A Dynamic Option Pricing Model with Economic Regime Shifts

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Abstract

This paper develops a dynamic option pricing model with economic regime shifts. Assuming the one-period logarithmic returns of the underlying asset follows a simple hidden Markov model, we derive analytical pricing formula for European options. It is found that option prices are highly dependent on the likelihood of the current economic regime, with additional price premium spelled out for the regimes with high volatilities. The so-called volatility smile or smirk observed from the market activities are well explained by our model. To elaborate the versatility of our model, we compare our model analysis to the findings developed by Hardy (2001), which is a special case of our general setting.

1 Introduction

With the experience of the financial market turmoil in the past 30 years, many investors believe that pricing of derivative securities should be further developed, as the observed market activities are not consistent with the implication of the standard pricing models. There is a strong consensus that asset returns exhibit higher volatilities when the economy is in contraction than they do when the economy is in expansion. A meaningful option pricing model should be able to capture this market phenomenon.

As surveyed by Bakshi, Cao and Chen (1997), existing option pricing models do not perform well. For example, in-the-money stock call options or out-of-the-money stock put options are usually over priced, while out-of-money calls and in-the-money puts are under prices by the Black-Scholes-Merton model. If the risk free rate is considered to be constant for a short time period, the volatility of asset returns underlying the options is the key component. Standard models set this

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quantity as a constant (Black and Scholes, 1973; Merton, 1973; Cox et al. 1979), while empirical analysis has proven that it changes over time. There have been several approaches to fixing this deficiency. Early models, such as Merton (1976), Hull and White (1990), and Heston (1993) allowed for jump diffusion processes and/or stochastic volatility to capture discontinuity in the underlying asset prices and to relax the constancy of the volatility parameter. A general auto-regressive and conditional heteroscedasticity (GARCH) process has been widely applied to volatility modeling (e.g., Duan 1995; Engle, 1982; Bollerslev, 1986). However, these models are so complex that parameter estimations and practical applications may be difficult.


The prevailing regimes of the economy are unobservable. The reasons for it are straightforward: the economy does not function in a way where all components of the economy are independent against each other. For example, a thriving economy tends to motivate investors to borrow more therefore increases the borrowing rate, while a stagnant economy tends to drive the interest rate down due to lacking demand for money. However, the economic activities driven by Federal Reserve may have impact on the interest rate on an opposite way. Therefore, on the surface that appears to be a growing economy may lead to a low interest rate, vice versa. Likewise in the financial market, what is observed (the signal of market trend) does not always indicate regimes of the financial market. There are periods of sustained increases in the price of securities during bear market and periods of sustained decreases in the price of securities during bull market, which are so called bear market rally and bull market rally, respectively.

In the literature of asset pricing, a number of researchers have assumed that the distribution of the underlying asset returns shift over time within a number of economic modes, which are
unobservable regimes introduced from financial shocks (e.g. Jeanne and Masson, 2000; Cerra, 2005; Hamilton, 2005), changes in consumer preference (e.g. Veronesi, 1999) and abrupt changes in government policies (Hamilton, 1989; Sims and Zha, 2006; Davig, 2004). In addition, consumer and investors' behaviors, such as decisions, consumption, savings and investment, may differ between an expansion and contraction economy. Norden and Schaller (1993) examine the U.S. stock market by testing a single regime in the stock market against three different alternative regimes and find strong evidence that there are regimes indeed in the US stock market. Assuming the existence of regimes, several researchers have considered how options prices should be changed. Yao, Zhang and Zhou (2003) simulate two regimes for the dynamic moving of the asset returns of European style options. The recent work by Fuh, Ho, Hu and Wang (2012) provide tree model to compute the price of European call options. However, None of these papers characterize the source of the volatility.

Hardy (2001) extended the Black-Scholes-Merton model by allowing the location parameters of asset returns to switch within a set of possible values. Using her model, volatility smile was well explained. However, the option pricing model was based on steady state probability distribution of regimes, and prediction of future economic regimes cannot be done, even though the posterior distribution of the regimes are changing over time. To develop a more applicable model, we allow the distribution of asset returns to switch instead of just the location parameters. This model setting essentially embraces Hardy’s model as a special case in which the two model will yield the same pricing mechanism if the posterior probability distribution of the regimes reach the steady state.

In this paper, we first derive analytical solutions for pricing European options. Then, we analyze TSE 300 index options to determine whether a regime-switching model is better than a standard pricing model at explaining the pricing inefficiency described before. An innovative finding out of this research is that options prices are dependent on the likelihood of the economic regime, with additional premium spelled out for high volatility regimes.

The remainder of the paper is organized as follows. Section 2 introduces modeling of the economic regimes and the underlying asset prices and corresponding empirical analysis. In Section 3, we derive the analytical formula for option prices and demonstrate the corresponding empirical analysis. Section 4 concludes the paper.

