Building Efficient Portfolios Sensitive to Market Volatility

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Abstract

This paper builds efficient diversified portfolios that are sensitive to market volatility. A two-factor model based on Black’s zero-beta CAPM is employed to measure sensitivity to cross-sectional market volatility. Based on U.S. stock returns in the sample period 1965 to 2015, aggregate portfolios are constructed by combining the CRSP market index with zero-investment portfolios that reflect different levels of market volatility risk. Strikingly, a substantially higher average return of almost 1.50 percent per month can be achieved relative to the benchmark CRSP market index of 0.89 percent, holding constant total risk. Further results show that an efficient frontier of zero-investment portfolios with different risk levels can be developed that outperforms zero-investment portfolios formed on size, value, and momentum investment strategies. These and other results lead us to conclude that market volatility is useful in building relatively efficient portfolios.

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This paper seeks to empirically construct efficient aggregate portfolios for U.S. stocks based on earlier theoretical work on the mean-variance investment parabola by Liu, Kolari, and Huang (2012). We especially thank the Teachers Retirement System of Texas for working with us to help develop real world applications of theoretical results in this earlier working paper, which was presented at the Midwest Finance Association 2012 meetings in New Orleans, Louisiana, Multinational Finance Society 2012 conference in Krakow, Poland, and Financial Management Association 2012 conference in Atlanta, Georgia (Best Paper in Investments award). The authors gratefully acknowledge helpful comments from conference participants and colleagues, including Ali Anari, Jaap Bos, Ivan Brick, Yong Chen, Lammertjan Dam, Serdar Dinc, Paige Fields, Markus Franke, Britt Harris, Jianhua Huang, Hogyu Jhang, Hagen Kim, Jin-Mo Kim, Johan Knif, Anestis Ladas, Scott Lee, Qi Li, Yishan Liu, Francisco Penaranda, Seppo Pynnonen, Katharina Schüller, Ben Sopranzetti, Ahmet Tuncez, David Veal, Yangru Wu, Yuzhao Zhang, Zhaozhong Zhong, Jian Yang, Nan Yang, Christopher Yost-Bremm, Zhao Xin, Jun Zhang, and Tony van Zijl.
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1 Introduction

In the well-known capital asset pricing model (CAPM) of Treynor (1961, 1962)-Sharpe (1964)-Lintner (1965)-Mossin (1966), the market portfolio is typically proxied using aggregate stock market indices. According to the Roll (1977) critique, the empirical estimation of the CAPM is subject to question due the possibility that aggregate market indices, such as the S&P 500 index and CRSP index, are not efficient portfolios. In this regard, a fairly large body of research finds that general market proxies are located far from the Markowitz (1959) mean-variance efficient frontier.\(^1\) If commonly-used aggregate market indices are not efficient, a major gap in the literature exists with respect to how to build more efficient aggregate portfolios.\(^2\)

Black (1972) proposed the zero-beta CAPM to circumvent the problem of identifying a unique tangent market portfolio on the minimum-variance parabola in line with CAPM theory. To do this, the single factor CAPM was recast in terms of any two orthogonal portfolios on the investment parabola – namely, an efficient portfolio and uncorrelated zero-beta portfolio. Unfortunately, he did not specify how to compute these paired portfolios in the real world.

In an effort to bridge the gap between efficient portfolio theory and practice, this paper utilizes a derivative zero-beta CAPM model (dubbed the ZCAPM) proposed by Liu, Kolari, and Huang (2012) to construct efficient portfolios relative to common market indexes. The key theoretical insight of the ZCAPM is a new geometry for identifying portfolios on the minimum-variance parabola. Rather than finding tangent points to the parabola, the ZCAPM posits that: (I) the average aggregate return of all assets lies approximately on the axis of symmetry that divides the parabola into two halves, and (II) the cross-sectional standard deviation of all assets’ returns determines the span

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\(^2\)Stambaugh (1982) tested the CAPM using alternative aggregate portfolios comprised of bonds, stocks, real estate, and consumer durables but found that the results were unchanged compared to using only stocks to proxy the market portfolio.
or width of the parabola. Proposition I provides theoretical support for earlier cited evidence on the inefficiency of aggregate market indices. Relevant to locating efficient portfolios, Proposition II suggests that it is possible to earn higher risk-adjusted returns than general stock market indices by taking into account cross-sectional market volatility. Intuitively, this higher risk premium arises from the combined effects of other risk factors such as size, value, momentum, etc. that contribute to the cross-sectional volatility of the market. Related to these propositions, the ZCAPM is a two-factor model in which asset returns are a function of market risk attributable to the average return on all assets in the market (i.e., beta risk) and their cross-sectional standard deviation (i.e., zeta risk). These two market factors can be readily estimated from available stock return data series.

Consistent with the ZCAPM’s geometry, to form more efficient aggregate portfolios, we combine the CRSP aggregate stock market index with zero-investment portfolios that capture different levels of risk associated with cross-sectional market volatility. Using U.S. stocks in the sample period 1965 to 2015, we begin by estimating the two-factor ZCAPM for each stock. Next, stocks are ranked on their zeta risk coefficient estimates (representing different volatility risk levels) and formed into 12 zero-investment portfolio slices that stratify zeta risk from high to low. For stocks in the top +1 percent/bottom −1 percent of estimated zeta risk coefficients, zero-investment portfolio returns average 2.30 percent in the post-formation month, which is exceptional relative to other zero-investment portfolios (e.g., momentum strategies). Supporting the two-factor ZCAPM, we find a strong positive and linear relation between zeta risk and one-month-ahead holding period returns. Higher (lower) portfolio returns are associated with higher (lower) zeta market risk.

We subsequently combine zero-investment portfolios sensitive to market volatility with the CRSP value-weighted index to form aggregate stock portfolios. As conjectured by the ZCAPM, the resultant aggregate portfolios trace out a mean-variance frontier that lies well above the CRSP market index. In line with Proposition I, the CRSP index lies approximately on the axis of symmetry. Strikingly, setting the standard deviation of returns equal to the CRSP index, an aggregate portfolio with a much higher average monthly return can be achieved of approximately 1.50 percent compared to 0.89 percent
for the CRSP index. We infer that this relatively efficient aggregate index would be a more appropriate proxy for the market portfolio than the commonly-used CRSP index in empirical tests of the CAPM. These and other results confirm the influence of cross-sectional market volatility on the span of the investment parabola in line with Proposition II.

We conclude that relatively efficient diversified portfolios can be created by means of the ZCAPM, a special case of Black’s zero-beta CAPM. As proposed by this two-factor model, average market returns and their cross-sectional volatility are two fundamental state variables that describe the investment opportunity set embodied in the shape of the minimum-variance parabola. An important implication of our findings is that asset pricing methods can be applied to investment portfolio management practice. Also, more efficient aggregate indices should enhance asset pricing models incorporating a market factor, which is typically proxied by the less efficient CRSP index. No previous studies to the authors’ knowledge apply an asset pricing model to the problem of constructing efficient portfolios.

The next section overviews the two-factor ZCAPM. Section 3 discusses its application to portfolio construction. Section 4 gives empirical results. The last section concludes.

