]}.
$$

$^{55}$As in Broadie and Kaya (2006), $\int_t^{t+h} \sqrt{V_u} \, dW_{L,u}$ is approximated with a Gaussian distribution of mean zero and variance $\Delta I_{t,h}$. Indeed, since the instantaneous variance is stochastic, the distribution of $\int_t^{t+h} \sqrt{V_u} \, dW_{L,u}$ is not truly Gaussian. Though, it would have been the case if the variance was a deterministic function of time. The variance of $\int_t^{t+h} \sqrt{V_u} \, dW_{L,u}$ is $E^V \left[ \int_t^{t+h} V_u \, du \right] = E^V \left[ \Delta I_{t,h} \right]$.

$^{56}$Note that the log-prices are simulated intraday but the observed log-price is used whenever it is available, that is at the end of each day. These intermediary log-prices are required in the computation of $\Delta QV_{t,t-h}$. 

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Note that a time aggregation of Equation (33) suggests the approximation

\[ Y_{t+\tau} - Y_t = \sum_{j=1}^{M} (Y_{t+jh} - Y_{t+(j-1)h}) \equiv \mu_{t+\tau} + \sigma_{t+\tau}^2 \epsilon_{t+\tau} \]

where \( \epsilon_{t+\tau} \) is a standard normal random variable,

\[ \mu_{t+\tau} = c_1 \tau + c_2 \tau I_{t+\tau} + \frac{\rho}{\sigma} (V_{t+\tau} - V_t) \]

\[ -\frac{\rho}{\sigma} \sum_{j=1}^{M} Z_{V_{t+jh}} I_{t+jh} \mathbb{1}_{\{N_{t+jh} - N_{t+(j-1)h} = 1\}} + \sum_{j=1}^{M} Z_{Y_{t+jh}} I_{t+jh} \mathbb{1}_{\{N_{Y_{t+jh}} - N_{Y_{t+(j-1)h}} = 1\}} \]

and \( \sigma_{t+\tau}^2 = (1 - \rho^2) \tau I_{t+\tau} \). The model option implied volatility \( IV_{t+\tau}(Y_{t+\tau}, V_{t+\tau}) \) is calculated based on Equation (27).\(^5\) Finally, the Euler discretization of Equation (10) provides an approximation of the option quadratic variation:\(^6\)

\[ \Delta OQV_{t+\tau} = h \sum_{j=0}^{M-1} \left( \left( \frac{\partial O_{t+jh}}{\partial y} (Y_{t+jh}, V_{t+jh}) \right)^2 + \sigma^2 \left( \frac{\partial O_{t+jh}}{\partial y} (Y_{t+jh}, V_{t+jh}) \right)^2 \right) \]

\[ + 2\sigma \rho \frac{\partial O_{t+jh}}{\partial y} (Y_{t+jh}, V_{t+jh}) \frac{\partial O_{t+jh}}{\partial y} (Y_{t+jh}, V_{t+jh}) \Delta I_{t+jh} \]

\[ + \sum_{j=1}^{M} \{ O_{t+jh} (Y_{t+jh}, V_{t+jh}) - O_{t+jh} (Y_{t+jh}, V_{t+jh}) \}^2 \].

Internet Appendix C.2.1 provides details about how these derivatives are computed. At the end of this stage, a simulated vector

\[ x_{t+\tau} = (Y_{t+\tau}, V_{t+\tau}, \Delta I_{t+\tau}, QV_{t+\tau}, O_{t+\tau}^{(i)}(Y_{t+\tau}, V_{t+\tau}), \Delta OQV_{t+\tau}^{(i)} : i \in \{1, 2, \ldots, n_i\}) \]

is obtained at the end of the day.

---

\(^5\)\(^5\)\(^6\)The log-equity process is only observable on a daily basis. However, \( \Delta OQV_{t+jh} \) requires intraday log-equity values. Therefore, it is convenient to treat intraday log-equity values as latent variables.
A Model Construction

A.1 General Properties of Moment Generating Functions

The jumps component are model using Cox processes. Moment generating function for such processes are involved in many proofs.

Lemma 1. Let \( \varphi^\mathbb{P}_X (a) = \mathbb{E}^\mathbb{P} [\exp (aX)] \) be the moment generating function of the random variable \( X \). Assume that \( \{X_k\}_{k=1}^\infty \) is a sequence of independent and identically distributed random variables. Then

\[
\varphi^\mathbb{P}_{\sum_{i=1}^n X_i} (a) = \prod_{i=1}^n \mathbb{E}^\mathbb{P} [\exp (aX_i)] = \left( \varphi^\mathbb{P}_{X_1} (a) \right)^n. \tag{36}
\]

If \( N \) is a Poisson distributed random variable of expectation \( \lambda \), then the law of iterated expectation implies that

\[
\varphi^\mathbb{P}_{\sum_{i=1}^N X_i} (a) = \mathbb{E}^\mathbb{P} \left[ \mathbb{E}^\mathbb{P} \left[ \exp \left( a \sum_{k=1}^N X_k \right) \right] N \right] = \sum_{n=0}^\infty \left( \varphi^\mathbb{P}_{X_1} (a) \right)^n \exp (\lambda) \frac{\lambda^n}{n!} = \exp \left( \lambda \varphi^\mathbb{P}_{X_1} (a) - 1 \right). \tag{37}
\]

If \( \{N_t\}_{t\geq 0} \) is a Cox process with predictable intensity \( \{\lambda_t\}_{t\geq 0} \), then

\[
\varphi^\mathbb{P}_{\sum_{i=1}^{N_t} X_i} (a) = \mathbb{E}^\mathbb{P} \left[ \mathbb{E}^\mathbb{P} \left[ \exp \left( a \sum_{i=1}^{N_t} X_i \right) \right] \int_0^t \lambda_u du \right] = \mathbb{E}^\mathbb{P} \left[ \exp \left( \left( \int_0^t \lambda_u du \right) (\varphi^\mathbb{P}_{X_1} (a) - 1) \right) \right]. \tag{38}
\]

Lemma 2. If \( X \sim \mathcal{N}(\mu, \sigma^2) \), then \( \varphi^\mathbb{P}_X (a) = \exp (\mu a + \frac{1}{2} a^2 \sigma^2) \).

Lemma 3. If \( X \) is exponentially distributed of expectation \( \mu \), then \( \varphi^\mathbb{P}_X (a) = (1 - a\mu)^{-1} \) provided that \( a < \frac{1}{\mu} \).

A.2 Exponential Martingales

Many proofs are based on properties of exponential martingales. The following three lemmas are the central pivots of many proofs.

Lemma 4. If \( \mathbb{E}^\mathbb{P} \left[ \int_0^t \Lambda^2_{V,u} \, du \right] < \infty \) and \( \mathbb{E}^\mathbb{P} \left[ \int_0^t \Lambda^2_{\perp,u} \, du \right] < \infty \), then

\[
\left\{ \exp \left( - \int_0^t \Lambda_{V,u} \, dW_{V,u} - \int_0^t \Lambda_{\perp,u} \, dW_{\perp,u} - \frac{1}{2} \int_0^t \left( \Lambda^2_{V,u} + \Lambda^2_{\perp,u} \right) \, du \right) \right\}_{t\geq 0}
\]

is a \( \mathbb{P} \)-martingale of expectation 1.

Lemma 5. For \( Z \in \{Y, V\}, \left\{ J_{Z,t} - \mu_Z \int_0^t \lambda_{Z,s} \, ds \right\}_{t\geq 0} \) is a \( \mathbb{P} \)-martingale.

Lemma 6. For \( X \in \{Y, V\}, \left\{ \exp \left( \Gamma_X J_{X,t} - \left( \varphi^\mathbb{P}_{Z_X} (\Gamma_X) - 1 \right) \int_0^t \lambda_{X,u} \, du \right) \right\}_{t\geq 0} \) is a \( \mathbb{P} \)-martingale where \( \varphi^\mathbb{P}_{Z_X} \) is the moment generating function of the jump size \( Z_X \).
A.2.1 Proofs

Proof of Lemma 4. The continuous process \( \{X_t\}_{t \geq 0} \) where

\[
X_t = - \int_0^t \Lambda_{V_{u-}} dW_{V_u} - \int_0^t \Lambda_{1_{u-}} dW_{1_u}
\]

is a \( \mathbb{P} \)-martingale. Its quadratic variation is \( [X, X]_t = \int_0^t \left( \Lambda_{V_{u-}}^2 + \Lambda_{1_{u-}}^2 \right) du \). Using Itô’s lemma,

\[
d \exp \left( X_t - \frac{1}{2} [X, X]_t \right) = \exp \left( X_0 - \frac{1}{2} [X, X]_0 \right) \exp \left( \int_0^t \left( X_s - \frac{1}{2} [X, X]_s \right) ds \right),
\]

Because \( \exp \left( X_t - \frac{1}{2} [X, X]_t \right) \) is a stochastic integral with respect to a martingale, \( \{\exp \left( X_t - \frac{1}{2} [X, X]_t \right)\}_{t \geq 0} \) is a \( \mathbb{P} \)-martingale. Since \( X_0 = 0 \), then

\[
\mathbb{E}^\mathbb{P} \left[ \exp \left( X_t - \frac{1}{2} [X, X]_t \right) \right] = \mathbb{E}^\mathbb{P} \left[ \exp \left( X_0 - \frac{1}{2} [X, X]_0 \right) \right] = 1.
\]

Proof of Lemma 5.

\[
\mathbb{E}^\mathbb{P}_{F_s} \left[ J_{Z,t} - J_{Z,s} - \mu Z \int_0^t \lambda_{Z_{u-}} du \right]
\]

\[
= \mathbb{E}^\mathbb{P}_{F_s} \left[ \mathbb{E}^\mathbb{P}_{F_s} \left[ J_{Z,t} - J_{Z,s} - \mu Z \int_0^t \lambda_{Z_{u-}} du \right] \right]
\]

\[
= \mathbb{E}^\mathbb{P}_{F_s} \left[ \sum_{n=N_{Z,t}+1}^{N_{Z,t}} Z_{Y,n} - \mu Z \int_0^t \lambda_{Z_{u-}} du \right]
\]

\[
= \mathbb{E}^\mathbb{P}_{F_s} \left[ \sum_{n=N_{Z,t}+1}^{N_{Z,t}} Z_{Y,n} - \mu Z \int_0^t \lambda_{Z_{u-}} du \right]
\]

\[
= \mathbb{E}^\mathbb{P}_{F_s} \left[ \sum_{n=0}^{\infty} n \mu Z \exp \left( - \int_0^t \lambda_{Z_{u-}} du \right) \left( \frac{\int_0^t \lambda_{Z_{u-}} du}{n!} \right)^n \right]
\]

\[
= \mathbb{E}^\mathbb{P}_{F_s} \left[ \mu Z \int_0^t \lambda_{Z_{u-}} du - \mu Z \int_0^t \lambda_{Z_{u-}} du \right]
\]

\[
= 0.
\]