2 Modeling the Economic Regimes and Underlying Asset Prices

2.1 Modeling the economic regimes

National Bureau of Economic Research (NBER) business dating committee retrospectively divides economic strength into two regimes, contraction and expansion, which correspond to substantial
changes in influential economic statistics, such as real GDP and employment rate. However, there are issues for us to use NBER analysis to characterize the economic strength over time. The role of thumb for the NBER analysis is that a specific regime is kept for a period of at least 6 months before it switches to the other regime. It is by this role that NBER analysis may incorrectly specify the economic strength if it has changed the mode only for a period of time less than 6 months. This is troublesome for option pricing, as accurate estimation of the current regime of the economic strength is an input to the option pricing model.

We develop a hidden Markov regime switching model using macroeconomic indicators to estimate the unobserved market regimes. Under the Markov chain regime-switching framework, switches amongst the regimes are indeterminate and stochastic. One cannot directly tell whether a switch derives from observations without any further study. Thus, model-implied posterior probability serves as an indicator of the market regimes. In this paper, we propose an asset-pricing model for financial securities that exhibits a dynamic pattern with economic indicators described by a Markov regime-switching process. The TSE 300 index plays as the driving force for a persistent change in volatility in our model. Based on the maximum likelihood, the inferred regimes turn out to be more than 80% coincident with the NBER business dating for the period of 1956-1999.

Denote the observed macroeconomic variables as $L_t$ and the unobserved economic strength as $M_t$, which can only take discrete values, 1, 2, 3, ..., $K$. We assume that

$$L_t = A_{M_t} + B_{M_t} \phi_t$$

where $A_{M_t}$, $B_{M_t}$ are regime-dependent. $\phi_t$ is a multivariate random variable with standard multivariate normal distribution and independent over time. There are $K$ distinct regimes and all the regimes follow the first order Markov chain with an initial regime distribution $q_0$ and a constant transition matrix $P = p_{ij}$, where $p_{ij}$ indicates that the transition probability of the market transfer from regime $i$ at time $t - 1$ to regime $j$ at time $t$. The transition matrix is given by the probability of $M_t$ depends on the probability of $M_{t-1}$; that is,

$$P = \begin{bmatrix}
    p_{11} & p_{12} & \cdots & p_{1K} \\
    p_{21} & p_{22} & \cdots & p_{2K} \\
    \vdots & \vdots & \ddots & \vdots \\
    p_{K1} & p_{K2} & \cdots & p_{KK}
\end{bmatrix}$$

where $Pr(M_t = j|M_{t-1} = i, M_{t-2} = k, \ldots) = Pr(M_t = j|M_{t-1} = i) = p_{ij}$ is the transition matrix probability from state $i$ to state $j$. The distribution of regimes at time $t$ can be updated by the Bayesian analysis with new information inflows from $t-1$ to $t$ under a multivariate regime-switching model.

The optimal number of regimes are selected by the Bayesian information criterion (BIC) (Schwarz, 1978):

$$BIC(K) = -2 \ln(L|K) + f(K) \ln(T)$$
where $K$ is the number of regimes and $L$ is the maximum likelihood of the parameters, given the number of regimes and the observed data. $T$ is the length of the observed data, and $f(K)$ is the number of parameters. The optimal number of regimes is the minimizer of $BIC(K)$.

Conditional on regimes at time $t + 1$, the stock returns follow the process:

$$\log \frac{S_{t+1}}{S_t} | M_{t+1} \sim N(\mu_{M_{t+1}}, \sigma^2_{M_{t+1}}),$$

where $S_t$ is the underlying stock price at time $t$. The stock return process $R_t = \log \frac{S_{t+1}}{S_t}$ is in one of the market regimes, $M_t = 1, 2, \cdots, K$. Therefore, the dynamic underlying asset price is characterized by the following process:

$$R_t = \mu_{M_t} + \sigma_{M_t} \epsilon_t,$$

where $R_t$, defined as $\log \frac{S_{t+1}}{S_t}$, is a vector of logarithmic returns from time $t - 1$ to $t$. $\mu_{M_t}$ is the regime-dependent mean returns and $\sigma_{M_t}$ is the regime-dependent volatility of the underlying asset returns. The unconditional expected asset return from $t - 1$ to $t$ is the expectation of regime-dependent expected conditional asset returns, with the prior probability of $p_t(m)$ at regime $m$ and time $t$:

$$E(R_t) = E(E(R_t|M_t)) = \sum_{m=1}^{K} (\mu_{M_t}) p_t(m),$$

and the variance-covariance matrix is:

$$V(R_t) = \sum_{m=1}^{K} [(E(R_t) - E(R_t|M_t = m))^2 + \sigma_{M_t}] p_t(m)$$

Although Hardy(2001) relaxed the naive normal distribution assumption of BSM model in the following model set up:

$$\log \frac{S_{t+1}}{S_t} | M_t \sim N(\mu_{M_t}, \sigma_{M_t}),$$

this model only allows the location of asset return parameters to switch within a set of values. Conditional on the current market regime $M_t$, the asset returns follow a mixture normal distribution in our model:

$$\log \frac{S_{t+1}}{S_t} | M_t \sim \sum_{m=1}^{K} p_m N_m(\mu_{M_t}, \sigma^2_{M_t}),$$

where $m$ is an index taking values from 1 to $K$. Therefore, our model allows the distribution of asset returns to switch over time.