2 Overview of the Two-Factor ZCAPM

Black’s (1972) now famous zero-beta CAPM is written as follows:

\[ E(\tilde{R}_i) = E(\tilde{R}_{ZM}) + \beta_{i,M}[E(\tilde{R}_M) - E(\tilde{R}_{ZM})], \quad (1) \]

where \( E(\tilde{R}_i) \) = the expected return on the \( i \)th asset, \( E(\tilde{R}_M) \) = the expected return on the market portfolio, \( E(\tilde{R}_{ZM}) \) = the expected return on the orthogonal, zero-beta portfolio, and \( \beta_{i,M} \) is the sensitivity or beta risk of asset \( i \) with respect to the market portfolio. Orthogonal portfolios \( M \) and \( ZM \) lie on the superior (efficient frontier) and inferior portions, respectively, of the Markowitz (1959) minimum-variance parabola. Roll (1980) shows that the market portfolio can be replaced by any market index that is an efficient portfolio. For each efficient index portfolio \( I \), a companion minimum-variance,
orthogonal portfolio \( ZI \) can be found. As such, the zero-beta CAPM can be expressed in the more general form as follows:

\[
E(\tilde{R}_i) = E(\tilde{R}_{ZI}) + \beta_{i,I}[E(\tilde{R}_I) - E(\tilde{R}_{ZI})],
\]

where \( \beta_{i,I} \) captures the beta risk of asset \( i \) with respect to efficient market index factor \( I \). Notice that the true market portfolio \( M \) can be constructed from a combination of any given minimum-variance portfolios \( I \) and \( ZI \). Furthermore, under the assumption that \( E(\tilde{R}_{ZM}) = R_f \), where \( R_f \) is the riskless rate, then \( E(\tilde{R}_I) = E(\tilde{R}_M) \), such that the traditional CAPM holds, or \( E(\tilde{R}_i) = E(\tilde{R}_f) + \beta_{i,M}[E(\tilde{R}_M) - R_f] \).

Liu, Kolari, and Huang (2012) derive a new geometry for the zero-beta CAPM that leads to a two-factor model referred to as the ZCAPM. Consistent with Black’s zero-beta CAPM, Figure 1 shows that, for any chosen expected return denoted \( E(\tilde{R}_X) \) on the \( Y \)-axis, two tangent rays to the minimum-variance parabola identify a pair of efficient index \( I \) (or \( I' \)) and orthogonal zero-beta index \( ZI \) (or \( ZI' \)) portfolios. Among the infinite possible orthogonal pairs of \( I \) and \( ZI \) portfolios, as shown in Figure 2, the ZCAPM focuses on the special case of two unique tangent index portfolios \( I^* \) and \( ZI^* \) that are identified based on rays extending from the expected return on the global, minimum-variance portfolio, or \( E(\tilde{R}_G) \), to the minimum-variance parabola.\(^3\) Using random matrix theory, they contend that in finite samples \( E(\tilde{R}_G) \approx E(\tilde{R}_a) \), where \( E(\tilde{R}_a) \) equals the average return on all assets in the market. If the average market return lies approximately on the axis of symmetry of the parabola to the right of the global, minimum-variance portfolio, a general market index such as the S&P 500 and CRSP indices will not be efficient, i.e., Proposition I. Subsequently, the authors solve for the expected returns on orthogonal indices \( I^* \) and \( ZI^* \) and obtain the following fixed formulas:

\[
E(\tilde{R}_{I^*}) = \frac{A}{C} + \sqrt{\frac{BC - A^2}{C^2}}
\]

\(^3\)Setting the return variances of \( I \) and \( ZI \) equal to one another, these two portfolios could also be located using both Merton’s (1972) and Roll’s (1980) well-known geometric approaches; however, they are less intuitively obvious and therefore have not drawn attention in previous literature.
where \( A, B, \) and \( C \) are well-known scalers that define the shape and location of the investment parabola. Based on the converging values of these scalars in large samples, they derive the following expected returns for indices \( I^* \) and \( ZI^* \):

\[
E(\tilde{R}_{I^*}) \approx E(\tilde{R}_a) + \tilde{\sigma}_a
\]

(5)

\[
E(\tilde{R}_{ZI^*}) \approx E(\tilde{R}_a) - \tilde{\sigma}_a,
\]

(6)

where the ratio \( \frac{A}{C} \) in equations (3) and (4) equals \( E(\tilde{R}_g) \approx E(\tilde{R}_a) \), and \( \tilde{\sigma}_a \) is the cross-sectional standard deviation of all assets in the market (i.e., market volatility). Substituting these definitions for \( E(\tilde{R}_{I^*}) \) and \( E(\tilde{R}_{ZI^*}) \) into the zero-beta CAPM in equation (2), the two-factor ZCAPM with no riskless asset becomes:

\[
E(\tilde{R}_i) = E(\tilde{R}_a) + Z_{i,a} \tilde{\sigma}_a,
\]

(7)

where the zeta risk parameter \( Z_{i,a} \) reflects the sensitivity of asset \( i \) to cross-sectional market volatility. Notice that \( Z_{i,a} = 1 \) for market index \( I^* \) but \( Z_{i,a} = -1 \) for zero-beta index \( ZI^* \). In this sense, market volatility associated with zeta risk can have two-sided, opposite effects on asset returns (e.g., higher market volatility increases \( E(\tilde{R}_I) \) but decreases \( E(\tilde{R}_{ZI}) \), all else the same). Additionally, cross-sectional market volatility defines the width or span of the minimum-variance parabola, i.e., Proposition II. Together, Propositions I and II imply that, as the parabola changes over time, assets within the investment opportunity set enveloped by the minimum-variance parabola are affected by common changes in both the average market return and the cross-sectional market volatility. In effect, these two market forces are underlying state variables that define the minimum-variance investment opportunity set available to investors.

The geometry of the two-factor ZCAPM suggests two ways to locate efficient portfolio \( I^* \). One way is the tangent line from the average return on a general market index (e.g., the
CRSP index) to the the minimum-variance parabola; however, this approach is infeasible due to the inability to estimate portfolios on the parabola. A second way to geometrically locate portfolio $I^*$ consistent with equation (5) is to move horizontally along the average return on a market index, or $E(\tilde{R}_a)$, and then vertically up by the cross-sectional standard deviation of market assets’ returns, or $\tilde{\sigma}_a$ (i.e., $Z_{i,a} = 1$). Inefficient portfolio Z$I^*$ can be located using a similar process with equation (6) but moving vertically down with respect to cross-sectional dispersion. As an empirical test of this alternative approach, in the next section we construct zero-investment, market-volatility mimicking portfolios, wherein zeta risk associated with $\tilde{\sigma}_a$ spans different volatility levels. Subsequently, if the proposed geometry holds, by adding CRSP index returns to these different market-volatility mimicking portfolios’ returns, we conjecture that a set of diversified aggregate portfolios can be obtained that together provide a relatively efficient return/risk frontier.