Proof of Lemma 6. From the moment generating function of a compound Poisson process with stochastic jump intensity of Equation (38),

\[
\mathbb{E}^\mathbb{P}_{F_s} \left[ \exp \left( \Gamma_X (J_{X,t} - J_{X,s}) - \left( \int_s^t \lambda_{X_{u-}} du \right) \left( \phi_{\mathbb{P}} (\Gamma_X) - 1 \right) \right) \right]
\]

\[
= \mathbb{E}^\mathbb{P}_{F_s} \left[ \exp \left( \Gamma_X \left( \sum_{n=N_{X,t}+1}^{N_{X,t}} Z_{X,n} \right) - \left( \int_s^t \lambda_{X_{u-}} du \right) \left( \phi_{\mathbb{P}} (\Gamma_X) - 1 \right) \right) \right]
\]

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\[
\mathbb{E}_F^P \left[ \exp \left( \int_s^t \lambda_{X,u} \, du \right) \left( \varphi_{Zs}^P (\Gamma_X) - 1 \right) - \left( \int_s^t \lambda_{X,u} \, du \right) \left( \varphi_{Zs}^P (\Gamma_X) - 1 \right) \right] = 0.
\]

For the log-equity price jumps,

\[
\varphi_{Zt}^P (\Gamma_Y) - 1 = \exp \left( \mu_Y \Gamma_Y + \frac{1}{2} \sigma_Y^2 \Gamma_Y^2 \right) - 1,
\]

while for the variance jumps,

\[
\varphi_{Zt}^P (\Gamma_V) - 1 = (1 - \Gamma_V \mu_V)^{-1} - 1
\]

provided that \( \Gamma_V < \mu_V^{-1} \).

\[\square\]

### A.3 Risk-Neutral Innovations

Lemma 7 justifies Equation (23). Lemma 8 determines the jump dynamics under \( \mathbb{Q} \). These are components of the risk-neutral version of the model presented in Appendices A.1 and A.2.

**Lemma 7.** For \( X \in \{Y, \perp\} \), the risk-neutral moment generating function of the Brownian increments \( W_{X,t} - W_{X,s} \) satisfies

\[
\varphi_{W_{X,t} - W_{X,s}}^\mathbb{Q} (a) = \mathbb{E}_F^P \left[ \exp \left( -a \int_s^t \lambda_{X,u} \, du + \frac{a^2 (t - s)}{2} \right) \right]
\]

which is the moment generating function of a Gaussian distribution of expectation \( \mathbb{E}_F^P \left[ -a \int_s^t \lambda_{X,u} \, du \right] \) and variance \( t - s \).

**Lemma 8.** For \( X \in \{Y, V\} \), the risk-neutral moment generating function of the jump increments \( J_{Y,t} - J_{Y,s} \) is given by

\[
\mathbb{E}_F^Q \left[ \exp \left( a (J_{X,t} - J_{X,s}) \right) \right] = \mathbb{E}_F^P \left[ \exp \left( \left( \varphi_{Zs}^P (a + \Gamma_X) - \varphi_{Zs}^P (\Gamma_X) \right) \int_s^t \lambda_{X,u} \, du \right) \right].
\]

In particular

\[
\varphi_{J_{Y,t} - J_{Y,s}}^\mathbb{Q} (a) = \mathbb{E}_F^Q \left[ \exp \left( a (J_{Y,t} - J_{Y,s}) \right) \right] = \mathbb{E}_F^P \left[ \exp \left( \left( \mu_Y (a + \Gamma_Y) + \frac{1}{2} (a + \Gamma_Y)^2 \sigma_Y^2 \right) - \exp \left( \mu_Y \Gamma_Y + \frac{1}{2} \Gamma_Y^2 \sigma_Y^2 \right) \right) \right]
\]

and

\[
\varphi_{J_{V,t} - J_{V,s}}^\mathbb{Q} (a) = \mathbb{E}_F^Q \left[ \exp \left( a (J_{V,t} - J_{V,s}) \right) \right] = \mathbb{E}_F^P \left[ \exp \left( \left( (1 - (a + \Gamma_V) \mu_{YV})^{-1} - (1 - \Gamma_V \mu_{YV})^{-1} \right) \int_s^t \lambda_{Y,u} \, du \right) \right]
\]

provided that \( \Gamma_V < \mu_{YV}^{-1} \) and \( a + \Gamma_V < \mu_{YV}^{-1} \).

A comparison of the \( \mathbb{P} \)– and the \( \mathbb{Q} \)–versions of the between the \( \mathbb{P} \)– and \( \mathbb{Q} \)–parameters:

\[
\varphi_{J_{Y,t} - J_{Y,s}}^P (a) = \mathbb{E}_F^P \left[ \exp \left( \int_s^t \lambda_{Y,u} \, du \left( \exp \left( \mu_Y a + \frac{1}{2} \sigma_Y^2 a^2 \right) - 1 \right) \right) \right],
\]

\[
\varphi_{J_{Y,t} - J_{Y,s}}^Q (a) = \mathbb{E}_F^P \left[ \exp \left( \mu_Y \Gamma_Y + \frac{1}{2} \Gamma_Y^2 \sigma_Y^2 \right) \int_s^t \lambda_{Y,u} \, du \frac{\exp \left( \mu_Y (a + \Gamma_Y) + \frac{1}{2} (a + \Gamma_Y)^2 \sigma_Y^2 \right)}{\exp \left( \mu_Y \Gamma_Y + \frac{1}{2} \Gamma_Y^2 \sigma_Y^2 \right)} - 1 \right] \right]
\]

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= \mathbb{E}_F^P \left[ \exp \left( \exp \left( \mu_Y + \frac{1}{2} \left( \Gamma_Y^2 \sigma_Y^2 \right) \right) - 1 \right) \right].

\varphi_{J_{\text{v}}, J_{\text{s}}} (a) = \mathbb{E}_F^P \left[ \exp \left( \int_s^{t'} \lambda_{\text{v}, \text{a}} - du \left( \frac{1 - \Gamma_Y \mu_y}{1 - (a + \Gamma_Y) \mu_y} - 1 \right) \right] .

\varphi_{\text{Q}_{J_{\text{v}}, J_{\text{s}}} (a)} = \mathbb{E}_F^P \left[ \exp \left( \int_s^{t'} \lambda_{\text{v}, \text{a}} - du \left( \frac{1 - \Gamma_Y \mu_y}{1 - (a + \Gamma_Y) \mu_y} - 1 \right) \right] .

A.3.1 Proofs

Proof of Lemma 7. The moment generating function of $W_{X,t} - W_{X,s}$ under $Q$ is

$$
\varphi_{\text{Q}_{W_{X,t} - W_{X,s}}} (a) = \mathbb{E}_F^P \left[ \exp \left( a \left( W_{X,t} - W_{X,s} \right) \right) \right].
$$

Because the variance jump intensity is independent of the other components inside the exponential function, the last expression is equal to

$$
\mathbb{E}_F^P \left[ \exp \left( a \left( W_{X,t} - W_{X,s} \right) \right) \right] = \mathbb{E}_F^P \left[ \exp \left( \Gamma_Y (J_{\text{v}, t} - J_{\text{v}, s}) \right) (1 - \mu_Y) \right].
$$

Lemma 6 implies that the last expectation is 1. Using the tower property of conditional expectations, conditioning on the log-equity price intensity, the previous expression becomes

$$
\mathbb{E}_F^P \left[ \exp \left( \Gamma_Y (J_{\text{v}, t} - J_{\text{v}, s}) \right) (1 - \mu_Y) \right].
$$

Finally, from Lemma 4,

$$
\varphi_{\text{Q}_{W_{Y}, W_{X,s}}} (a) = \mathbb{E}_F^P \left[ \exp \left( a \left( W_{Y,t} - W_{X,s} \right) \right) \right] = \mathbb{E}_F^P \left[ \exp \left( \int_s^{t'} (a - \lambda_{\text{X}, \text{a}}) dW_{\text{X,u}} - \frac{1}{2} \int_s^{t'} \lambda_{\text{X}, \text{a}}^2 dW_{\text{X,u}} + \frac{1}{2} \int_s^{t'} \lambda_{\text{X}, \text{a}}^2 dW_{\text{X,u}} + \frac{1}{2} \int_s^{t'} (a - \lambda_{\text{X}, \text{a}})^2 du \right] .
$$
Proof of Lemma 8.

\[
E^P_{J_t} \left[ \exp \left( a (J_{X,t} - J_{X,s}) \right) \right] = E^P_{J_t} \left[ \exp \left( - \int_s^t \Lambda_{ur} \, dW_{ur} - \int_s^t \Lambda_{ur} \, dW_{ur} - \frac{1}{2} \int_s^t \left( \Lambda_{ur}^2 + \Lambda_{ur}^2 \right) \, du \right) \right].
\]

Using the tower property of conditional expectations, the last expression becomes

\[
E^P_{J_t} \left[ \exp \left( - \int_s^t \Lambda_{ur} \, dW_{ur} - \int_s^t \Lambda_{ur} \, dW_{ur} - \frac{1}{2} \int_s^t \left( \Lambda_{ur}^2 + \Lambda_{ur}^2 \right) \, du \right) \right] E^P_{J_t} \left[ \exp \left( a (J_{X,t} - J_{X,s}) \right) \right] \times E^P_{J_t} \left[ \exp \left( \frac{1}{2} \int_s^t \left( \Lambda_{ur}^2 + \Lambda_{ur}^2 \right) \, du \right) \right] \times E^P_{J_t} \left[ \int_s^t \Lambda_{ur} \, du \right].
\]

Lemma 4 implies that the first term vanishes. Therefore,

\[
E^P_{J_t} \left[ \exp \left( a (J_{X,t} - J_{X,s}) \right) \right] = E^P_{J_t} \left[ \exp \left( (a + \Gamma_X) (J_{X,t} - J_{X,s}) - (\varphi_{Z_X} (\Gamma_X) - 1) \int_s^t \Lambda_{ur} \, du \right) \right]
\]

\[
= E^P_{J_t} \left[ \exp \left( (a + \Gamma_X) (J_{X,t} - J_{X,s}) - (\varphi_{Z_X} (a + \Gamma_X) - 1) \int_s^t \Lambda_{ur} \, du \right) \right]
\]

\[
= E^P_{J_t} \left[ \exp \left( (a + \Gamma_X) (J_{X,t} - J_{X,s}) - (\varphi_{Z_X} (a + \Gamma_X) - 1) \int_s^t \Lambda_{ur} \, du \right) \right]
\]

\[
= E^P_{J_t} \left[ \exp \left( (a + \Gamma_X) (J_{X,t} - J_{X,s}) - \varphi_{Z_X} (a + \Gamma_X) \int_s^t \Lambda_{ur} \, du \right) \right]
\]

\[
= E^P_{J_t} \left[ \exp \left( \varphi_{Z_X} (a + \Gamma_X) \int_s^t \Lambda_{ur} \, du \right) \right].
\]

\[
\square
\]

A.4 Log-Equity Price Drift

This section provides a detailed proof of the log-equity price drift term described in Appendix B.1.