### 2.2 Empirical Analysis of the Underlying Asset Pricing Model

Hardy’s study (2001) is what we focus in this paper; we utilize the same dataset that Hardy uses in order to make a direct comparison between her model and ours. The data contain TSE 300 total
return Index from 1956 to 1999. Although S&P/TSX composite index has replaced the name of TSE 300 since 2002, we keep using TSE 300 in this paper to make our comparison straightforward.

2.2.1 TSE 300 Total Return Index Macro Level Regime Switching Model

For ease of exposition, TSE 300 total return index is the only macro level factor that characterizes market regimes. Using Bayes Information Criteria (BIC) to determine the optimal number of regimes, as Table 1 shows, this paper chooses 2 market regimes to derive all of the following results.

<table>
<thead>
<tr>
<th>Regime number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Likelihood</td>
<td>886.5</td>
<td>924.3</td>
<td>932.4</td>
<td>939.19</td>
<td>944.7</td>
</tr>
<tr>
<td>BIC</td>
<td>-1760.4</td>
<td>-1804.7</td>
<td>-1777</td>
<td>-1734.3</td>
<td>-1676.4</td>
</tr>
</tbody>
</table>

Figure 1 illustrates monthly total returns of TSE 300 index and implied posterior probabilities of bull regimes, where the left y-axis measures the scale of monthly total returns of TSE 300 and the right y-axis represents implied market regimes.

Figure 1: Monthly Total Returns of TSE 300 and Implied Market Regimes

The estimations of bear and bull regimes highly match the empirical data. During a recession, the volatility in bear regimes is much higher than which in bull markets. From 1956 to 1999, a
few significant financial recessions occurred, which include the Eisenhower Recession in 1958, 1973 stock market crash, 1979 oil crash, and 1997 Asian financial crisis. Our model-implied posterior probabilities accurately capture those economic downturns: all of the financial recessions above during that period are characterized.

The monthly returns of TSE 300 index reflect volatility clustering in Figure 1. The amplitude of the index differs over time, inferring that the risk level also has a time variation. A large vibration of returns in one period is more likely to result in great turbulences in the following periods, and vice versa. This is consistent with the characteristics of transition probability matrix, which contains a higher probability of retaining compared to transition:

\[
P = \begin{bmatrix}
0.8068 & 0.1932 \\
0.0364 & 0.9636
\end{bmatrix}
\]

2.2.2 Model Estimation and Prediction

Given \( M_t \) is a latent variable, Expectation and Maximization algorithm (e.g., Dempster, 1976) is used to carry out the model estimation. The EM algorithm is efficient for estimating models that have missing data or unobservable latent variables. The estimation is an interaction process between expectation and maximization. The E-step is used to estimate the missing data on regimes based on observed data and current estimates by calculating the expected log likelihood with the updating missing data. The M-step is to maximize the log likelihood function based on the missing data on regimes found in the E-step. Estimated model parameters are displayed in Table 2.
The regime-dependent means are all statistically significant (Table 2). It shows that the bull regime contains a negative mean and a high volatility, while the bear regime consists of a positive mean and a low volatility. The estimated parameters from the macro level model do not have obvious differences from which in Hardy’s model, as the parameters are estimated using maximum likelihood estimation in Hardy’s study. The estimated parameters of the single regime model are calculated as a benchmark of our macro level model. The mean in the single regime model is higher than which with the bear regime and lower than which using the bull regime; same for the volatility.

The procedure of fitting the model is also described in this section. Since the model possesses predictive power, the information set is prominent in terms of prediction. The performance of the model tends to better when the information becomes more completed. We provide three fitting procedures with different information sets: one period ahead, two periods ahead and three periods ahead. The fitting procedure can be generalized as follow.

- The one-period model fitted values are:
  \[ E[\ln(S_t|F_{t-1})] = \ln S_{t-1} + E[R_t(m)|F_{t-1}] \]
  \[ = \ln S_{t-1} + \sum_{m=1}^M q_t(m)R_t(m) \]

- The two-period model fitted values are:
  \[ E[\ln(S_t|F_{t-2})] = \ln S_{t-2} + E[R_{t-1}(m)|F_{t-2}] + E[R_t|F_{t-2}] \]
  \[ = E[\ln S_{t-1}|F_{t-2}] + E[R_t|F_{t-2}] \]
  \[ = E[\ln S_{t-1}|F_{t-2}] + \sum_{m=1}^M q_t(m)R_t(m) \]
The three-period model fitted values are:

$$E[\ln(S_t|F_{t-3})] = \ln S_{t-3} + E[R_{t-2}(m)|F_{t-3}] + E[R_{t-1}|F_{t-3}] + E[R_t(m)|F_{t-3}]$$

$$= E[\ln S_{t-1}|F_{t-3}] + E[R_t|F_{t-3}]$$

$$= E[\ln S_{t-1}|F_{t-3}] + \sum_{m=1}^{M} q_t(m) R_t(m)$$

(12) (13) (14)

where $S_t$ is the stock price at time $t$, $R_t(m)$ is the asset returns at regime $m$ and time $t$, and $q_t(m)$ is the posterior probability at regime $m$ and time $t$ and $T$ is the transition matrix.