Introducing the possibility of long and short positions in a riskless asset, the two-factor ZCAPM becomes:

$$E(\tilde{R}_i) - R_f = \beta_{i,a}[E(\tilde{R}_a) - R_f] + Z_{i,a}\tilde{\sigma}_a, \tag{8}$$

where the beta risk parameter $\beta_{i,a}$ measures the sensitivity of asset $i$ to average market excess returns as opposed to market portfolio excess returns, and other notation is as before. For indices $I^*$ and Z$I^*$, $\beta_{I^*,a} = \beta_{ZI^*,a} = 1$, $Z_{I^*,a} = 1$, and $Z_{ZI^*,a} = -1$, such that we obtain equations (5) and (6), respectively. Importantly, according to the two-factor model in equation (8), asset prices are a function of market risk comprised of two components: (1) beta risk related to average market returns, and (2) zeta risk associated with the cross-sectional market volatility of returns. An advantage of this model is that the average market return $E(\tilde{R}_a)$ and cross-sectional standard deviation of all assets $\tilde{\sigma}_a$ can be readily estimated for a population of stock returns. Also, cross-sectional market volatility is widely recognized as a macroeconomic state variable that proxies fundamental economic

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4Accurate estimation of the return covariance matrix for diversified portfolios with a large number of assets is not possible because: (1) its elements are time-dependent unobservable quantities, and (2) insufficient observations are typically available (e.g., a matrix dimensioned for 5,000 assets requires more than 5,000 observations for each asset to avoid singularity).

5It is generally accepted that multi-factors, including size, value, momentum, profit, capital investment, liquidity, etc., are needed to better approximate market risk related to the market portfolio. Hence, $\beta$ in multi-factor models is not associated with the market portfolio per se but an average market return index.
variables. In general, the ZCAPM embodies a conditional multi-factor representation of equilibrium expected returns with the moments of the stock market return and volatility as state variables.

An interesting insight from the two-factor ZCAPM is that market volatility can have two-sided, opposite effects on asset returns. To see this, suppose that the cross-sectional distribution of asset returns widens due to an increase in $\tilde{\sigma}_a$. All else the same, assets with returns less than the expected market return would experience decreasing returns, whereas assets with returns greater than the expected market return would have increasing returns. Conversely, if market volatility decreases rather than increases, then the former assets would experience increasing returns, and the latter assets increasing returns. Extending this simple intuition to the ZCAPM in equation (8), given an asset’s return exceeds its expected return defined by its beta risk (i.e., $\beta_{i,a}[E(\tilde{R}_a) - R_f]$), if $Z_{i,a}$ is positive for the asset, increased (decreased) market volatility will tend to push up (down) its excess return. The opposite effect occurs if an asset’s $Z_{i,a}$ coefficient is negative, as its excess return will tend to decrease (increase) in response to increased (decreased) market volatility. Since cross-sectional market volatility can be substantial in the real world, the impact of zeta risk on expected asset returns should be economically significant.

In the real world, the magnitudes of observed values of $\tilde{R}_a$ and $\tilde{\sigma}_a$ will change in each period, as average market returns and cross-sectional market volatilities fluctuate over time, thereby changing the level and span, respectively, of the minimum-variance parabola. Also, the direction of zeta risk (i.e., sign of $Z_{i,a}$) on an asset is not observable at time $t$. In an effort to capture the impacts of time-varying changes in the level and span of the parabola on asset returns over time, Liu et al. propose the following empirical version of the two-factor ZCAPM:

$$\tilde{R}_{it} - R_{ft} = \alpha_i + \beta_{i,a}(\tilde{R}_{at} - R_{ft}) + Z_{i,a}D_{it}\tilde{\sigma}_{at} + \tilde{u}_{it}, \quad t = 1, \ldots, T \quad (9)$$

where $D_{it}$ is a signal variable taking values $+1$ and $-1$ to indicate, respectively, positive

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and negative market volatility effects on the $i$th asset’s returns at time $t$, and $\tilde{u}_{it} \sim \text{iid } \mathcal{N}(0, \sigma_i^2)$. Unobservable signal variables $D_{it}$ are assumed to be random variables with binomial distribution $+1$ or $-1$ and associated probabilities $p_i$ and $1-p_i$, respectively. To compute a maximum likelihood estimate of $p_i$, the authors use an expectation-maximization (EM) regression approach. This widely-used regression method requires that $\alpha_i$ be set to zero as an approximation to avoid different local maximization points of the likelihood function. As such, the two-factor ZCAPM is:

$$\tilde{R}_{it} - R_{ft} = \beta_i R_{at} - \beta_i R_{ft} + Z_{i,a}^* \tilde{\sigma}_{at} + \tilde{u}_{it}, \quad (10)$$

where $Z_{i,a}^* = Z_{i,a}(2\hat{p}_i - 1)$.

Appendix A contains a step-by-step description of how to estimate this two-factor model using EM regression methods. Relevant to the present paper, the most important parameter for portfolio construction purposes is the zeta risk coefficient $Z_{i,a}^*$ associated with market volatility. If $Z_{i,a}^*$ is positive (negative) in sign, the stock has positive (negative) sensitivity to market volatility.

3 Two-Factor Model Application to Portfolio Construction

We use all U.S. stocks in the intersection of the NYSE, AMEX, and NASDAQ return files from the Center for Research in Security Prices (CRSP) in the period January 1965 to December 2015. Relatively small firms with market capitalization in the bottom 10 percent at year end are dropped in the next year. Daily CRSP value-weighted returns and Treasury bill rates are used to proxy mean market returns ($\tilde{R}_{at}$) and riskless returns ($R_{ft}$), respectively. To proxy the cross-sectional standard deviation of assets’ returns (i.e., market volatility), value-weighted returns for $i = 1, ..., n$ assets on day $t$ are used to compute

$$\tilde{\sigma}_{at} = \sqrt{\frac{n}{n-1} \sum_{i=1}^{n} a_{it-1} (\tilde{R}_{it} - \tilde{R}_{at})^2}, \quad (11)$$
where $a_{it} =$ the market value weight for asset $i$. Liu et al. show that $\tilde{\sigma}_{at}$ is a biased but consistent estimator of $\sigma_{at}$. Because the bias in the estimator $\tilde{\sigma}_{at}$ is a constant proportion of $\sigma_{at}$, it can be used as a reasonable proxy for cross-sectional market volatility. U.S. stock return evidence in their paper using $\tilde{\sigma}_{at}$ in empirical ZCAPM estimations supports this inference. Within our sample period, the mean monthly return (and standard deviation of these returns) for the value-weighted CRSP index is 0.89 (4.49), and the mean cross-sectional standard deviation of monthly returns (and standard deviation of this dispersion) for CRSP stocks is 8.29 (2.75) percent. Also, the estimated correlation coefficient between mean monthly excess market returns (subtracting the one-month U.S. Treasury bill rate) and monthly cross-sectional standard deviations of market returns in the sample period is quite low at only 0.07. The average Treasury bill rate is 0.44 percent per month.