Lemma 9. Under the risk-neutral measure, the log-equity price is characterized by

\[
dY_t = \alpha_{Y_t}^Q \, dt + \sqrt{V_t} \, dW_{Y_t}^Q + dJ_{Y_t}^Q,
\]

\[
\alpha_{Y_t}^Q = r - q - \frac{1}{2} \sqrt{V_t} - \xi_Y \lambda_{Y_t}^Q,
\]

\[
\xi_Y^Q = \varphi_{Z_Y} (1) - 1 = \exp \left( \mu_Y + \frac{1}{2} \sigma_Y^2 \right) - 1,
\]

\[
\lambda_{Y_t}^Q = \varphi_{Z_Y} (\Gamma_Y) \lambda_{Y_t}^Q = \exp \left( \mu_Y \Gamma_Y + \frac{1}{2} \Gamma_Y \sigma_Y^2 \right) \lambda_{Y_t}^Q.
\]

Proof of Lemma 9. Using Itô’s lemma,

\[
\exp (Y_t) - \exp (Y_0) = \int_0^t \exp (Y_{u-}) \, dY_u + \frac{1}{2} \int_0^t \exp (Y_{u-}) \, d [Y, Y]_u^c + \sum_{0 < u \leq t} \left( \exp (Y_{u-}) - \exp (Y_{u-}) - \exp (Y_{u-}) \Delta Y_u \right).
\]

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The last term is equal to

\[
\sum_{0 < u \leq t} \exp(Y_{u^-}) \{ \exp(Y_u - Y_{u^-}) - 1 \} - \sum_{0 < u \leq t} \{ \exp(Y_{u^-}) \left( J_{Y_u}^Q - J_{Y_{u^-}}^Q \right) \}
\]

\[
= \sum_{0 < u \leq t} \exp(Y_{u^-}) \{ \exp(J_{Y_u}^Q) - J_{Y_{u^-}}^Q \} - \sum_{0 < u \leq t} \{ \exp(Y_{u^-}) \left( J_{Y_u}^Q - J_{Y_{u^-}}^Q \right) \}
\]

\[
= \sum_{0 < u \leq t} \exp(Y_{u^-} - J_{Y_{u^-}}^Q) \{ \exp(J_{Y_u}^Q) - \exp(J_{Y_{u^-}}^Q) \}
\]

\[
- \sum_{0 < u \leq t} \{ \exp(Y_{u^-}) \left( J_{Y_u}^Q - J_{Y_{u^-}}^Q \right) \}.
\]

Therefore, by substituting Equation (25) in the first term of \( \exp(Y_t) - \exp(Y_0) \), we obtain

\[
\exp(Y_t) - \exp(Y_0) = \int_0^t \exp(Y_u) \alpha_u^Q \, du + \int_0^t \exp(Y_u) \sqrt{V_u} \, dW_{Yu} + \int_0^t \exp(Y_u) \, dJ_{Yu}^Q \\
+ \frac{1}{2} \int_0^t \exp(Y_u) \, V_u \, du + \int_0^t \exp(Y_u - J_{Y_{u^-}}^Q) \, d \exp(J_{Y_u}^Q) - \int_0^t \exp(Y_u) \, dJ_{Yu}^Q \\
= \int_0^t \exp(Y_u) \left( \alpha_u^Q + \frac{1}{2} V_u \right) \, du + \int_0^t \exp(Y_u) \sqrt{V_u} \, dW_{Yu} \\
+ \int_0^t \exp(Y_u - J_{Y_{u^-}}^Q) \, d \exp(J_{Y_u}^Q).
\]

Lemma 6 states that \( \{ \exp(J_{Y_u}^Q - (\varphi_{Z_u}^Q(1) - 1) \int_0^u \lambda_{Yu}^Q \, du) \}_{u \geq 0} \) is a \( Q \)-martingale, where \( \varphi_{Z_u}^Q \) is the moment generating function of the log-equity price jump size. Therefore, if

\[
\xi_u^Q = \varphi_{Z_u}^Q(1) - 1 = \exp \left( \mu_Y + \frac{1}{2} \sigma_Y^2 \right) - 1,
\]

then

\[
\exp(Y_t) - \exp(Y_0) = \int_0^t \exp(Y_u) \left( \alpha_u^Q + \frac{1}{2} V_u \right) \, du + \int_0^t \exp(Y_u) \sqrt{V_u} \, dW_{Yu} \\
+ \int_0^t \exp(Y_u - J_{Y_{u^-}}^Q) \, d \exp \left( J_{Yu}^Q - \xi_u^Q \int_0^u \lambda_{Yu}^Q \, ds \right) \exp \left( \xi_u^Q \int_0^u \lambda_{Yu}^Q \, ds \right) \\
= \int_0^t \exp(Y_u) \left( \alpha_u^Q + \frac{1}{2} V_u \right) \, du + \int_0^t \exp(Y_u) \sqrt{V_u} \, dW_{Yu} \\
+ \int_0^t \exp(Y_u - J_{Y_{u^-}}^Q) \, d \exp \left( J_{Yu}^Q - \xi_u^Q \int_0^u \lambda_{Yu}^Q \, ds \right) \exp \left( \xi_u^Q \int_0^u \lambda_{Yu}^Q \, ds \right) dJ_{Yu}^Q \\
= \int_0^t \exp(Y_u) \left( \alpha_u^Q + \frac{1}{2} V_u + \xi_u^Q \lambda_{Yu}^Q \right) \, du + \int_0^t \exp(Y_u) \sqrt{V_u} \, dW_{Yu} \\
+ \int_0^t \exp(Y_u - J_{Y_{u^-}}^Q) \, d \exp \left( \xi_u^Q \int_0^u \lambda_{Yu}^Q \, ds \right) \exp \left( J_{Yu}^Q - \xi_u^Q \int_0^u \lambda_{Yu}^Q \, ds \right) dJ_{Yu}^Q.
\]

Note that the last two terms are martingales. Because the discount factor and the dividend correction are both continuous, i.e.,

\[
D_t = \exp \left( -(r - q)t \right),
\]

(39)
integration by part implies that

\[
D_t \exp (Y_t) - D_0 \exp (Y_0) = \int_0^t D_u d \exp (Y_u) - \int_0^t (r - q) D_u \exp (Y_u) du
\]

\[
= \int_0^t D_u \exp (Y_u) \left( \alpha_u - \frac{1}{2} V_u + \xi_u \lambda_u - r + q \right) du
\]

\[
+ \int_0^t D_u \exp (Y_u) \sqrt{V_u} dW_u
\]

\[
+ \int_0^t D_u \exp (Y_u) - J_u \exp \left( \xi_u \int_0^u \lambda_u du \right) d \exp \left( J_u - \xi_u \int_0^u \lambda_u du \right).
\]

The pricing theory implied that \(\{D_t \exp (Y_t)\}_{t \geq 0}\) be a \(\mathbb{Q}\)-martingale. Therefore, the drift component must be nil, that is

\[
\alpha_u = r - q - \frac{1}{2} V_u - \xi_u \lambda_u.
\]

\[\Box\]

**Lemma 10.** Under the \(\mathbb{P}\)-measure, the returns’ are characterized by

\[
dY = \alpha \, dt + \sqrt{V_t} dW_t + dJ_t,
\]

\[
\alpha_u = r - q - \frac{1}{2} V_u + \sqrt{V_u} \lambda_u - (\xi_u - \xi) \lambda_u - (\xi_u - \xi) \lambda_u - (\xi_u - \xi) \lambda_u - (\xi_u - \xi) \lambda_u - (\xi_u - \xi) \lambda_u - (\xi_u - \xi) \lambda_u.
\]

\[
\xi_u = \varphi \, (\Gamma Y) - 1,
\]

\[
\xi_u = \varphi \, (\Gamma Y) - 1,
\]

\[
\xi_u = \varphi \, (1 + \Gamma Y) - 1,
\]

\[
\xi_u = \varphi \, (1 + \Gamma Y) - 1,
\]

**Proof of Lemma 10.** Let

\[
Z_t = \left( - \int_0^t \lambda_{Vt} dW_{Vt} - \int_0^t \lambda_{Wt} dW_{Wt} - \frac{1}{2} \int_0^t \Lambda_{Vt}^2 + \Lambda_{Wt}^2 \right) dt + \Gamma J_t - \xi - \xi \int_0^t \lambda_{Vt} du + \Gamma J_t - \xi - \xi \int_0^t \lambda_{Vt} du
\]

be associated to the Radon-Nikodym derivative (21) and \(D_t \exp (-r \, q \, t)\) be the combination of the discount factor and the dividend yield. Since the discounted price must be a \(\mathbb{Q}\)-martingale,

\[
1 = E_{\mathbb{P}} \left[ \frac{D_t \exp (Y_t)}{D_t \exp (Y)} \right] = E_{\mathbb{P}} \left[ \frac{D_t \exp (Y_t + Z_t)}{D_t \exp (Y + Z_t)} \right]
\]

which means that \(\{D_t \exp (Y_t + Z_t)\}_{t \geq 0}\) is a \(\mathbb{P}\)-martingale. Note that

\[
Y_t + Z_t = Y_0 + Z_0 + \int_0^t \left( \alpha_u - \frac{1}{2} \Lambda_{Vt}^2 + \Lambda_{Wt}^2 \right) du
\]

\[
+ \int_0^t \left( \rho \sqrt{V_u} - \lambda_{Vt} \right) dW_{Vt} + \int_0^t \left( \sqrt{1 - \rho^2} \sqrt{V_u} - \lambda_{Wt} \right) dW_{Wt}
\]