Figure 2: Model Fitted Returns, Hardy(2001) Fitted Returns, and Actual Returns
Figure 2 and Table 2 illustrate the comparison between the macro level regime-switching parameters and Hardy’s parameters (2001). We plot an empirical growth returns \( \sum_{m=1}^{M} q_t(m)R_t(m) \) fitting process, using three-period models, to visualize our study.

The solid line in Figure 2 represents actual returns from 1956 to 1999 and the dashed line poetries one period fitted values. The actual returns are more volatile than the fitted values. The fitted values capture dynamic movements of the actual returns. We demonstrate it using prediction accurate ratio in the following section.

In a word, compared to Hardy’s model, our model performs better in catching dynamics of underlying securities at a macro level. Considering we include only one macro factor, TSE 300 index, more reasonable macro factors may generate a more precise result in dynamic market movements characterization.

2.2.3 Model Measurements

Fitness level is one of the major concerns when selecting models. In this section, we use different measuring approaches to select the best model.

Root Mean Square Errors (RMSE) measures the difference between estimated values by a model and observed values of estimators, which serves as a common way of selecting better-fitted models. Fitness level becomes higher when RMSE converges to zero. RMSE is formulated as:

\[
RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{x}_i - x_i)^2}
\]  

where \( \hat{x}_i \) is the model predicted values of the estimator while \( x_i \) is the observed actual values of the estimator. \( n \) is total number of the observations. The fitted price values are in equation (1) and (2):

\[
E[\ln(S_t|F_{t-1})] = \ln S_{t-1} + E[R_t(m)|F_{t-1}]
\]

\[
= \ln S_{t-1} + \sum_{m=1}^{M} q_t(m)R_t(m)
\]

<table>
<thead>
<tr>
<th></th>
<th>Hardy’s</th>
<th>Our Model</th>
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<tbody>
<tr>
<td>One Period</td>
<td>0.0020</td>
<td>0.0019</td>
</tr>
<tr>
<td>Two Period</td>
<td>0.0044</td>
<td>0.0039</td>
</tr>
<tr>
<td>Three Period</td>
<td>0.0064</td>
<td>0.0056</td>
</tr>
</tbody>
</table>
We compare RMSEs of Hardy’s Model (2001) and ours during 1956 and 1999. As the number of periods increases, the fitness levels of both our model and Hardy’s model decrease due to the loss of information. Our model has smaller RMSEs than Hardy’s model over all three periods, which indicates that our model has a better fit than the alternative.

Prediction accurate rate (PAR), which measures the movement direction of asset returns, is another criterion to evaluate performances of a model. When the model predicted returns move at the same direction (up or down) as the actual returns at the same time point, the PAR increases. The higher PAR is, the better prediction power the model has. The expression for PAR is:

\[ x_t = \begin{cases} 
1 & \text{if } \hat{R}_t \times R_t > 0; \\
0 & \text{if } \hat{R}_t \times R_t \leq 0.
\end{cases} \]

and

\[ PAR = \frac{\sum_{t=1}^{n} x_t}{n} \times 100\% \]

where \( R_t \) is the actual returns and \( \hat{R}_t = \sum_{m=1}^{K} q_t(m)R_t(m) \) is the model estimated returns. \( x_t \) is a logical number taking 1 when \( R_t \) and \( \hat{R}_t \) have same sign, and taking 0 when \( R_t \) and \( \hat{R}_t \) have opposite sign. PAR is the rate when \( R_t \) and \( \hat{R}_t \) have the same sign.

It closely relates to investors in practice. Investors care the movement direction of the security returns. Intuitively, they buy at this period when the security price is predicted to move up at the following period and vise versa, which ends up with profits. Therefore, a higher rate of accurate prediction is what an ideal model should have. Table 4 shows prediction accurate rates of the two candidate models.

<table>
<thead>
<tr>
<th></th>
<th>Hardy’s Model</th>
<th>Our Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>PAR</td>
<td>60.84%</td>
<td>64.07%</td>
</tr>
</tbody>
</table>

Table 4 indicates that our model exhibits a higher PAR than Hardy’s model. In other words, if investors simply follow the model’s predicted directions to form a strategy of selling and purchasing, they would make profits with a probability of 64%.