Under Proposition I of the two-factor ZCAPM, the CRSP index return should be a reasonable proxy for one state variable (i.e., the axis of symmetry) defining the shape of the minimum-variance parabola. According to Proposition II, the cross-sectional standard deviation of CRSP index returns $\sigma_{at}$ is a second state variable needed to define the span of the parabola. As discussed in the previous section, to construct portfolios with higher return/risk profiles than the CRSP index, we can combine the CRSP index with zero-investment, market-volatility mimicking portfolios.\footnote{Ang, Hodrick, Xing, and Zhang (2006) construct a volatility mimicking factor based on innovations in the VIX (i.e., the Chicago Board Options Exchange market volatility index). The VIX attempts to measure the market’s expectation of 30-day volatility, which is a time-series rather than cross-sectional volatility measure.} The following steps are used to construct a variety of zero-investment portfolios with different zeta risk levels:

1. Times-series regressions are estimated for all individual stocks using the two-factor ZCAPM in equation (10) with daily returns for the initial 12-month estimation period January to December 1964.

2. All stocks are ranked in terms of their estimated zeta coefficient $\hat{Z}_{i,a}^*$. 

3. A total of 24 portfolios are formed from high positive to low negative zeta risk.

4. Using these portfolios, 12 slices are formed with one long and one short portfolio in each slice. Slices 1 and 2 are zero-investment portfolios that are long/short in stocks with high zeta market risk $\hat{Z}_{i,a}^*$ in the top +1 percent/bottom −1 percent and top
+2 percent/bottom −2 percent of all sample stocks, respectively. Slices 3 to 12 are comprised of long/short stocks in incremental +5/− 5 percent portfolios (i.e., slice 3 has the top +5/− 5 percent stocks with relatively high zeta risk coefficients, slice 4 includes the next +5/− 5 percent of stocks according to their zeta risk coefficients, ..., slice 12 contains the bottom +5/− 5 percent stocks with zeta risk coefficients near zero).

5. For all stocks in each long/short zeta risk slice, we compute the one-month-ahead, equal-weighted portfolio holding period return $R_{pt+1}$ in January 1965.\(^8\) This out-of-sample approach mitigates data snooping and other potential problems associated with evaluating the practical usefulness of asset pricing models (see Simin (2008) and Ferson, Nallareddy, and Xie (2013)).

6. The above process is repeated by rolling the estimation period in step 1 forward one month at a time to rebalance the mimicking portfolios until the last one-month-ahead holding period stock return is computed in December 2015. The time-series standard deviation of these out-of-sample stock returns in the sample period is computed also.

Importantly, the 12 zeta risk slices or portfolios span a wide range of market risk associated with cross-sectional market volatility. If there is a relation between the zeta risk loadings and one-month-ahead returns, average returns should increase from low for slice 12 to high for slice 1.

Lastly, consistent with ZCAPM geometry, we build aggregate zeta risk portfolios by combining each of the above zero-investment, zeta-risk portfolios (slices) with the CRSP value-weighted index (denoted portfolio \(a\)). Aggregate portfolios have weights for all CRSP stocks that add to one. Monthly returns are computed for these aggregate zeta risk portfolios from January 1965 to December 2015.\(^9\) It should be noted that these aggregate stock portfolios contain short positions. In this regard, previous studies have shown that short positions are needed to create more efficient portfolios (e.g., Pulley (1981), Levy (1983), Kallberg and Ziemba (1983), Kroll, Levy, and Markowitz (1984), \(^8\)These returns should be primarily attributable to zeta risk, as the betas of diversified long and short portfolios will be near one and cancel out for the most part.
\(^9\)Aggregate returns are computed by simply adding the CRSP return to the zero-investment portfolio return.
Green and Hollifield (1992), Jagannathan and Ma (2003), and Levy and Ritov (2010)). However, a criticism of this approach is that short positions are not always available to investors in many small stocks. For this reason, we also examine the average performance of the 24 zeta risk portfolios assuming they are held only as long positions. Moreover, we construct 24 long aggregate portfolios by investing 50 percent of funds in the CRSP index and 50 percent in each zeta risk portfolio, which enables insights into the effects of short positions on aggregate portfolios.

4 Empirical Results

The average return and risk results for the 24 zero-investment, market-volatility mimicking portfolios created for 12 different zeta risk levels or slices are shown in Table 1. For each zero-investment portfolio in the sample period January 1965 to December 2015, the table shows the average zeta risk coefficient for constituent stocks denoted $Z_{pt}^*$ (i.e., 12-month zeta risk estimates are rolled forward one month at a time to yield 600 estimates for each portfolio), their average one-month-ahead returns denoted $\bar{R}_{pt+1}$ (for 600 months), and the standard deviations of these estimates in parentheses. For each zero-investment portfolio, the average zeta risk and monthly return estimates for the long and short portfolios (and standard deviations in parentheses) are shown also. Notably, for stocks in slice 1 containing the top +1 percent/bottom −1 percent of estimated zeta risk coefficients, undiversified portfolio returns average 2.67 percent in the post-formation month. This average one-month-ahead return performance surpasses published results in momentum studies that have attracted considerable attention for high zero-investment portfolio returns.\(^{10}\)

As shown in Table 1, for stocks in zeta risk slices 1 to 12, average one-month holding

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\(^{10}\)For example, upon experimenting with different undiversified momentum strategies, Jegadeesh and Titman (1993, p. 70) obtain an abnormally high average monthly return of 1.49 percent per month on momentum portfolios created from 12 months of past returns and 3-month holding periods for U.S. stocks. In another paper, Jegadeesh and Titman (2001, p. 704) examine different subperiods for U.S. stocks and report relatively high 6-month momentum profits of 1.65 percent in the period 1990 to 1998 for small U.S. stocks. For the longer period 1965 to 1998, the highest momentum profit was 1.42 percent. Due to these strong return results, momentum strategies have attracted the attention of both academics and practitioners. It should be noted that Daniel (2011) finds poor momentum performance known as “crashes” after the year 2000. In this regard, Barroso and Santa-Clara (2015) and Daniel and Moskowitz (2016) recommend risk management strategies to mitigate momentum crashes.
period returns in our sample period gradually decrease from 2.67 percent for slice 1 to 0.08 percent for slice 12. Also, the corresponding Sharpe ratios decline from 0.26 for slice 1 to -0.31 for slice 12, which compare favorably (in absolute value terms) to the CRSP index at 0.10. As average zeta risk decreases from 12.99 for slice 1 to 0.16 for slice 12, average monthly holding period returns and Sharpe ratios decline in step with diminishing market risk. The close relation between zeta risk and out-of-sample stock returns lends empirical support for the two-factor ZCAPM as an asset pricing model that captures common market risk. In this respect, Fama and MacBeth (1973, p. 613) have observed, "As a normative theory the model only has content if there is some relation between future returns and estimates of risk that can be made on the basis of current information."

Turning to the construction details of the zeta risk portfolio slices in Table 1, the estimated values of $\bar{Z}_{pt}^{*}$ for long and short stocks in the zero-investment portfolios for zeta risk slices 1 to 12 range from 4.25 to $-0.50$ and $-8.74$ to $-0.67$, respectively. Also, the average one-month-ahead returns for long and short stocks in zeta risk slices 1 to 12 range from approximately 1.61 percent to 1.16 percent and $-1.06$ percent to 1.08 percent, respectively. For both long and short slices, total risk as measured by the standard deviation of returns gradually decreases from slice 1 to 12. Notice that long slices 1 and 2 earn approximately 1.61 percent and 1.70 percent, respectively, which are almost two times the average return on the CRSP index equal to 0.89 percent; however, their total risks as measured by the standard deviation of monthly returns in the sample period are higher at 8.76 percent and 7.80 percent compared to 4.49 percent for the CRSP index. Holding total risk approximately constant, long slices 9 to 12 have higher average returns of 1.20 percent, 1.12 percent, 1.09 percent, and 1.16 percent, respectively, than the CRSP index and similar standard deviations (i.e, 4.52 percent, 4.59 percent, 4.59 percent, and 4.64 percent). Also notice that short slices 1 and 2 have relatively high negative zeta risk that results in negative post-formation average portfolio returns of $-1.06$ percent and $-0.55$ percent, respectively.