\[
+ (1 + \Gamma Y) \int_0^t dJ_{Vt} + \Gamma V \int_0^t dJ_{Vt}.
\]
Ito’s lemma implies that
\[
\exp(Y_t + Z_t) - \exp(Y_0 + Z_0) = \int_0^t \exp(Y_{u^-} + Z_{u^-}) d(Y_u + Z_u)
\]
\[
+ \frac{1}{2} \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( V_{u^-} - 2\sqrt{V_{u^-}} \Lambda_{Y_{u^-}} + \Lambda_{V_{u^-}}^2 + \Lambda_{\bot_{u^-}}^2 \right) du
\]
\[
+ \sum_{0 < u \leq t} \{ \exp(Y_u + Z_u) - \exp(Y_{u^-} + Z_{u^-}) - \exp(Y_{u^-} + Z_{u^-}) \Delta(Y_u + Z_u) \}
\]
Replacing the first expression in the second one leads to
\[
\exp(Y_t + Z_t) - \exp(Y_0 + Z_0)
= \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( 2 \Lambda_{V_{u^-}} \right) du
\]
\[
+ \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( 1 - \rho^2 \sqrt{V_{u^-}} - \Lambda_{\bot_{u^-}} \right) dW_{\bot_{u^-}}
\]
\[
+ \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( 1 + \Gamma_Y \right) dJ_{Y_{u^-}} + \sum_{0 < u \leq t} \{ \exp(Y_u + Z_u) - \exp(Y_{u^-} + Z_{u^-}) - \exp(Y_{u^-} + Z_{u^-}) \Delta(Y_u + Z_u) \}
\]
The last term can be rewritten as
\[
\sum_{0 < u \leq t} \exp(Y_{u^-} + Z_{u^-}) \{ \exp(\Delta Y_u + \Delta Z_u) - 1 - \Delta(Y_u + Z_u) \}
\]
\[
= \sum_{0 < u \leq t} \exp(Y_{u^-} + Z_{u^-}) \left[ \exp((1 + \Gamma_Y) \Delta J_{Y_{u^-}} + \Gamma_Y \Delta J_{V_{u^-}}) - 1 - (1 + \Gamma_Y) \Delta J_{Y_{u^-}} - \Gamma_Y \Delta J_{V_{u^-}} \right]
\]
\[
= \sum_{0 < u \leq t} \left[ \exp(Y_{u^-} + Z_{u^-}) - (1 + \Gamma_Y) \Delta J_{Y_{u^-}} - \Gamma_Y J_{V_{u^-}} \right]
\]
\[
- \sum_{0 < u \leq t} \exp(Y_{u^-} + Z_{u^-}) \{ (1 + \Gamma_Y) \Delta J_{Y_{u^-}} + \Gamma_Y \Delta J_{V_{u^-}} \}
\]
\[
= \int_0^t \exp(Y_{u^-} + Z_{u^-}) - (1 + \Gamma_Y) J_{Y_{u^-}} - \Gamma_Y J_{V_{u^-}} \right) dJ_{Y_{u^-}} - \int_0^t \exp(Y_{u^-} + Z_{u^-}) \Gamma_Y dJ_{V_{u^-}}.
\]
Therefore,
\[
\exp(Y_t + Z_t) - \exp(Y_0 + Z_0)
= \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( 2 \Lambda_{V_{u^-}} \right) du
\]
\[
+ \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( 1 - \rho^2 \sqrt{V_{u^-}} - \Lambda_{\bot_{u^-}} \right) dW_{\bot_{u^-}}
\]
\[
+ \int_0^t \exp(Y_{u^-} + Z_{u^-}) \left( 1 + \Gamma_Y \right) dJ_{Y_{u^-}}
\]
\[
+ \sum_{0 < u \leq t} \{ \exp(Y_u + Z_u) - \exp(Y_{u^-} + Z_{u^-}) - \exp(Y_{u^-} + Z_{u^-}) \Delta(Y_u + Z_u) \}.
\]
\[ + \int_0^t \exp (Y_u - Z_u - (1 + \Gamma Y) J_{Y,u} - \Gamma V J_{V,u}) \, d \exp ((1 + \Gamma Y) J_{Y,u} + \Gamma V J_{V,u}) \, . \]

We need to construct a martingale out of the last term. From Lemma 6, \( \{M_t\}_{t \geq 0} \) is a \( \mathbb{P} \)-martingale where

\[ M_t = \exp \left( (1 + \Gamma Y) J_{Y,t} + \Gamma V J_{V,t} - \zeta_Y^p \int_0^t \lambda_{Y,u} \, du - \zeta_V^p \int_0^t \lambda_{V,u} \, du \right) . \]

Indeed,

\[
\mathbb{E}_{\mathbb{F}_t} \left[ \exp \left( (1 + \Gamma Y) (J_{Y,t} - J_{Y,s}) + \Gamma V (J_{V,t} - J_{V,s}) - \zeta_Y^p \left( \int_s^t \lambda_{Y,u} \, du \right) - \zeta_V^p \left( \int_s^t \lambda_{V,u} \, du \right) \right) \right] = 1. \]

Moreover,

\[
d \exp ((1 + \Gamma Y) J_{Y,t} + \Gamma V J_{V,t}) = dM_t \exp \left( \zeta_Y^p \int_0^t \lambda_{Y,u} \, du + \zeta_V^p \int_0^t \lambda_{V,u} \, du \right)
\]

\[
= M_t \cdot d \exp \left( \zeta_Y^p \int_0^t \lambda_{Y,u} \, du + \zeta_V^p \int_0^t \lambda_{V,u} \, du \right) + \exp \left( \zeta_Y^p \int_0^t \lambda_{Y,u} \, du + \zeta_V^p \int_0^t \lambda_{V,u} \, du \right) \, dM_t
\]

\[
= \left( \zeta_Y^p \lambda_{Y,t} + \zeta_V^p \lambda_{V,t} \right) M_t \cdot \exp \left( \zeta_Y^p \int_0^t \lambda_{Y,u} \, du + \zeta_V^p \int_0^t \lambda_{V,u} \, du \right) \, dt
\]

\[
+ \exp \left( \zeta_Y^p \int_0^t \lambda_{Y,u} \, du + \zeta_V^p \int_0^t \lambda_{V,u} \, du \right) \, dM_t
\]

\[
= \left( \zeta_Y^p \lambda_{Y,t} + \zeta_V^p \lambda_{V,t} \right) \exp ((1 + \Gamma Y) J_{Y,t} + \Gamma V J_{V,t}) \, dt
\]

Therefore,

\[
\exp (Y_t + Z_t) - \exp (Y_0 + Z_0)
\]

\[
= \int_0^t \exp (Y_u - Z_u) \left( \alpha_u^p - \zeta_Y^p \lambda_{Y,u} - \zeta_V^p \lambda_{V,u} + \frac{1}{2} V_u - \sqrt{V_u} \Lambda_{V,u} \right) \, du
\]

\[
+ \int_0^t \exp (Y_u - Z_u) \left( \rho \sqrt{V_u} \right) \, dW_{V,u}
\]

\[
+ \int_0^t \exp (Y_u - Z_u) \left( \sqrt{1 - \rho^2} \sqrt{V_u} \right) \, dW_{L,u}
\]

\[
+ \int_0^t \exp (Y_u - Z_u) \left( \zeta_Y^p \lambda_{Y,u} + \zeta_V^p \lambda_{V,u} \right) \, dM_u
\]

\[
= \int_0^t \exp (Y_u - Z_u) \left( \alpha_u^p - \zeta_Y^p \lambda_{Y,u} - \zeta_V^p \lambda_{V,u} + \frac{1}{2} V_u - \sqrt{V_u} \Lambda_{V,u} + \zeta_Y^p \lambda_{Y,u} + \zeta_V^p \lambda_{V,u} \right) \, du
\]

\[
+ \int_0^t \exp (Y_u - Z_u) \left( \rho \sqrt{V_u} \right) \, dW_{V,u}
\]

\[
+ \int_0^t \exp (Y_u - Z_u) \left( \sqrt{1 - \rho^2} \sqrt{V_u} \right) \, dW_{L,u}
\]

\[
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\[ + \int_0^t \exp \left( Y_u - Z_u - (1 + \Gamma_Y) J_{Y_u} - \frac{1}{2} V_u^2 + \zeta_Y \int_0^u \lambda_{Y,s} \, ds + \zeta_Y^0 \int_0^u \lambda_{Y,s} \, ds \right) \, dM_u. \]

Since

\[ D_t \exp (Y_t + Z_t) - D_0 \exp (Y_0 + Z_0) = \int_0^t D_u d \exp (Y_u + Z_u) + \int_0^t (q - r) D_u \exp (Y_u + Z_u) \, du, \]

then

\[ \begin{aligned}
&= \int_0^t D_u \exp (Y_u + Z_u) \left( \alpha_u - \xi_0^Y \lambda_{Y,u} - \xi_0^Z \lambda_{Z,u} + \frac{1}{2} V_u^2 - \sqrt{V_u} \lambda_{Y,u} + q - r \right) \, du \\
&+ \int_0^t D_u \exp (Y_u + Z_u) (\rho \sqrt{V_u} - \Lambda_{Z,u}) \, dW_u \\
&+ \int_0^t D_u \exp (Y_u + Z_u) (\sqrt{1 - \rho^2} \sqrt{V_u} - \Lambda_{Y,u}) \, dW_u \\
&+ \int_0^t D_u \exp (Y_u + Z_u) (1 + \Gamma_Y) J_{Y,u} - \Gamma_{V,Y} \lambda_{Y,u} + \zeta_Y \int_0^u \lambda_{Y,s} \, ds + \zeta_Y^0 \int_0^u \lambda_{Y,s} \, ds \right) \, dM_u. 
\end{aligned} \]

Finally, as \( D_t \exp (Y_t + Z_t) \) needs to be a \( \mathbb{P} \)-martingale, the drift term must be nil, implying that

\[ \alpha_u = r - q - \frac{1}{2} V_u + \sqrt{V_u} \lambda_{Y,u} + (\xi_Y - \zeta_Y) \lambda_{Y,u} + (\xi_Y^0 - \zeta_Y^0) \lambda_{Y,u}. \]

\( \square \)

**Corollary 11.**

\[ E^P_\tau [\exp(Y_T)] = \exp(Y_T) E^P_\tau \left[ \exp \left( \int_0^T m_u^P \, du \right) \right] \]

where

\[ m_u^P = \alpha_u + \frac{1}{2} V_u - dW_u + \left( \varphi_{Z_u} \left( 1 \right) - 1 \right) \lambda_{Y,u} \]

\[ = r - q + \sqrt{V_u} \lambda_{Y,u} + (\xi_Y - \zeta_Y) \lambda_{Y,u} + (\varphi_{Z_u} \left( 1 \right) - 1) \lambda_{Y,u} \]

\[ = r - q + \sqrt{V_u} \lambda_{Y,u} + (\varphi_{Z_u} \left( 1 \right) - 1) \lambda_{Y,u}. \]

**Proof.** Starting from Equation (1),

\[ E^P_\tau [\exp(Y_T)] \]

\[ = \exp(Y_T) E^P_\tau \left[ \exp \left( \int_0^T \alpha_u \, du + \int_0^T \sqrt{V_u} \, dW_u + \int_0^T J_u \right) \right] \]

\[ = \exp(Y_T) E^P_\tau \left[ \exp \left( \int_0^T \alpha_u \, du + \frac{1}{2} \int_0^T V_u \, du + \left( \varphi_{Z_u} \left( 1 \right) - 1 \right) \int_0^T \lambda_{Y,u} \, du \right) \right] \]

\[ = \exp(Y_T) \times \left[ E^P_\tau \left[ \exp \left( \int_0^T \alpha_u \, du + \frac{1}{2} \int_0^T V_u \, du + \left( \varphi_{Z_u} \left( 1 \right) - 1 \right) \int_0^T \lambda_{Y,u} \, du \right) \right] \right] \]

\[ = \exp(Y_T) \exp \left( \int_0^T \left( \alpha_u + \frac{1}{2} V_u \, du + \left( \varphi_{Z_u} \left( 1 \right) - 1 \right) \lambda_{Y,u} \right) \right). \]
Under our modelling framework, it is straightforward to compute

\[
\exp(\mathbf{Y}_T) \mathbb{E}_t^P \left[ \exp \left( \int_t^T m_{a_t}^p \, du \right) \right]
\]