3 The option pricing model

3.1 One Period Model

Under fundamental theorem of asset pricing, there is a risk neutral measurement that the expected future payoffs in all the markets states are discounted as the asset’s price if the market is complete
and arbitrage free. In such a risk neutral world, the asset returns follow mixture normal distributions; Conditional on regimes $M_t$, one period ($T = 1$, $\Delta t = 1$) regime-dependent asset returns are

$$R_u \sim N((r - \frac{1}{2}\sigma_1^2)\Delta t, \sigma_1^2\Delta t)$$ (18)

$$R_d \sim N((r - \frac{1}{2}\sigma_2^2)\Delta t, \sigma_2^2\Delta t)$$ (19)

where $R_u$ and $R_d$ are the asset returns in regime 1 and regime 2. $r$ is the risk free rate, $\sigma_1$ and $\sigma_2$ are the volatility of the asset return in regime 1 and regime 2. $\Delta t$ is the step length while $T = 1$ means that it’s an one period model.

The option price is defined as

$$C_0 = e^{-rT}E[(S_1 - K)^+]$$ (20)

$$= p_1C^u_0 + p_2C^d_0$$ (21)

$$= e^{-rT}([p_1E(S^u_1 - K)^+|M = 1] + p_2[E(S^d_1 - K)^+|M = 2]$$ (22)

where $S_1$ is the underlying asset price at time $T = 1$ and $K$ is the option strike price, $p_1$ and $p_2 = 1 - p_1$ are the prior probabilities. $C^u_0$ and $C^d_0$ are the option prices at time $t = 0$ conditional on regime 1 (bull) and regime 2 (bear), respectively. By the Black-Sholes-Merton formula, the one period option price model can be derived as:

$$C_0 = p_1[S_0N(d^u_1) - e^{-rT}KN(d^u_2)] + p_2[S_0N(d^d_1) - e^{-rT}KN(d^d_2)]$$ (23)

where $d^u_1, d^u_2, d^d_1, d^d_2$ are defined

$$d^u_1 = \frac{ln(S_0/K) + (r\Delta t + \frac{1}{2}\sigma_1^2\Delta t)}{\sigma_1\sqrt{\Delta t}}$$ (24)

$$d^u_2 = \frac{ln(S_0/K) + (r\Delta t - \frac{1}{2}\sigma_1^2\Delta t)}{\sigma_1\sqrt{\Delta t}}$$ (25)

$$d^d_1 = \frac{ln(S_0/K) + (r\Delta t + \frac{1}{2}\sigma_2^2\Delta t)}{\sigma_2\sqrt{\Delta t}}$$ (26)

$$d^d_2 = \frac{ln(S_0/K) + (r\Delta t - \frac{1}{2}\sigma_2^2\Delta t)}{\sigma_2\sqrt{\Delta t}}$$ (27)

### 3.2 Two Period Model and N period Model

As the one period model has delivered the basic idea of the mixture normal regime switching option pricing model, we move one step forward to two period model and generalize it to N period model. When $T=2$, we need to define a transition probability matrix:
$$p_{ij} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where $p_{ij}$ is the probability of transferring state $i$ to state $j$. For example, if we are standing at state 1 at time $t$, the probability of staying at state 1 at time $t+1$ is $p_{11}$; the probability of transferring to state 2 at time $t+1$ is $p_{12}$. And $p_{11} + p_{12} = 1; p_{21} + p_{22} = 1$.

Basing on the assumption that optimal number of regimes is 2, there are 3 unique events at time $T = 2$: starting from time $T = 0$, market goes up twice; market either goes up and down or goes down and up; market goes down twice. Conditional on regime $M_t$, two period($T = 2, \Delta t = 1$) regime-depended asset return at $T = 2$ follows:

$$R^1_u + R^2_u \sim N(2(r - \frac{1}{2}\sigma_1^2)\Delta t, 2\sigma_1^2\Delta t) \quad (28)$$

$$(R^1_u + R^2_d) \text{ or } (R^2_u + R^1_d) \sim N((2r - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2)\Delta t, (\sigma_1^2 + \sigma_2^2)\Delta t) \quad (29)$$

$$R^1_d + R^2_d \sim N(2(r - \frac{1}{2}\sigma_2^2)\Delta t, 2\sigma_2^2\Delta t), \quad (30)$$

where $R^i_t$, $i \in (u,d)$, $t \in (1 \cdots T)$ are independent among each other, thus the covariance of any of two items in $R^i_t$ is zero.