Using data in Table 1, Figure 3 shows the relationship between average zeta risk in the estimation period, or $\bar{Z}_{pt}^{*}$, and average one-month-ahead returns, or $\bar{R}_{pt+1}$, for the 12 zero-
investment mimicking portfolios based on zeta risk slices. A strong positive and linear relation is clearly evident among the 12 zeta risk portfolios, further supporting our earlier inference that zeta risk is predictive of one-month-ahead holding period returns. Figure 4 plots average one-month-ahead returns for each portfolio with respect to their time-series standard deviation in the sample period. Again, a strong positive and linear relation is obvious. We also show the locations in this risk/return space for the popular zero-investment portfolios $SMB$, $HML$, and $MOM$. The $SMB$ portfolio is noticeably below the zeta risk portfolio with similar return standard deviation, but the $HML$ and $MOM$ portfolios have average risk-return profiles close to our zeta risk portfolios. Also, $MOM$ has a higher average return and risk profile compared to $SMB$ and $HML$. In this regard, previous researchers provide little or no theoretical foundation for these anomalous factors.

The evidence in Figure 3 suggests that these popular multi-factors mimic long/short zeta risk portfolios. That is, $SMB$, $HML$, and $MOM$ are zero-investment portfolios that pick up market risk information related to cross-sectional market volatility.

We next report the results for aggregate stock portfolios based on combining the CRSP value-weighted index (i.e., portfolio $a$) with each of the zero-investment, market-volatility mimicking portfolios. Aggregate portfolios have weights for all CRSP stocks that add to one. Monthly returns are computed for aggregate zeta risk portfolios from January 1965 to December 2015. Figure 5 shows the mean-variance relationship between average returns for aggregate zeta risk portfolios and their time-series standard deviation. As shown there, the aggregate portfolios trace out a smooth frontier that closely approximates a hyperbola (or parabola in variance-of-returns terms). This hyperbola serves as a benchmark to evaluate the relative efficiency of other portfolios. Relevant to Proposition I, a square marks the location of the CRSP index. Also, the locations of different combinations of the CRSP index plus zero-investment portfolios $SMB$, $HML$, and $MOM$ are shown.

Consistent with Proposition I, in Figure 5 the CRSP index lies well below the frontier of aggregate zeta risk portfolios at an interior location proximate to its axis of symmetry. Setting the standard deviation of returns equal to that for the CRSP index, a much higher average monthly return can be achieved of almost 1.50 percent compared to 0.89 percent for the CRSP index. This large (almost twofold) disparity in average returns provides
compelling evidence for the notion that the CRSP index is not efficient. More formally, using these aggregate 12 zeta risk portfolios as test assets, we apply the Gibbons, Ross, and Shanken (GRS) (1989) test for whether the CRSP index is efficient. Given $N = 12$ test assets and $T = 612$ months, the $F$ statistic is 7.41, which is highly significant at less than the 1 percent level. Thus, the GRS test fails to accept the null hypothesis that the CRSP index is efficient. We also compute the GRS test for whether a multi-factor model such as the familiar Carhart (1997) four-factor model explains our 12 portfolios’ returns. Given $N = 12$ test assets, $L = 4$ factors and $T = 612$ months, the $F$ statistic is 5.43, which is highly significant at less than the 1 percent level. Thus, the GRS test fails to accept the null hypothesis that the Carhart model explains these portfolios.

With respect to aggregate portfolios containing the CRSP index plus $SMB$, $HML$, and $MOM$, Figure 5 shows that most of these aggregate portfolios lie well beneath the zeta risk hyperbola and therefore are not efficient. One exception is the aggregate portfolio comprised of CRSP + $HML + MOM$, which lies near the zeta risk frontier. Also, the aggregate portfolio CRSP + $HML$ lies fairly close to the aggregate zeta risk hyperbola. While none of these aggregate portfolios is located above the aggregate zeta risk hyperbola and therefore is more efficient, it is apparent that these multi-factors capture some amount of zeta risk associated with cross-sectional market volatility.

Finally, Figure 6 graphically displays the average performances of the 24 long zeta risk portfolios. As shown there, the basic shape of an hyperbola is traced by these portfolios. The CRSP index lies slightly below the hyperbola and represents that average performance of all stocks in the 24 portfolios. Figure 7 combines each of the 24 zeta risk portfolios (i.e., 50 percent weight) with the CRSP index (i.e., 50 percent weight). These aggregate portfolios are entirely long portfolios with no short positions. Because zeta risk captures cross-sectional market volatility, taking this market risk into account in portfolio construction yields aggregate zeta risk portfolios with average returns that are substantially higher than the CRSP index consistent with Proposition I. Also, supporting Proposition II, incorporating different levels of zeta risk, diversified portfolios can be created that span increasing levels of average returns as total risk increases, thereby tracing out a frontier that envelopes other less efficient portfolios. Comparing the fron-
tiers in Figures 5 and 7 allows some insight into the effects of short positions on aggregate portfolio efficiency. Consistent with prior literature cited earlier, short positions visibly increase mean-variance efficiency, especially for higher total risk portfolios.

5 Robustness Analyses

In unreported results for comparison purposes, we repeated the above analyses by constructing zero-investment mimicking portfolios based on beta coefficient estimates for the size and value factors of Fama and French (1992, 1993, 1995) in addition to the momentum factor of Jegadeesh and Titman (1993). Using daily data for one year in each estimation period downloaded from Kenneth French’s website, we estimated the Carhart (1997) four-factor model, wherein the market, size ($SMB$), value ($HML$), and momentum ($MOM$) factors were proxied by the CRSP value-weighted index, small minus big firms’ stock returns, high book-to-market equity minus low book-to-market equity firms’ stock returns, and high past return minus low past return firms’ stock returns, respectively. Stocks were sorted to form 12 risk slices based on their estimated size beta coefficients (i.e., $\hat{\beta}_{SMB}^i$). Similarly, 12 risk slices were formed based on value beta coefficients (i.e., $\hat{\beta}_{HML}^i$) as well as momentum beta coefficients (i.e., $\hat{\beta}_{MOM}^i$). Equal weights for each stock in any given beta slice were used as before.