Therefore,

\[
m_{a_t}^p = \alpha_{a_t}^p + \frac{1}{2} V_{a_t}^p \, du + \left( \varphi_{Z_t}^p (1) - 1 \right) \lambda_{Y,a}^p
\]

\[
r - q - \frac{1}{2} V_{a_t}^p + \sqrt{V_{a_t}^p} \lambda_{Y,a}^p + \left( \xi_{Y,Y}^p - \xi_{Y}^p \right) \lambda_{Y,a}^p + \left( \xi_{Y}^p - \xi_{Y}^p \right) \lambda_{Y,a}^p
\]

\[
+ \frac{1}{2} V_{a_t}^p \, du + \left( \varphi_{Z_t}^p (1) - 1 \right) \lambda_{Y,a}^p
\]

\[
r - q + \sqrt{V_{a_t}^p} \lambda_{Y,a}^p + \left( \xi_{Y,Y}^p - \xi_{Y}^p + \varphi_{Z_t}^p (1) - 1 \right) \lambda_{Y,a}^p + \left( \xi_{Y}^p - \xi_{Y}^p \right) \lambda_{Y,a}^p
\]

\[
r - q + \sqrt{V_{a_t}^p} \lambda_{Y,a}^p + \left( \varphi_{Z_t}^p (1) - \varphi_{Z_t}^p (1 + \Gamma_Y) + \varphi_{Z_t}^p (1) - 1 \right) \lambda_{Y,a}^p + \left( \varphi_{Z_t}^p (1) - \varphi_{Z_t}^p (\Gamma_Y) \right) \lambda_{Y,a}^p.
\]

\( \square \)

B Integrated Risk Premiums

B.1 Equity

In the spirit of Bardgett et al. (2015), and consistent with Bollerslev and Todorov (2011) and Aït-Sahalia et al. (2015), we can divide the integrated equity risk premium in two components: a diffusive and a jump risk premiums. In our framework,

\[
IERP(t, T) = \frac{1}{T-t} \left( \mathbb{E}_t^P [Y_T - Y_t] - \mathbb{E}_t^Q [Y_T - Y_t] \right)
\]

and this \( IERP \) can be decomposed into two components:

\[
IERP(t, T) = IERP_{\text{Diffusion}}(t, T) + IERP_{\text{Jump}}(t, T)
\]

where

\[
IERP_{\text{Diffusion}}(t, T) = \frac{1}{T-t} \eta_y \mathbb{E}_t^P \left[ \int_t^T V_s \, ds \right]
\]

and

\[
IERP_{\text{Jump}}(t, T) = \frac{1}{T-t} \gamma_y \left( \lambda_{Y,0}(T-t) + \lambda_{Y,1} \mathbb{E}_t^P \left[ \int_t^T V_s \, ds \right] \right).
\]

Under our modelling framework, it is straightforward to compute \( \mathbb{E}_t^P \left[ \int_t^T V_s \, ds \right] \). First, we know that

\[
\mathbb{E}_t^P \left[ \int_t^T V_s \, ds \right] = \int_t^T \mathbb{E}_t^P [V_s] \, ds
\]

by Tonelli’s theorem. Then, by taking the expectation on both side of the SDE, we can find the following ODE:

\[
y' = \kappa (\theta - y) \, dt + \lambda_{Y,0} \mu_V
\]

where \( y = \mathbb{E}_t^P [V_s] \). The solution of the ordinary differential equation is therefore given by

\[
\mathbb{E}_t^P [V_s] = A + \exp \left( -\kappa (s-t) \right) \left( V_t - A \right), \quad A = \frac{\lambda_{Y,0} \mu_V + \theta \kappa}{\kappa}.
\]

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Finally,
\[ \mathbb{E}_t^T \left[ \int_t^T V_s \, ds \right] = A(T - t) + \frac{1}{k} \left( 1 - \exp \left( -\kappa (T - t) \right) \right) (V_t - A). \]

### B.2 Variance

The integrated variance risk premium is given by the following expression:
\[ \text{IVRP}(t, T) = \frac{1}{T - t} \left( \mathbb{E}_t^T \left[ \Delta QV_{t, T} \right] - \mathbb{E}_t^Q \left[ \Delta QV_{t, T} \right] \right). \]

Again, we can divide the integrated variance risk premium in two components: a diffusive and a jump risk premiums,
\[ \text{IVRP}(t, T) = \text{IVRP}^{\text{Diffusion}}(t, T) + \text{IVRP}^{\text{Jump}}(t, T) \]

where
\[ \text{IVRP}^{\text{Diffusion}}(t, T) = \frac{1}{T - t} \left( \lambda_{Y,0}(T - t) + \lambda_{Y,1} \mathbb{E}_t^P \left[ \int_t^T V_s \, ds \right] \right) \left( (\mu_Y)^2 + (\sigma_Y)^2 \right) \]
and
\[ \text{IVRP}^{\text{Jump}}(t, T) = -\frac{1}{T - t} \left( \lambda_{Y,0}(T - t) + \lambda_{Y,1} \mathbb{E}_t^Q \left[ \int_t^T V_s \, ds \right] \right) \left( (\mu_Y)^2 + (\sigma_Y)^2 \right). \]

### C Option Pricing

#### C.1 Moment Generating Function of the Log-Equity Price Variation

**Lemma 12.** The risk-neutral moment generating function of \( Y_T \) satisfies
\[ \varphi_{Y_t|Y_T} (u, Y_t, V_t) = \mathbb{E}^Q \left[ \exp (uY_T) | Y_t, V_t \right] = \exp \left( \mathcal{A}(u, t, T) + u Y_t + C(u, t, T) V_t \right). \]

where \( C(u, t, T) \) and \( \mathcal{A}(u, t, T) \) are provided at Equations (28) and (29) respectively.

**Proof.** The proof is based on Filipovic and Mayerhofer (2009) and Duffie et al. (2000).

According to Duffie et al. (2000), \( \mathcal{A}(u, t, T), \mathcal{B}(u, t, T) \) and \( C(u, t, T) \) satisfy the complex-valued ordinary differential equations (ODEs)
\[ \mathcal{A}'(u; t, T) = r - \left[ \frac{\sigma^2}{2} - \lambda_{Y,0}^Q \left( \varphi_{Z_t}^Q \left( 1 - 1 \right) \right) \right] \mathcal{B}(u; t, T) \]
\[ - \lambda_{Y,0}^Q \left( \varphi_{Z_t}^Q \left( \mathcal{C}(u; t, T) \right) - 1 \right), \]
\[ \mathcal{B}'(u; t, T) = 0, \]
\[ \mathcal{C}'(u; t, T) = \frac{1}{2} \lambda_{Y,1}^Q \left( \varphi_{Z_t}^Q \left( 1 - 1 \right) \right) \mathcal{B}(u; t, T) \]
\[ - \lambda_{Y,1}^Q \left( \varphi_{Z_t}^Q \left( u - 1 \right) \right), \]
with initial conditions \( \mathcal{A}(u; T, T) = 0, \mathcal{B}(u; T, T) = u \) and \( C(u; T, T) = 0 \). The second ODE is obvious: since \( \mathcal{B}(u; T, T) = u \) and \( \mathcal{B}'(u; T, T) = 0 \), it means that \( \mathcal{B}(u; t, T) = u \). Then, from this equation, we
can try to find the solution to the third ODE (i.e., \(C(u, t, T)\)). As in Filipovic and Mayerhofer (2009), the solution to this ODE is given by Equation (28). Finally, the expression for \(A(u, t, T)\) provided at Equation (29) is obtained from a simple integration since there are no \(A\) in the right hand side of Equation (40).

\[\square\]

### C.2 Quadratic Variation of Option Prices

The option price is a function of time, returns and variance: \(O_t = O_t(Y_t, V_t)\). Assume that \(O\) is twice continuously differentiable. Itô’s formula implies that

\[
O_t(Y_t, V_t) - O_0(Y_0, V_0) = \int_0^t \frac{\partial O_u}{\partial u} (Y_{u-}, V_{u-}) \, du + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \, \alpha_u \, du + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \, \rho \sqrt{V_{u-}} \, dW_{V, u} + \int_0^t \frac{\partial^2 O_u}{\partial y^2} (Y_{u-}, V_{u-}) \, \sqrt{1 - \rho^2} \sqrt{V_{u-}} \, dW_{\perp, u} + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \, \kappa (\theta - V_{u-}) \, du + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \, \sigma \sqrt{V_{u-}} \, dW_{V, u} + \int_0^t \frac{\partial O_u}{\partial v} (Y_{u-}, V_{u-}) \, dJ_{V, u} + \int_0^t \frac{\partial^2 O_u}{\partial v^2} (Y_{u-}, V_{u-}) \, \sigma^2 V_{u-} \, du + \int_0^t \frac{\partial^2 O_u}{\partial v^2} (Y_{u-}, V_{u-}) \, \sigma \rho V_{u-} \, dW_{\perp, u} + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \, dJ_{V, u} + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \, dJ_{V, u}.
\]

Replacing Equation (1) in the latter leads to

\[
O_t(Y_t, V_t) - O_0(Y_0, V_0) = \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \sum_{0 < u \leq t} \{ \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) (Y_u - Y_{u-}) \} - \sum_{0 < u \leq t} \left\{ \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) (V_u - V_{u-}) \right\}.
\]

Then,

\[
O_t(Y_t, V_t) - O_0(Y_0, V_0) = \int_0^t \left\{ \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) + \alpha_u \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) + \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \kappa (\theta - V_{u-}) \right\} \, du + \int_0^t \left\{ \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) + \frac{\partial O_u}{\partial v} (Y_{u-}, V_{u-}) \sigma^2 + \frac{\partial^2 O_u}{\partial v^2} (Y_{u-}, V_{u-}) \sigma \rho \right\} \, dJ_{V, u} + \int_0^t \left\{ \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \right\} \sqrt{V_{u-}} \, dW_{V, u} + \int_0^t \frac{\partial O_u}{\partial y} (Y_{u-}, V_{u-}) \sqrt{1 - \rho^2} \sqrt{V_{u-}} \, dW_{\perp, u} + \sum_{0 < u \leq t} \{ O_u(Y_u, V_u) - O_u(Y_{u-}, V_{u-}) \}.
\]