Starting from

$$C_0 = e^{-rT}E[(S_T - K)^+], \quad T = 2, \quad (31)$$

two periods model can be expressed as

$$C_0 = p_1p_{11}[S_0 N(d^u_1) - e^{-rT}KN(d^u_2)] \quad (32)$$

$$+ (p_1p_{12} + p_2p_{21})[S_0 N(d^d_1) - e^{-rT}KN(d^d_2)] \quad (33)$$

$$+ p_2p_{22}[S_0 N(d^d_1) - e^{-rT}KN(d^d_2)]], \quad (34)$$

13
where $d_{1u}^{uu}$, $d_{2u}^{uu}$, $d_{1u}^{ud}$, $d_{2u}^{ud}$, $d_{1d}^{dd}$, and $d_{2d}^{dd}$ are defined as

\[
d_{1u}^{uu} = \frac{\ln(S_0/K) + (2r\Delta t + \sigma_1^2 \Delta t)}{\sqrt{2\sigma_1^2 \Delta t}}
\]

(35)

\[
d_{2u}^{uu} = \frac{\ln(S_0/K) + (2r\Delta t - \sigma_1^2 \Delta t)}{\sqrt{2\sigma_1^2 \Delta t}}
\]

(36)

\[
d_{1u}^{ud} = \frac{\ln(S_0/K) + (2r\Delta t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \Delta t)}{\sqrt{(\sigma_1^2 + \sigma_2^2) \Delta t}}
\]

(37)

\[
d_{2u}^{ud} = \frac{\ln(S_0/K) + (2r\Delta t - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \Delta t)}{\sqrt{(\sigma_1^2 + \sigma_2^2) \Delta t}}
\]

(38)

\[
d_{1d}^{dd} = \frac{\ln(S_0/K) + (2r\Delta t + \sigma_2^2 \Delta t)}{\sqrt{2\sigma_2^2 \Delta t}}
\]

(39)

\[
d_{2d}^{dd} = \frac{\ln(S_0/K) + (2r\Delta t - \sigma_2^2 \Delta t)}{\sqrt{2\sigma_2^2 \Delta t}}
\]

(40)

Thus we can generalize our model into $N$ period. As we can see from equation (16) to (18), the differences come from two parts as the number of period is increasing: the probability part and the $d$ parts.

Firstly, let’s look at the $d$ part. Assuming among time period $N$, there are $n$ period with market going up. Then there are $N-n$ time periods with market going down. In such a case

\[
d_{1u}^{uu,(N-n)d} = \frac{\ln(S_0/K) + Nr\Delta t + \frac{1}{2}[n\sigma_1^2 + (N-n)\sigma_2^2] \Delta t}{\sqrt{(n\sigma_1^2 + (N-n)\sigma_2^2) \Delta t}}
\]

(41)

\[
d_{2u}^{uu,(N-n)d} = \frac{\ln(S_0/K) + Nr\Delta t - \frac{1}{2}[n\sigma_1^2 + (N-n)\sigma_2^2] \Delta t}{\sqrt{(n\sigma_1^2 + (N-n)\sigma_2^2) \Delta t}}
\]

(42)

Then the probability function is derived in the following section.

### 3.3 Probability Function

Following the option pricing model derived above, we need a generalized probability function. The probability function is derived mainly based on Hardy’s study (2001).

Let $R$ be the total sojourn in regime 1 from time 1 to time $T$, $R \in \{0, 1, \cdots, T\}$. $R_t$ be the total sojourn in regime 1 from time $t + 1$ to $T$. Thus, $R_0 = R$. Denote the probability $P(R_t = r)$ as the probability of the total sojourn is $r$ from time $t + 1$ to time $T$, $r \in \{0, 1, \cdots, T - t\}$, and $t \in \{1, 2, \cdots, T - 1\}$. Conditional on the regime at time $t$, the probability of total number of sojourn in regime 1 from time $t + 1$ to $T$ is defined as $P(R_t = r | M_t)$. After defining all the denotations, the generalized probability function needs to be summarized.
As we have shown in the finite time periods example, the probability function is nothing about a series of products of transition probability from one state to the other. Clearly, there are some rules to follow. Let’s start thinking about time $T$ from the backward way and initialize the probability function at the end time period $T$. Standing at $T - 1$, there is only one more time period to move, either to regime 1 or regime 2. Therefore, $r$ can take 0, which means regime 2 at time $T$ or 1, which means regime 1 at time $T$. Other than the two values, the probability function is zero. Conditional on the regimes at time $T - 1$, the probability function can be shown as:

\begin{align}
P(R_{T-1} = 0|M_{T-1} = 1) &= p_{12} \quad (43) \\
P(R_{T-1} = 0|M_{T-1} = 2) &= p_{22}, \quad (44)
\end{align}

when $r = 0$,

\begin{align}
P(R_{T-1} = 1|M_{T-1} = 1) &= p_{11} \quad (45) \\
P(R_{T-1} = 1|M_{T-1} = 2) &= p_{21}, \quad (46)
\end{align}

when $r = 1$, and

\begin{align}
P(R_{T-1} = r|M_{T-1} = 2) &= 0, \quad (47)
\end{align}

when $r \geq 2$.