In sum, we found no relation between average one-month-ahead stock returns and average factor loadings for $\bar{\beta}_{SMB}^{pt}$ and $\bar{\beta}_{MOM}^{pt}$. A positive relation was found between one-month-ahead returns and $\bar{\beta}_{HML}^{pt}$ portfolio estimates. However, one-month-ahead returns were unrelated to the time-series standard deviations for 12 zero-investment mimicking portfolios based on all three of these multi-factors. In general, each of these multi-factors had higher average returns per unit total risk than zero-investment mimicking portfolios based on size, value, and momentum beta risks. Importantly, the results for aggregate portfolios formed by combining the CRSP index with each of these zero-investment beta-risk portfolios traced out flat lines approximately on the axis of symmetry in Figure 5 (using the CRSP index to benchmark the axis of symmetry). Thus, these multi-factors are not useful in constructing relatively efficient aggregate portfolios. To conserve space,
6 Conclusion

The Roll critique argues that, if a good proxy for the market portfolio is not available, the CAPM cannot be empirically tested. In this regard, many studies have found that popular aggregate market indices are not efficient portfolios. However, no studies to our knowledge have demonstrated methods for constructing more efficient aggregate market portfolios. Recognizing this gap in the literature, this paper sought to build efficient diversified portfolios that are sensitive to market volatility. Based on Black’s (1972) zero-beta CAPM, the two-factor ZCAPM of Liu, Kolari and Huang (2012) posits the following portfolio propositions about the shape of the minimum-variance parabola: (I) general market indices have average returns that lie approximately on the axis of symmetry of the investment parabola and therefore are not efficient portfolios, and (II) the span or width of the parabola is defined by the cross-sectional standard deviation of all assets’ returns (i.e., market volatility). According to this two-factor model, expected asset returns are a function of beta risk related to average market returns and zeta risk associated with cross-sectional market volatility. Regarding the latter factor, market volatility can have two-sided, opposite effects on asset returns. Given average market returns, as market volatility increases, assets in the upper (lower) half of the parabola experience increasing (decreasing) returns, all else the same.

Applying the two-factor ZAPM to U.S. stocks in the sample period January 1965 to December 2015, zero-investment, market-volatility mimicking portfolios were constructed that contain stocks with different levels of zeta market risk (i.e., sensitivity to cross-sectional market volatility). Stocks were ranked on their zeta risk estimates and formed into 12 zero-investment mimicking portfolios that stratify zeta risk into slices ranging from high to low. Notably, for stocks in the top +1 percent/bottom −1 percent of estimated zeta risk coefficients, portfolio returns averaged 2.67 percent in the post-formation month, which is exceptional in comparison to previously published zero-investment portfolio results (e.g., momentum strategies). More generally, supporting the ZCAPM’s ability to
capture market risk, a strong positive and linear relation between zeta risk and one-month-ahead holding period returns was documented.

Using the geometry of ZCAPM portfolio theory, we combined the CRSP value-weighted index with zero-investment, market-volatility mimicking portfolios to form aggregate zeta risk portfolios. The latter aggregate portfolios traced out an upward sloping (efficient) portion of the investment parabola. Supporting Proposition I that common stock market indices are not mean-variance efficient portfolios, the resultant aggregate zeta risk portfolios dominate the CRSP index, which lies proximate to the axis of symmetry of the investment parabola. Supporting Proposition II that return dispersion defines the width of the investment parabola, upon fixing average total risk equal to the CRSP index’s standard deviation of returns in the sample period, average aggregate zeta risk portfolio returns were almost two times greater than average CRSP index returns. We infer that this relatively efficient aggregate index would be a more appropriate proxy for the market portfolio than the commonly-used CRSP index in empirical tests of the CAPPM. Interestingly, zero-investment portfolios based on size, value, and momentum factors exhibited average return/risk performance similar to different zeta risk portfolios. As such, we infer that these multi-factors appear to be capture information related to cross-sectional market volatility (i.e., zeta risk).

We conclude that relatively efficient diversified portfolios can be formed by taking into account market volatility. As proposed by ZCAPM geometry, efficient portfolios can be attained by moving horizontally along the investment parabola’s axis of symmetry via an aggregate stock index and then utilizing cross-sectional volatility to move vertically up (down) toward more (less) efficient portfolios. An important implication of our findings is that asset pricing methods can be applied to investment portfolio management practice. Also, our findings have implications for asset pricing analyses. Future research is recommended that substitutes efficient aggregate portfolios for the CRSP index to investigate their significance in cross-sectional empirical tests of the price of market beta risk. Most asset pricing studies find the beta is not priced in the cross-section of stock returns. In view of the Roll critique, it is possible that a more efficient aggregate index would be priced in such tests. Since multi-factor models typically include the market factor, a more
significant market factor would enhance most asset pricing models.
References


Appendix

A Expectation-Maximization (EM) Regression Method

In this appendix we provide a step-by-step guide to estimating the two-factor ZCAPM using the expectation-maximization (EM) regression method. From equation (9) in the text, this two-factor model can be stated as:

\[ \tilde{R}_{it} - R_{ft} = \alpha_i + \beta_{i,a}(\tilde{R}_{at} - R_{ft}) + Z_{i,a}D_{it}\tilde{\sigma}_{at} + \tilde{u}_{it}, \quad t = 1, \ldots, T \quad (A.1) \]

where \( D_{it} \) is a signal variable taking values +1 and −1 to indicate, respectively, positive and negative market volatility effects on the \( i \)th asset’s returns at time \( t \), and \( \tilde{u}_{it} \sim \text{iid } N(0, \sigma_i^2) \). As mentioned in the text, signal variables \( D_{it} \) are unobservable, independent random dummy variables:

\[
D_{it} = \begin{cases} 
1 & \text{with probability } p_i \\
-1 & \text{with probability } 1 - p_i,
\end{cases} \quad (A.2)
\]

where \( p_i (1 - p_i) \) is the probability of a positive (negative) market volatility effect. Because \( D_{it} \) is not known from observed data, we use Bayes’ rule to compute the probabilities of being ±1 (see equation (A.9) below).

Parameters in the two-factor ZCAPM as represented by equations (A.1) and (A.2) can be collectively denoted \( \theta_i = (\beta_{i,a}, Z_{i,a}, \sigma_i^2, p_i) \). If signal variables \( D_{it} \) were observable, then least squares regression could be used to estimate the first three elements of \( \theta_i \), and \( p_i \) could be estimated by computing the sample frequency of \( D_{it} \) taking the value 1. However, valid statistical inference for the model should only use observed data, not unobservables. A solution to this problem is to estimate the parameters by maximizing the likelihood of observed data. Since the likelihood of observed data has a complicated expression, the familiar expectation-maximization (EM) algorithm can be employed to compute the maximum likelihood estimate (Dempster, Laird, and Rubin (1977)).

In the EM algorithm, the observed data are considered to be part of the complete data, which include unobserved “data” or latent variables. In general, the complete data
likelihood has a simple expression but cannot be used for valid statistical inference due to dependence on latent variables that are unobservable. In order to make valid inferences, the latent variable distribution should be integrated out from the complete data likelihood. The EM algorithm has the advantage of avoiding such integration and still yields valid inferences. EM iterates between an E-step and M-step until convergence is reached. The E-step calculates the conditional expectation of the complete data log-likelihood given the observed data and the current tentative estimates of the parameter values. The M-step maximizes the conditional expectation obtained from the E-step. Under regularity conditions, the EM algorithm converges to the maximum likelihood estimate (Wu, 1983).