Finally, the quadratic variation is

\[
[O, O]_t.
\]
\[
\int_0^t \left( \frac{\partial O_u}{\partial y} (Y_u, V_u) + \sigma \frac{\partial O_u}{\partial v} (Y_u, V_u) \right)^2 V_u \, du + \int_t^\infty \left( \frac{\partial O_u}{\partial y} (Y_u, V_u) \right)^2 (1 - \rho^2) V_u \, du \\
+ \sum_{0 < u \leq T} [O_u (Y_u, V_u) - O_u (Y_u, V_u)]^2
\]

\[
= \int_0^t \left( \frac{\partial O_u}{\partial y} (Y_u, V_u) \right)^2 V_u \, du + \sigma \int_t^\infty \frac{\partial O_u}{\partial v} (Y_u, V_u) \frac{\partial O_u}{\partial v} (Y_u, V_u) + \sigma^2 \left( \frac{\partial O_u}{\partial v} (Y_u, V_u) \right)^2 V_u \, du \\
+ \sum_{0 < u \leq T} [O_u (Y_u, V_u) - O_u (Y_u, V_u)]^2.
\]

C.2.1 Derivative Calculation for ΔQOV

Starting from Equation (27),

\[
\frac{\partial}{\partial y} C_i(y, v) = \exp (y - q(T - t)) \left( P_1(y, v) + \frac{\partial P_1}{\partial x} (y, v) \right) - K \exp (-r(T - t)) \frac{\partial P_2}{\partial y} (y, v)
\]

and

\[
\frac{\partial}{\partial v} C_i(y, v) = \exp (y - q(T - t)) \frac{\partial P_1}{\partial v} (y, v) - K \exp (-r(T - t)) \frac{\partial P_2}{\partial v} (y, v).
\]

The derivatives \( \frac{\partial}{\partial y} P_1 (y, v, k; t, T) \) and \( \frac{\partial}{\partial v} P_2 (y, v, k; t, T) \) can be calculated by computing the derivative of the integrand.\(^{50}\) For \( x \in \{y, v\}, \)

\[
\frac{\partial}{\partial x} P_1 (y, v) = \frac{\partial}{\partial x} \left[ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left( \frac{1}{u} \exp (-iuk - y) \varphi_{Y, Y, Y} (u1, 1, y, v) \right) \, du \right] \\
= \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp (-iuk - y) \varphi_{Y, Y, Y} (u1, 1, y, v) \right] \, du \\
= \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp (-iuk - y) \exp (\mathcal{A} (u1 + 1, t, T) + (u1 + 1) y + C (u1 + 1, T) v) \right] \, du
\]

where the last expression comes from the specific shape of the moment generating function. Because \( \exp (a + ib) = \exp (a) \times (\cos b + i \sin b) \), the last expression becomes

\[
\frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp \left( \mathrm{Re} [\mathcal{A} (u1 + 1, t, T)] + \mathrm{Re} [C (u1 + 1, t, T)] v + \mathrm{Re} (\mathcal{A} (u1 + 1, t, T) + C (u1 + 1, t, T)] v + iu \right) \, du \right] \\
= \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp \left( \mathrm{Re} [\mathcal{A} (u1 + 1, t, T)] + \mathrm{Re} [C (u1 + 1, t, T)] v + iu \right) \, du \right] \\
= \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp \left( \mathrm{Re} [\mathcal{A} (u1 + 1, t, T)] + \mathrm{Re} [C (u1 + 1, t, T)] v + iu \right) \, du \right] \\
= \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp \left( \mathrm{Re} [\mathcal{A} (u1 + 1, t, T)] + \mathrm{Re} [C (u1 + 1, t, T)] v + iu \right) \, du \right] \\
= \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left[ \frac{1}{u} \exp \left( \mathrm{Re} [\mathcal{A} (u1 + 1, t, T)] + \mathrm{Re} [C (u1 + 1, t, T)] v + iu \right) \, du \right]
\]

\(^{50}\)Note that Leibniz integral rule is used here. One should verify that the integrand is a Lebesgue-integrable function of \( x \) for each \( u \), that for almost all \( x \), the derivative of the integrand (w.r.t. \( x \)) exists for all \( x \), and that there is an integrable function \( \theta \) such that \( \frac{\partial}{\partial x} \left( e^{i\mathcal{A} (u1 + 1, t, T)} \right) \leq \theta(u) \) for all \( x \) and almost every \( u \).
If $x = y$, then
\[
\frac{\partial P_1}{\partial y}(y, v) = \frac{1}{\pi} \int_0^{\infty} \left( \exp(\text{Re} [\alpha (ui + 1, t, T) + \text{Re} [\gamma (ui + 1, t, T) v]) \times \cos (\text{Im} [\alpha (ui + 1, t, T) + \text{Im} [\gamma (ui + 1, t, T) v] + u (y - k)]) \right) du.
\]

For $x = v$,
\[
\frac{\partial P_1}{\partial v}(y, v) = \frac{1}{\pi} \int_0^{\infty} \left( \frac{1}{\sqrt{u}} \left( \begin{array}{c} \text{Re} [C (ui + 1, t, T)] \exp (\text{Re} [A (ui + 1, t, T)] + \text{Re} [C (ui + 1, t, T) v]) \\ \text{Im} [C (ui + 1, t, T)] \exp (\text{Re} [A (ui + 1, t, T)] + \text{Re} [C (ui + 1, t, T) v]) \end{array} \right) \times \cos (\text{Im} [\mathcal{A} (ui + 1, t, T)] + \text{Im} [\mathcal{C} (ui + 1, t, T) v] + u (y - k)) \right) du.
\]

**D Simulation-Based Results: Additional Results**

**D.1 Filtering Tests**

Figures 5 to 9 show an example of the filtered values based on different data sources. For this experiment, $M$ is set to 5. In general, using the five different data sources yields accurate filtered values (with narrower confidence intervals). Instantaneous variance and log-equity jumps are more precisely filtered when using the realized option variance. When only log-equity values are used, the various filtered values are very imprecise. This observation is consistent with Table 2: the error measures are higher when we only consider the log-equity values in the filtering step.

**D.2 Option Data: How Much Is Enough?**

We use the simulation experiment of Section 4.1, but this time, fifteen implied volatilities are observed on each day. These European OTM options have a maturity of 30, 90 and 150 business days, and call-equivalent deltas of 0.20, 0.35, 0.50, 0.65 and 0.80 respectively. The filter is run using $M = 5$.

The idea behind this test is to only use part of the option data available and select a limited number of maturities or call-equivalent deltas. From this subsample, we apply the filter and compare filtered values to real simulated values and to filtered values computed using the whole dataset.

Table 13 shows RMSE for various filtered values. These error measures are computed with respect to true values and filtered means again.

Regarding the option maturity, it has a marginal impact on the results. However, according to Panel A, it seems that the use of two maturities is better than one. Therefore, adding more options helps the filtering of latent quantities, regardless of the maturity of the option. According to Panel B of Table 13, OTM put options filter the jump components more adequately.

**D.3 Option Quadratic Variation Increments Approximation**

One of the key features of the filter proposed in Section 3 is that a few number of intraday prices are required to compute $ROV$. We test this approximation by studying the impact of $M$ on the computation of $ROV$.

Using the parameters given in Table 1, we simulate 500 paths at a frequency of 1/1,560 over a day and compute the realized option variance for a call along each path. Next, the simulated $ROV$ is compared to approximations of this quantity, denoted by $A_{OQV}$, which are computed with a lower number of intraday observations $M \in \{1; 2; 3; 5; 10; 1,560\}$. The maturity of the options used in this exercise is 30
Figure 5: **Filtered instantaneous variance using various data sources.**

Variance path’s means obtained by the particle filters, true spot variance path, and 95% and confidence intervals computed using empirical quantiles (from particle filters) are shown in this figure using different data sources. The mean of the filtered density (obtained using particle filters) is taken as the filtered instantaneous variance. $Y$ means daily log-equtity value, $RV$ means realized variance, $BV$ means bipower variation, $IV$ means implied volatility here, and $ROV$ means realized option variance.
Figure 6: Filtered quadratic variation using various data sources.
Quadratic variation path’s means obtained by the particle filters, true quadratic variation path, and 95% and confidence intervals computed using empirical quantiles (from particle filters) are shown in this figure using different data sources. \( Y \) means daily log-equity value, \( RV \) means realized variance, \( BV \) means bipower variation, \( IV \) means implied volatility here, and \( ROV \) means realized option variance.
Figure 7: Filtered integrated variance using various data sources.
Integrated variance path’s means obtained by the particle filters, true integrated variance path, and 95% and confidence intervals computed using empirical quantiles (from particle filters) are shown in this figure using different data sources. $Y$ means daily log-equity value, $RV$ means realized variance, $BV$ means bipower variation, $IV$ means implied volatility here, and $ROV$ means realized option variance.
Figure 8: **Filtered log-equity jumps using various data sources.**
Return jump path’s means obtained by the particle filters and true return jump path are shown in this figure using different data sources. $Y$ means daily log-equity value, $RV$ means realized variance, $BV$ means bipower variation, $IV$ means implied volatility here, and $ROV$ means realized option variance.
Figure 9: Filtered variance jumps using various data sources.

Variance jump path’s means obtained by the particle filters and true variance jump path are shown in this figure using different data sources. \( Y \) means daily returns, \( RV \) means realized variance, \( BV \) means bipower variation, \( IV \) means implied volatility here, and \( ROV \) means realized option variance.
Table 13: RMSE for various quantities across 100 simulated paths when using only a limited number of options.

### Panel A: Maturities

<table>
<thead>
<tr>
<th>Maturity</th>
<th>RMSE (true)</th>
<th>RMSE (filtered)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTM = 30</td>
<td>5.9196</td>
<td>0.0212</td>
</tr>
<tr>
<td>DTM = 90</td>
<td>6.3944</td>
<td>0.0202</td>
</tr>
<tr>
<td>DTM = 150</td>
<td>6.6394</td>
<td>0.0198</td>
</tr>
</tbody>
</table>

### Panel B: Moneyness

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>RMSE (true)</th>
<th>RMSE (filtered)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δε = 0.20</td>
<td>6.7777</td>
<td>0.0271</td>
</tr>
<tr>
<td>Δε = 0.35</td>
<td>6.8674</td>
<td>0.0265</td>
</tr>
<tr>
<td>Δε = 0.50</td>
<td>6.9247</td>
<td>0.0246</td>
</tr>
</tbody>
</table>

Quantities were multiplied by 1,000. Filtered values are computed as the mean of resampled particles obtained via a SIR particle filter with M = 5 using Y, RV, BV, I and ROV and every maturities and moneyness. A maximum of fifteen implied volatilities are observed on each day: these European OTM options have a maturity of 30, 90 and 150 business days, and call-equivalent deltas of 0.20, 0.35, 0.50, 0.65 and 0.80 respectively. In the table, DTM means day to maturity and Δε = 0.20 represents call-equivalent delta.

As a measure of the quality of the approximation, we compute the RMSE and relative RMSE result from comparing the ROV computed at the highest frequency with that of the approximation. We also regress ΔOQV on ROV and employ the R-squared of this regression as a measure of the quality of the fit.