Following the Procedure:

\begin{align}
Pr[R_t = r|M_t = 1] &= p_{11} Pr[R_{t+1} = r - 1|M_{t+1} = 1] + p_{12} Pr[R_{t+1} = r|M_{t+1} = 2] \quad (48) \\
Pr[R_t = r|M_t = 2] &= p_{21} Pr[R_{t+1} = r - 1|M_{t+1} = 1] + p_{22} Pr[R_{t+1} = r|M_{t+1} = 2], \quad (49)
\end{align}

with

\begin{align}
Pr[R_{T-1} = 1|M_{T-1} = 1] &= p_{11} \quad (50) \\
Pr[R_{T-1} = 0|M_{T-1} = 1] &= p_{12} \quad (51) \\
Pr[R_{T-1} = 1|M_{T-1} = 2] &= p_{21} \quad (52) \\
Pr[R_{T-1} = 0|M_{T-1} = 2] &= p_{22} \quad (53)
\end{align}

Thus, the generalized probability function can be expressed as:

\begin{align}
Pr(R_0 = r) &= p_1 Pr(R_0 = r|M_{-1} = 1) + p_2 Pr(R_0 = r|M_{-1} = 2) \quad (54)
\end{align}
3.4 Empirical Analysis of the Option Pricing Model

3.4.1 Comparisons between TSE 300 Macro Level Option Pricing Model and Hardy Option Pricing Model

Our model produces the same result in option prices as Hardy did in 2001. TSE 300 index serves as an example: using the estimated parameters and the steady-state probability condition of Hardy’s parameters from Table 2, our put options prices under the strike prices of 80, 100 and 120, as Table 5 shows, completely match Hardy’s work, which implies that Hardy’s model can be regarded as one special case of our model.

Table 5: Examples of One Year Put Option Prices under Hardy’s estimated Parameters (2001)

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Option Price</th>
<th>Strike Price</th>
<th>Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.232</td>
<td>80</td>
<td>0.232</td>
</tr>
<tr>
<td>100</td>
<td>3.275</td>
<td>100</td>
<td>3.275</td>
</tr>
<tr>
<td>120</td>
<td>14.876</td>
<td>120</td>
<td>14.876</td>
</tr>
</tbody>
</table>

However, the steady-state probability is applicable only if the market was in a steady state. The posterior probabilities of market regimes vary dramatically overtime, which leads to significant changes in option prices. Table 6 and Figure 3 explain how option prices change with posterior probabilities.

Table 6: Call Option Prices(S=100,K=100,T=3 Months) with different Posterior Probabilities

<table>
<thead>
<tr>
<th>Posterior Probability (bear, bull)</th>
<th>3-month call option prices</th>
<th>3-month put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.0, 0.0)</td>
<td>5.1499</td>
<td>3.6611</td>
</tr>
<tr>
<td>(0.8, 0.2)</td>
<td>4.8000</td>
<td>3.3112</td>
</tr>
<tr>
<td>(0.6, 0.4)</td>
<td>4.4502</td>
<td>2.9614</td>
</tr>
<tr>
<td>(0.5, 0.5)</td>
<td>4.2753</td>
<td>2.7865</td>
</tr>
<tr>
<td>(0.4, 0.6)</td>
<td>4.1003</td>
<td>2.6116</td>
</tr>
<tr>
<td>(0.2, 0.8)</td>
<td>3.7505</td>
<td>2.2617</td>
</tr>
<tr>
<td>(0.0, 1.0)</td>
<td>3.4007</td>
<td>1.9119</td>
</tr>
</tbody>
</table>

Hardy’s Model with Steady States (2001)

<table>
<thead>
<tr>
<th>Posterior Probability (bear, bull)</th>
<th>3-month call option prices</th>
<th>3-month put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1501, 0.8499)</td>
<td>3.6632</td>
<td>2.1744</td>
</tr>
</tbody>
</table>
According to Table 6, the option prices under a steady state are disparate from those under the market with a high posterior probability. Figure 3 shows that call option prices change with bear regime posterior probabilities, from 0 to 1. The option prices diverge in a substantial magnitude with different posterior probability distributions. When the bear market regime posterior probability is 0, the call option price is $3.40; when the bear market posterior probability is 1, the call option price is $5.15; when the market stays at the so-called steady states, the call option price is $3.66.

The option prices vary in a substantial amount with different posterior probabilities. Thus, a reasonable estimation of posterior probabilities is essential for determining the option price. The simple assumption that market remains at a steady state does not lead to satisfactory results.

### 3.4.2 Comparisons between TSE 300 Macro Level Option Pricing model and BSM Model

We compare the outcomes from the macro level option-pricing model with those from BSM model and find an interesting phenomenon in the financial market. The volatility of TSE 300 index is 0.0449, and the initial stock price is 1000, with a maturity date of 3 months. We look at two extreme cases: when the posterior probability for bear market is 100 % and when the posterior probability for bear market is 0%. We elaborate the interesting phenomenon using Figure 4: When the posterior probability of bear market is substantially high (Figure 4a and 4b), the BSM model
exhibits a lower option price compared to the macro level model. When the posterior probability of bull market is substantially high (Figure 4c and Figure 4d), the BSM model exhibits a higher option price than which from the macro level model.