Applying the EM algorithm to the two-factor ZCAPM, for the $i$th asset, the complete data set is $\{\tilde{R}_{it}, D_{it}\}$, and the incomplete, observed data set is $\{\tilde{R}_{it}\}$. The complete data likelihood of the model is:

$$L(\theta_i) = \prod_{t=1}^{T} P(\tilde{R}_{it}|D_{it})P(D_{it}), \quad \text{(A.3)}$$

where

$$P(D_{it}) = p_i^{I(D_{it}=1)}(1 - p_i)^{I(D_{it}=-1)},$$

$$P(\tilde{R}_{it}|D_{it}) = \frac{1}{\sqrt{2\pi\sigma_i^2}}\exp\left\{-\frac{\left(\tilde{R}_{it} - R_{ft} - \alpha_i - \beta_{i,a}(\tilde{R}_{at} - R_{ft}) - Z_{i,a}D_{it}\tilde{\sigma}_{at}\right)^2}{2\sigma_i^2}\right\}. \quad \text{(A.4)}$$

Denote the current guess of the parameters as $\theta'_i = (\alpha'_i, \beta'_{i,a}, Z'_{i,a}, (\sigma'_i)^2, p'_i)$. The E-step computes

$$Q_i(\theta_i, \theta'_i) = E\left[\log\{L(\theta_i)\}|\{\tilde{R}_{it}\}, \theta'_i\right]. \quad \text{(A.5)}$$

Define $\hat{p}_{it} = P(D_{it} = 1|\tilde{R}_{it}, \theta'_i)$, which can be computed using equation (A.9). It follows that

$$\tilde{D}_{it} \equiv E(D_{it}|\tilde{R}_{it}, \theta'_i) = P(D_{it} = 1|\tilde{R}_{it}, \theta'_i) - P(D_{it} = -1|\tilde{R}_{it}, \theta'_i) = \hat{p}_{it} - (1 - \hat{p}_{it}) = 2\hat{p}_{it} - 1.$$
Rewriting equation (A.5), we have

\[
Q_i(\theta_i, \theta'_i) = \sum_{t=1}^{T} \left\{ \left[ \tilde{p}_{it} \log p_i + (1 - \tilde{p}_{it}) \log(1 - p_i) \right] - \frac{1}{2} \log [2\pi \sigma_i^2] \right. \\
\left. - \frac{(\bar{R}_{it} - R_{ft} - \alpha_i - \beta_{i,a}(\bar{R}_{at} - R_{ft}) - Z_{i,a} \hat{D}_{it} \tilde{\sigma}_{at})^2}{2\sigma_i^2} - \frac{Z_{i,a}^2 (1 - \hat{D}_{it} \tilde{\sigma}_{at})^2}{2\sigma_i^2} \right\}.
\]

(A.7)

The M-step obtains an update of \( \theta_i \) by solving the following optimization problem:

\[
\text{Max}_{\theta_i} Q_i(\theta_i, \theta'_i).
\]

(A.8)

We should mention that the above maximization problem yields different local maximization points of the likelihood function for different initial input values of the mispricing term \( \alpha_i \). We avoid this pitfall by setting \( \alpha_i = 0 \), such that no mispricing error is allowed.\(^{11}\)

To compute \( \hat{p}_{it} = P(D_{it} = 1 | \bar{R}_{it}, \theta'_i) \), Bayes’ rule can be applied:

\[
\hat{p}_{it} = \frac{P(\bar{R}_{it} | D_{it} = 1, \theta'_i) \cdot P(D_{it} = 1)}{P(\bar{R}_{it} | D_{it} = 1, \theta'_i) \cdot P(D_{it} = 1) + P(\bar{R}_{it} | D_{it} = -1, \theta'_i) \cdot P(D_{it} = -1)},
\]

(A.9)

where

\[
P(\bar{R}_{it} | D_{it} = \pm 1, \theta'_i) = \frac{1}{\sqrt{2\pi(\sigma'_i)^2}} \exp \left\{ \frac{(\bar{R}_{it} - R_{ft} - \beta'_{i,a}(\bar{R}_{at} - R_{ft}) \mp Z'_{i,a} \tilde{\sigma}_{at})^2}{2(\sigma'_i)^2} \right\}.
\]

(A.10)

The above two equations can be rewritten as equations (A.13) and (A.14) defined below. Note that, when \( \theta' \) reaches values at the convergence of the EM algorithm, \( \hat{p}_{it} \) gives the estimated probability of a positive market volatility effect at time \( t \). Inspecting the \( Q(\theta, \theta') \) function given in equation (A.7) reveals that the parameters are separated and thus the optimization in equation (A.8) has closed-form solutions. The first order conditions give the equations (A.15), (A.16), and (A.17) shown below that provide updated estimates of parameters \( \hat{\beta}_{i,a}, \hat{Z}_{i,a}, \hat{p}_{i}, \) and \( \hat{\sigma}_{i}^2 \). The availability of closed-form solutions for the optimization problem is an important reason for using the EM algorithm instead of

\(^{11}\)Liu et al. (2012) show that setting \( \alpha_i = 0 \) has little or no effect on cross-sectional tests of the CAPM market model, Fama and French’s three-factor model, and Carhart’s four-factor model. Hence, this restriction does not appear to affect cross-sectional tests of factors in asset pricing models for the most part.
directly maximizing the observed data likelihood.

For clarity purposes, the following step-by-step procedure is provided for implementing the EM algorithm.

(1) Initialize the set of parameters \( \hat{\theta}_i^0 = (\hat{\beta}_i^0, \hat{Z}_{i,a}^0, (\hat{\sigma}_i^0)^2, \hat{p}_i^0) \) for the \( i \)-th asset. To begin, estimate the simple OLS regression with no mispricing error:

\[
\tilde{R}_{it} - R_{ft} = \beta_{i,a}(\tilde{R}_{at} - R_{ft}) + \tilde{\epsilon}_{it}, \quad t = 1, \ldots, T \tag{A.11}
\]

and set \( D_{it}^0 = 1 \) when \( \tilde{\epsilon}_{it} > 0 \) or \( D_{it}^0 = -1 \) when \( \tilde{\epsilon}_{it} < 0 \) to compute the initial probability \( \hat{p}_i^0 = \frac{1}{T} \sum_{t=1}^{T} D_{it}^0 \). Next, estimate the two-factor ZCAPM regression:

\[
\tilde{R}_{it} - R_{ft} = \beta_{i,a}(\tilde{R}_{at} - R_{ft}) + Z_{i,a} D_{it}^0 \tilde{\sigma}_{at} + \tilde{\epsilon}_{it}, \quad t = 1, \ldots, T \tag{A.12}
\]

to obtain \( \hat{\beta}_{i,a}^0, \hat{Z}_{i,a}^0, \) and the initial variance \( (\hat{\sigma}_i^0)^2 = \frac{1}{T} \sum_{t=1}^{T} \tilde{\epsilon}_{it}^2 \). Set these initials as the current estimates of the parameters, that is, set \( \theta_i^0 = \hat{\theta}_i^0 \).