Table 14 shows the root mean square errors (RMSE), relative RMSE, and regression R-squareds for different levels of moneyness. As expected, we find that errors decrease as M becomes larger and that with as few as M = 5 observations per day the approximation provides satisfactory results. Note that the presence of jumps has an apparent impact on the performance of the approximation, which is not surprising as these rare events introduce more variation in the estimation.

To visualize the performance of the approximation, Figure 10 plots ROV and their approximations ΔOQV for days when jumps are absent (top panels) and present (bottom panels), respectively. If ΔOQV constitutes a good approximation of ROV, all the points should be aligned on the diagonal, as it is the case for M = 1,560. Note that ΔOQV approaches to ROV for values of M = 3 and higher.

### E Exploring Realized Option Variances Empirically: Additional Material

#### E.1 Data

Figure 11 presents the time series associated with some of these variables. It shows in the first three panels the daily time series of the previous three variables and in the last panel the realized option variance time series for the ATM option with a maturity of 30 business days. We observe that all series

---

60 We run the following regression: ΔOQVi = β0 + βiROVi + εi, and compute the R-squared associated with it. The higher the R-squared, the better the quality of the approximation.

---
Figure 10: Realized option variance against option quadratic variance increments for a call-equivalent delta of 0.50 and a maturity of 30 business days, on days when jumps are absent (top panels) and present (bottom panels).

This figure is generated using 500 simulated paths of one day and a European call option that has a call-equivalent delta of 0.50 and a maturity of 30 business days. ROVs are compared to six different values for which various $M$ are used (i.e. 1, 2, 3, 5, 10 and 1,560). Parameters of Table 1 are used.
Table 14: RMSE, relative RMSE and regression $R^2$-squared for $\Delta OQV$ approximation across 500 days.

Panel A: RMSE

<table>
<thead>
<tr>
<th>$\Delta e$</th>
<th>Jumps are absent</th>
<th>Jumps are present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.851 2.794 1.295 0.563</td>
<td>21.925 35.893 55.659 62.941 30.808</td>
</tr>
<tr>
<td>0.35</td>
<td>0.106 1.599 0.713 0.295</td>
<td>16.286 28.140 45.653 58.119 28.371</td>
</tr>
<tr>
<td>0.5</td>
<td>0.211 1.719 0.856 0.429</td>
<td>14.131 22.775 32.613 37.994 20.383</td>
</tr>
<tr>
<td>0.65</td>
<td>0.057 0.381 0.137</td>
<td>11.956 15.303 20.103 20.851 12.772</td>
</tr>
<tr>
<td>0.8</td>
<td>0.172 0.455 0.220</td>
<td>8.960 13.258 18.675 19.392 11.282</td>
</tr>
<tr>
<td>1.0</td>
<td>0.146 0.433 0.348 0.126</td>
<td>0.725 1.714 2.924 1.890 0.787</td>
</tr>
</tbody>
</table>

Panel B: Relative RMSE

<table>
<thead>
<tr>
<th>$\Delta e$</th>
<th>Jumps are absent</th>
<th>Jumps are present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.205 0.151 0.121 0.141 0.175</td>
<td>0.289 0.240 0.205 0.176 0.199</td>
</tr>
<tr>
<td>0.35</td>
<td>0.081 0.068 0.076 0.089</td>
<td>0.209 0.179 0.158 0.144 0.159</td>
</tr>
<tr>
<td>0.5</td>
<td>0.077 0.058 0.052 0.059 0.065</td>
<td>0.141 0.118 0.103 0.108 0.120</td>
</tr>
<tr>
<td>0.65</td>
<td>0.057 0.046 0.043 0.046 0.049</td>
<td>0.111 0.091 0.077 0.077 0.089</td>
</tr>
<tr>
<td>0.8</td>
<td>0.043 0.039 0.038 0.040 0.040</td>
<td>0.091 0.079 0.069 0.070 0.078</td>
</tr>
<tr>
<td>1.0</td>
<td>0.038 0.038 0.036 0.037 0.038</td>
<td>0.026 0.027 0.027 0.025 0.024</td>
</tr>
</tbody>
</table>

Panel C: Regression $R^2$-squared

<table>
<thead>
<tr>
<th>$\Delta e$</th>
<th>Jumps are absent</th>
<th>Jumps are present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.439 0.475 0.532 0.585 0.573</td>
<td>0.019 0.092 0.864 0.916 0.918</td>
</tr>
<tr>
<td>0.35</td>
<td>0.867 0.855 0.889 0.899</td>
<td>0.953 0.935 0.910 0.926 0.928</td>
</tr>
<tr>
<td>0.5</td>
<td>0.944 0.932 0.922 0.930 0.941</td>
<td>0.964 0.957 0.952 0.960 0.956</td>
</tr>
<tr>
<td>0.65</td>
<td>0.967 0.957 0.946 0.954 0.963</td>
<td>0.977 0.981 0.981 0.988 0.981</td>
</tr>
<tr>
<td>0.8</td>
<td>0.978 0.968 0.958 0.965 0.973</td>
<td>0.987 0.985 0.984 0.989 0.985</td>
</tr>
<tr>
<td>1.0</td>
<td>0.981 0.970 0.960 0.969 0.976</td>
<td>1.000 1.000 1.000 1.000 1.000</td>
</tr>
</tbody>
</table>

The real $ROV$ value is computed using option prices at a frequency of 1/1,560. The error corresponds to the difference between $\Delta OQV$ and $ROV$. 0.20, 0.35, 0.50, 0.65 and 0.80 correspond to the different call-equivalent deltas considered. The OTM option maturity is 30 business days. $ROV$s are compared to six different values for which various $M$ are used (i.e., 1, 2, 3, 5, 10 and 1,560). To compute the $R^2$-squared, we run the following regression: $\Delta OQV_i = \beta_0 + \beta_1 ROV_i + \varepsilon_i$.

increase during the financial crisis period and exhibit sporadic spikes across the sample following the flash crash episode (May 6, 2010) and the downgrade of U.S. debt (August 5, 2011). This commonality shows the degree of interdependence across different markets in response to important events. We now proceed to explore in more detail the full panel of realized option variances.

Table 15 exhibits descriptive statistics of realized variance, bipower variation and realized option variance.

E.2 Understanding Realized Option Variances

Figure 12 shows the first six in-sample principal components (PCs) of the realized option variance surface.

Figure 13 shows the surface induced by realized option variances on July 6, 2004. As it is clear from the figure, the surface has an inverted U-shape, in which the highest values are observed for options that are at-the-money and the lowest for those that are out-of-the-money.
Figure 11: **Realized volatility, square root of the bipower variation, ATM option implied volatility for options with maturity closest to 30 business days and ATM realized option volatility (square root of ROV) for options with maturity closest to 30 business days.**

Annualized realized variance and bipower variation are computed from intraday S&P futures prices. Since futures are quarterly contracts, we build a time series from these data by rolling contracts two weeks before expiration. We follow Zhang et al. (2005) and compute a microstructure-noise robust estimate of the daily realized variance as the average of $RV$ and $BV$ estimates based on different subsets of prices. To compute the daily realized variance of an option, we employ tick-by-tick Level I quote data from options provided by Tick Data. From the data, we construct one-minute midquote series and compute the daily variation according to Equation (11). We again follow Zhang et al. (2005) and compute a microstructure-noise robust estimate of the daily realized option variance. Each time series is displayed from July 2004 to December 2012.
Table 15: Descriptive statistics of realized variance, bipower variation and realized option variance.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>S.D.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>10%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>RV × 1000</td>
<td>27.523</td>
<td>75.247</td>
<td>13.846</td>
<td>320.527</td>
<td>3.990</td>
<td>53.757</td>
</tr>
<tr>
<td>BV × 1000</td>
<td>25.783</td>
<td>72.283</td>
<td>13.371</td>
<td>294.644</td>
<td>3.591</td>
<td>49.767</td>
</tr>
<tr>
<td>ROV, DTM = 30, Δe = 0.20</td>
<td>24.001</td>
<td>83.350</td>
<td>8.827</td>
<td>104.327</td>
<td>0.792</td>
<td>55.478</td>
</tr>
<tr>
<td>ROV, DTM = 30, Δe = 0.35</td>
<td>33.794</td>
<td>87.637</td>
<td>8.314</td>
<td>92.257</td>
<td>2.832</td>
<td>68.863</td>
</tr>
<tr>
<td>ROV, DTM = 30, Δe = 0.50</td>
<td>29.858</td>
<td>67.238</td>
<td>9.305</td>
<td>130.376</td>
<td>4.279</td>
<td>59.223</td>
</tr>
<tr>
<td>ROV, DTM = 30, Δe = 0.65</td>
<td>19.484</td>
<td>46.732</td>
<td>6.753</td>
<td>66.278</td>
<td>1.307</td>
<td>44.229</td>
</tr>
<tr>
<td>ROV, DTM = 30, Δe = 0.80</td>
<td>17.434</td>
<td>78.232</td>
<td>10.151</td>
<td>141.495</td>
<td>2.510</td>
<td>53.110</td>
</tr>
<tr>
<td>ROV, DTM = 30, Δe = 0.90</td>
<td>25.184</td>
<td>53.298</td>
<td>8.336</td>
<td>109.830</td>
<td>3.671</td>
<td>47.970</td>
</tr>
<tr>
<td>ROV, DTM = 90, Δe = 0.20</td>
<td>26.705</td>
<td>77.032</td>
<td>10.151</td>
<td>141.495</td>
<td>2.510</td>
<td>53.110</td>
</tr>
<tr>
<td>ROV, DTM = 90, Δe = 0.35</td>
<td>34.453</td>
<td>74.512</td>
<td>8.202</td>
<td>97.525</td>
<td>5.806</td>
<td>66.649</td>
</tr>
<tr>
<td>ROV, DTM = 90, Δe = 0.50</td>
<td>25.184</td>
<td>53.298</td>
<td>8.336</td>
<td>109.830</td>
<td>3.671</td>
<td>47.970</td>
</tr>
<tr>
<td>ROV, DTM = 90, Δe = 0.65</td>
<td>15.027</td>
<td>55.173</td>
<td>16.408</td>
<td>363.312</td>
<td>1.272</td>
<td>30.695</td>
</tr>
</tbody>
</table>

S.D. stands for standard deviation. 10% and 90% represent the 10% and 90% quantiles of empirical samples respectively. RV and BV are given on an annualized basis.

Figure 12: Principal components of the realized option volatility surface.
The figure shows the first six principal components extracted from the S&P 500 realized option variance surface from July 2004 to December 2012. On each day, realized option variances are interpolated from a locally smoothing quadratic regression surface estimated on every available realized option volatility. Over the call-equivalent delta dimension, we use the grid of values 0.1, 0.2, ..., 0.9; over the tenor dimension, we employ maturities of 30 and 90 business days. These partitions require a total of 18 realized option volatilities per day.
Figure 13: Realized option variances and fitted surface as a function of both the option Black and Scholes (1973) call-equivalent delta and business days to maturity.