(a) Call Option (Bear Regime)  
(b) Call Option (Bull Regime)  
(c) Put Option (Bear Regime)  
(d) Put Option (Bull Regime)  

Figure 4: Option Prices across Strike Prices and Regimes under BSM and Macro Level Model

BSM model cannot capture high volatilities in bear market and low volatilities in bull market, causing it misprice the options in dynamic market states. BSM model undervalues call options when call options are out of money, and overvalue call options when they are in the money. In bear market, volatility is higher, so the option prices are likely to be higher and BSM model underestimates the option prices. Our model can capture the high volatility and low volatility in different market regimes.
3.4.3 Volatility Smile

As an aftermath of the crash in 1987, the option traded in American started exhibiting volatility smile/smirk. It shows different patterns across markets. The implied volatility given an expiration date should be constant against different strike prices. However, it yields a smile curve.

![Image of implied volatility smile across models](image)

Figure 5: Implied volatility smile across models

Our model is able to explain the volatility smile. Figure 5 illustrates that our model exhibits the volatility smile, which is consistent with empirical data. We display the volatility smile in three different cases.

1. When the posterior probability of bull regime is 100%
2. When the posterior probability of bull regime is 0%
3. When the market remains at so-called “steady states”

The first case ($P(bull) = 100\%$) captures the extreme case that investors are 100% positive about the bull market of next time period, while the second case ($P(bull) = 0$) describes the opposite case that investors are fully sure that the bear market dominates the next time period. Practically, the perception of investors implied by the model stays somewhere in between. The third case is a special one when the market remains at so-called a steady state”. In our model, the posterior probabilities for steady states are $P(bull) = 85.15\%$ and $P(bear) = 15.85\%$

Figure 5 indicates the comparison of the implied volatility between BSM model and ours. BSM implied volatility is a straight line across different strike prices. The higher curve is the implied
volatility under 100% agreement of bear market while the lower curve is the implied volatility under 100% belief of bull market. The curve in the middle is the implied volatility under steady states ($P(\text{bull}) = 84.15\%$). The implied volatilities under the model exhibit the smallest volatility when the strike price equals $1000e^{rT}$, where annual risk free rate is $r = 6\%$ and $T$ is 3 months.

Figure 6: Implied Volatility against Different Maturity Dates

Figure 6 shows the disparity of implied volatilities across different strike prices and different maturity dates. The upper surface is the implied volatility under the perception of 100% bear market while the lower surface is the implied volatility under the cognition of 100% bull market. The surface in the middle represents BSM implied volatility.

Based on Figure 6, the two implied volatility surfaces under our model exhibits a smile curve given a specific maturity date with different strike prices. The implied volatilities are relatively higher when the option is deeply in the money and out of the money compared to when the option is at the money. As maturity dates become longer, the smile curve tends to shade out. These findings are consistent with the empirical data.

4 Conclusion

Our study contributes in both theoretical frameworks and practical applications. On one hand, it makes a breakthrough in dynamic option pricing with economic regime shifts. On the other hand, it conducts practical implications from perspectives of real-world valuation and investment.
management: First, it provides a potential general pricing approach. Considering that the derivative market has developed to an advanced stage and the financial market has experienced turmoil in recent years, an increasing number of investors argue that a sophisticated and practical valuation model is necessary. Our model provides an analytical solution for European options with high feasibility and applicability. Due to the flexibility, our model is also able to price for American options, which will show in our future research\(^1\). Second, our model helps to formulate an optimized profitable investment strategy for both individual investors and fund managers, as it captures the market regime-dependent pricing approach for options in a more precise way. With a strong prediction capability, our model is applicable in both broad and narrow senses.

This paper develops a discrete time lattice-based mixture normal option pricing model using a Markov chain regime-switching framework. Our analytical pricing formula derives for European options. By comparing BSM model and Hardy’s model to ours, this study presents a comprehensive and critical empirical analysis in option pricing. One main drawback of BSM model, which is its incapacity to react to switches in market regimes, makes our regime-depended model more attractive. Hardy develops an option-pricing model based on a steady state regime probability distribution with no prediction power, while our model remedies this pitfall by allowing the distribution of asset returns to switch. The results suggest that, first, option prices bear a premium in a highly volatile market and vice versa; second, our model is able to explain volatility smile or smirk using empirical examples. For practical implications, this paper proposes a general option-pricing method from a global perspective and meanwhile offers constructive suggestions in investment. To conclude, the capacity of capturing dynamics of the financial market, the foresightedness in option pricing under regime-switching conditions and the innovation of transforming financial theories into real-world applications equip this study with a high ability to promote progresses in both academy and the real market.

\(^{1}\)Although an analytical solution might not be solved for American option, a simulation algorithm can be generated based on the framework.
References