Realized option variances are computed using the daily variation according to Equation (11). The example is given for July 6, 2004. The fitted surface uses the locally smoothing quadratic regression method.

F Option Pricing Implications

F.0.1 In-Sample Assessment

We first assess the ability of both parameter sets to fit historical implied volatilities. To perform this exercise, we use the relative implied volatility root mean square error (RIVRMSE), defined as follows:

\[
\text{RIVRMSE} = \sqrt{\frac{1}{\sum_k O_k} \sum_i \frac{1}{\sum_k O_k} \left( IV_{k,t,i}(Y_{t,k}, \hat{V}_{t,k}) - \sigma_{BS_{k,t,i}}^2 \right)^2},
\]

where \( IV \) is the model implied volatility, \( \hat{V}_{t,k} \) is the filtered instantaneous variance on day \( k \), and \( O_k \) represents the number of options in a subset of all the options available on day \( k \). Here, \( \sigma_{BS} \) denotes the Black-Scholes implied volatility associated with the observed option price.

The leftmost columns of Panel A (Table 16) show the RIVRMSE for the panel of 21,400 options employed in the estimation. We observe that the fit provided by the parameter set without ROV's is more accurate on average than the one that includes these variances – an RIVRMSE of 24.56 with ROV vs. 23.45 without. This result stems from the fact that maximization of the likelihood in the former case backs parameters that better fit IVs. However, the average fit is not that different, so the addition of ROVs does not substantially deteriorate the overall fit of option prices.

In order to identify which contracts would benefit more from the inclusion of ROVs, we assess the goodness of fit of both parameter sets for these variances. To this end, we use a criterion similar to the RIVRMSE in Equation (20) and compute the relative realized option variance root mean square error (RROVRMSE), defined as:

\[
\text{RROVRMSE} = \sqrt{\frac{1}{\sum_k O_k} \sum_i \sum_k \frac{1}{\sum_k O_k} \left( \Delta OQV_{(k-1)r,k,t,i} - ROV_{(k-1)r,k,t,i} \right)^2},
\]

where \( \Delta OQV \) is the option quadratic variation increment computed from the model and ROV is the
observed realized option variance.

Rightmost columns of Panel A (Table 16) exhibit the RROVRMSE by moneyness and maturity. As expected, the average fit of these quantities is lower when ROVs are included in the information set. Nonetheless, it is remarkable to observe that the overall RROVRMSE is about 3 times lower when ROVs is included and that there are significant differences across contracts. Whereas error differences for call options are almost fourfold, those differences for put options are about twofold. The differences across contracts suggest that ROVs bring more information from OTM calls than from OTM puts, which goes in line with the importance of the coefficient associated with call contracts in explaining future index variation as discussed in Subsection 5.3.

We continue with the in-sample analysis by looking at one-day ahead predicted option prices from both sets of parameters. To analyze the fit of implied volatilities, we enlarge our estimation sample of 21,400 options to include all ATM and OTM options available for the S&P 500 index in OptionMetrics between July 2004 and December 2012. We restrict our analysis to maturities of at least one week and at most one year. As before, observations violating no-arbitrage restrictions are excluded. Our new sample is composed of a total of 401,081 contracts. Option prices are converted to implied volatilities with the Black and Scholes’s (1973) pricing formula. To compute model-predicted implied volatilities on day $t$, we calculate one-day ahead expectations of model variables for day $t$ using the filter’s predictive distribution resulting from day $t-1$. Using observed and predicted implied volatilities, we compute RIVRMSEs according to maturity, moneyness, and year, which provides a better picture of the fit associated with each parameter set.

The overall RIVRMSEs (leftmost columns of Panel B in Table 16) are very close on average for both parameter sets, with lower values of RIVRMSE observed when ROVs are included (32.79 against 32.87). We use the Diebold and Mariano (1995, DM henceforth) test to see if the apparent predictive superiority of ROV based forecasts is not particular to this sample. Using both RIVRMSE time series, we compare their forecasting accuracy and test for:

$$H_0 : \mathbb{E}[d_t] = 0, \forall t \quad H_1 : \mathbb{E}[d_t] > 0, \forall t$$

where $d_t = \text{RIVRMSE}_{\text{without ROV}} - \text{RIVRMSE}_{\text{with ROV}}$ is the time-$t$ loss differential between the forecast produced without ROV and the one including it. The DM test statistic is 5.32 and is significant at a 1% level, confirming that there exists a differential between the two forecasts and that the one based on ROV information produces more accurate results on average. A closer scrutiny of the data reveals that including ROVs is especially helpful in the pricing of options with long maturities, OTM calls, and options for the years before 2008.

Finally, we carry out a similar exercise with realized option variances and analyze one-day ahead forecasts for both parameter sets. For this exercise, we restrict our enlarged sample to those options for which ROV data is available in the TickData database. Out of 401,081 options, 282,534 contracts were included in our analysis. As reported in the rightmost columns of Panel B, Table 16, we observe again that, across different option characteristics and years, RROVRMSE are lower for the parameter set that was obtained with ROVs. Similar to the RIVRMSE case, we apply the DM test to both RROVRMSE series and obtain a value of 13.63, confirming statistically the important differences between the two sets of parameters.

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61 We continue using ATM and OTM options to keep a comparable sample with that employed in our previous analyses.
62 The lag in the Diebold and Mariano (1995) is selected as the first partial autocorrelation that is within confidence bounds. The estimated lag in this exercise is 2.
63 We repeated our analysis of RIVRMSE with the sample of 282,534 contracts and found similar results.
Table 16: In-sample performance (2004–2012).

Panel A: In-sample option pricing and realized option variance performances, in terms of RIVRMSE and RROVRMSE.

<table>
<thead>
<tr>
<th></th>
<th>RIVRMSE</th>
<th>RROVRMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>With ROV</td>
<td>Without ROV</td>
</tr>
<tr>
<td>DTM = 30, Δ^e = 0.20</td>
<td>19.13</td>
<td>17.62</td>
</tr>
<tr>
<td>DTM = 30, Δ^e = 0.35</td>
<td>25.05</td>
<td>23.04</td>
</tr>
<tr>
<td>DTM = 30, Δ^e = 0.50</td>
<td>28.51</td>
<td>27.93</td>
</tr>
<tr>
<td>DTM = 30, Δ^e = 0.65</td>
<td>27.27</td>
<td>25.94</td>
</tr>
<tr>
<td>DTM = 90, Δ^e = 0.20</td>
<td>21.97</td>
<td>20.01</td>
</tr>
<tr>
<td>DTM = 90, Δ^e = 0.35</td>
<td>23.37</td>
<td>22.90</td>
</tr>
<tr>
<td>DTM = 90, Δ^e = 0.50</td>
<td>25.23</td>
<td>23.16</td>
</tr>
<tr>
<td>DTM = 90, Δ^e = 0.65</td>
<td>25.17</td>
<td>22.40</td>
</tr>
<tr>
<td>DTM = 90, Δ^e = 0.80</td>
<td>23.57</td>
<td>20.69</td>
</tr>
<tr>
<td>All</td>
<td>24.56</td>
<td>23.45</td>
</tr>
</tbody>
</table>

Panel B: One-day ahead in-sample performances, in terms of RIVRMSE and RROVRMSE.

<table>
<thead>
<tr>
<th></th>
<th>RIVRMSE</th>
<th>RROVRMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>With ROV</td>
<td>Without ROV</td>
</tr>
<tr>
<td>DTM &lt; 60</td>
<td>26.78</td>
<td>25.39</td>
</tr>
<tr>
<td>60 ≤ DTM &lt; 120</td>
<td>26.20</td>
<td>24.24</td>
</tr>
<tr>
<td>120 ≤ DTM &lt; 180</td>
<td>26.07</td>
<td>25.08</td>
</tr>
<tr>
<td>180 ≤ DTM</td>
<td>51.76</td>
<td>55.28</td>
</tr>
<tr>
<td>Δ^e &lt; 0.20</td>
<td>23.25</td>
<td>28.56</td>
</tr>
<tr>
<td>0.20 ≤ Δ^e &lt; 0.35</td>
<td>29.55</td>
<td>31.11</td>
</tr>
<tr>
<td>0.35 ≤ Δ^e &lt; 0.50</td>
<td>34.52</td>
<td>35.51</td>
</tr>
<tr>
<td>0.50 ≤ Δ^e &lt; 0.65</td>
<td>41.42</td>
<td>41.24</td>
</tr>
<tr>
<td>0.65 ≤ Δ^e &lt; 0.80</td>
<td>37.41</td>
<td>36.83</td>
</tr>
<tr>
<td>0.80 ≤ Δ^e</td>
<td>31.74</td>
<td>29.62</td>
</tr>
<tr>
<td>2004</td>
<td>25.73</td>
<td>28.52</td>
</tr>
<tr>
<td>2005</td>
<td>28.44</td>
<td>33.71</td>
</tr>
<tr>
<td>2006</td>
<td>26.58</td>
<td>32.05</td>
</tr>
<tr>
<td>2007</td>
<td>29.29</td>
<td>31.87</td>
</tr>
<tr>
<td>2008</td>
<td>31.17</td>
<td>30.56</td>
</tr>
<tr>
<td>2009</td>
<td>38.47</td>
<td>35.74</td>
</tr>
<tr>
<td>2010</td>
<td>34.99</td>
<td>33.49</td>
</tr>
<tr>
<td>2011</td>
<td>35.33</td>
<td>34.38</td>
</tr>
<tr>
<td>2012</td>
<td>31.68</td>
<td>31.39</td>
</tr>
<tr>
<td>All</td>
<td>32.79</td>
<td>32.87</td>
</tr>
</tbody>
</table>

In Panel A, only options used for estimation are employed. In Panel B, the sample of S&P 500 options is acquired via OptionMetrics. Options violating arbitrage conditions were discarded. In Panel C, the sample of S&P 500 options is acquired via Tick Data. Options violating arbitrage conditions were discarded. Model prices and ROVs are calculated by using the parameters of Table 10. The relative implied volatility root mean square error (RIVRMSE) is computed as follows:

\[ \text{RIVRMSE} = \sqrt{\frac{1}{\sum_i \sum_k O_k} \sum_k \sum_{i=1}^{O_k} \left( \frac{IV_{k_i}(Y_{k_i}, \hat{V}_{k_i}) - \sigma_{k_i}^{RIV}}{\sigma_{k_i}^{RIV}} \right)^2} \]

where IV is the model implied volatility, \( \hat{V}_{k_i} \) is the filtered instantaneous variance on day k, and \( O_i \) represents the number of options in a subset of all the options available on day k. The relative realized option variance root mean square error (RROVRMSE) is computed similarly: \( IV_{k_i}(Y_{k_i}, \hat{V}_{k_i}) \) is replaced by \( \Delta OOV_{(k-1)xk_2} \) and \( \sigma_{k_i}^{RIV} \) by ROV_{(k-1)xk_2}. RIVRMSEs and RROVRMSEs are given in percentage.