(2) Compute:

\[
\hat{p}_it = \frac{\eta_{it}^+ p_i'}{\eta_{it}^+ p_i' + \eta_{it}^- (1 - p_i')}, \tag{A.13}
\]

with

\[
\eta_{it}^\pm = \exp\left\{ -\frac{(\tilde{R}_{it} - R_{ft} - \beta_{i,a}'(\tilde{R}_{at} - R_{ft}) \mp Z_{i,a}' \tilde{\sigma}_{at})^2}{2(\sigma_i')^2} \right\}. \tag{A.14}
\]

(3) Solve the following linear equations to get updated estimates of \( \hat{\beta}_{i,a} \) and \( \hat{Z}_{i,a} \):

\[
\left( \sum_{t=1}^{T} (\tilde{R}_{at} - R_{ft})^2 \right) \beta_{i,a} + \left( \sum_{t=1}^{T} \hat{D}_{it}(\tilde{R}_{at} - R_{ft}) \tilde{\sigma}_{at} \right) Z_{i,a} = \sum_{t=1}^{T} (\tilde{R}_{it} - R_{ft})(\tilde{R}_{at} - R_{ft}),
\]

\[
\left( \sum_{t=1}^{T} \hat{D}_{it}(\tilde{R}_{at} - R_{ft}) \tilde{\sigma}_{at} \right) \beta_{i,a} + \left( \sum_{t=1}^{T} \tilde{\sigma}_{at}^2 \right) Z_{i,a} = \sum_{t=1}^{T} \hat{D}_{it}(\tilde{R}_{it} - R_{ft}) \tilde{\sigma}_{it} \tag{A.15}
\]

where \( \hat{D}_{it} = 2\hat{p}_it - 1 \) with \( \hat{p}_it \) obtained from step (2).
(4) Compute the updated variance:

\[
\hat{\sigma}_i^2 = \frac{1}{T} \sum_{t=1}^{T} \{ (\tilde{R}_{it} - R_{ft} - \hat{\beta}_{i,a}(\tilde{R}_{at} - R_{ft}) - \hat{\beta}_{i,a}(\tilde{D}_it)\tilde{\sigma}_{at})^2 + Z_{i,a}^2(1 - \hat{D}_it)^2\tilde{\sigma}_{at}^2 \}. \tag{A.16}
\]

(5) Compute the updated probability:

\[
\hat{p}_i = \frac{1}{T} \sum_{t=1}^{T} \hat{p}_{it}. \tag{A.17}
\]

(6) Define the vector

\[
\Delta \theta = \left( \begin{array}{c}
\left| \hat{\beta}_{i,a} - \beta'_{i,a} \right|, \\
\left| \tilde{Z}_{i,a} - Z'_{i,a} \right|, \\
\left| (\hat{\sigma}_i)^2 - (\sigma'_i)^2 \right|, \\
\left| \hat{p}_i - p'_i \right|
\end{array} \right), \tag{A.18}
\]

and declare convergence when \(\text{Max}(\Delta \theta) < 0.1\%\). If convergence is not reached, update \(\theta'_i\) to the recent values of \(\theta_i\) obtained from steps (3) to (5), and repeat steps (2) to (5).

Given an estimate of probability \(\hat{p}_i\) related to signal variable \(D_{it}\) in sample period \(t = 1, \cdots, T\), the two-factor ZCAPM in equation (A.1) becomes:

\[
\tilde{R}_{it} - R_{ft} = \beta_{i,a}(\tilde{R}_{at} - R_{ft}) + Z^*_{i,a}\tilde{\sigma}_{at} + \tilde{u}_{it}, \tag{A.19}
\]

where \(Z^*_{i,a} = Z_{i,a}(2\hat{p}_i - 1)\) (i.e., equation (10) in the text).
Figure 1: Geometry of the zero-beta CAPM
Figure 2: Identification of unique orthogonal ZCAPM portfolios $I^*$ and $ZI^*$
Figure 3: Relation between average one-month-ahead returns for 12 zero-investment, market-volatility mimicking portfolios and their average zeta risk in the sample period January 1965 to December 2015
Figure 4: Relation between average one-month-ahead returns and the time-series standard deviation of these returns for 12 zero-investment, market-volatility mimicking portfolios (based on zeta risk) compared to the zero-investment, multi-factors \(SMB, HML,\) and \(MOM\) from Kenneth French’s website in the sample period January 1965 to December 2015.
Figure 5: Frontier estimate based on average one-month-ahead returns and the time-series standard deviation of these returns for aggregate portfolios constructed by the CRSP market index plus 12 zero-investment, market-volatility mimicking portfolios (based on zeta risk) compared to the CRSP market index plus zero-investment, multi-factors $SMB$, $HML$, and $MOM$ in the sample period January 1965 to December 2015.
Figure 6: Frontier estimate based on average one-month-ahead returns and the time-series standard deviation of these returns for 24 long zeta risk portfolios compared to the CRSP market index in the sample period January 1965 to December 2015.
Figure 7: Frontier estimate based on average one-month-ahead returns and the time-series standard deviation of these returns for 24 aggregate portfolios formed by investing 50 percent of funds in the CRSP market index and 50 percent of funds in 24 long zeta risk portfolios in the sample period January 1965 to December 2015
Table 1: Construction details for the zero-investment, market-volatility mimicking portfolios formed on slices of zeta risk estimated from the two-factor ZCAPM in the sample period January 1965 to December 2015

This table reports construction details of the zero-investment, market-volatility mimicking portfolios formed on slices of zeta risk associated with cross-sectional market volatility in the two-factor ZCAPM. As described in the text, we estimate this empirical model via the expectation-maximization (EM) regression method using 12 months of daily data. Stocks are sorted in the estimation period based on their estimated \( Z_{i,a}^* \) coefficients. Two zero-investment mimicking portfolios are formed based on stocks in the top +1 percent/bottom −1 percent and top +2 percent/bottom −2 percent of estimated \( Z_{i,a}^* \) coefficients (i.e., slices 1 and 2, respectively), in addition to 10 zero-investment portfolios based on stocks in progressive +5/−5 percent slices of estimated \( Z_{it}^* \) coefficients (i.e., slices 3 to 12). Long \( \bar{Z}_{pt}^* \), short \( \bar{Z}_{pt}^* \), and \( \bar{Z}_{pt}^* \) are the average of positive, negative, and positive minus negative \( Z_{it}^* \) coefficients, respectively, for stocks in each zeta risk slice for 600 monthly rolling one-year estimation periods. Long \( \bar{R}_{pt+1} \), short \( \bar{R}_{pt+1} \), and \( \bar{R}_{pt+1} \) are the average post-formation, one-month-ahead holding period returns in percent terms for stocks in long zeta risk, short zeta risk, and zero-investment zeta risk portfolios, respectively, for 600 months in the sample period January 1965 to December 2015. Standard deviations of these values are shown in parentheses. For comparison purposes, Sharpe ratios for each slice are shown as well as descriptive statistics for the CRSP index.

<table>
<thead>
<tr>
<th>Portfolios</th>
<th>Long ( Z_{pt}^* )</th>
<th>Short ( Z_{pt}^* )</th>
<th>Long ( \bar{Z}_{pt}^* )</th>
<th>Short ( \bar{Z}_{pt}^* )</th>
<th>Long ( \bar{R}_{pt+1} )</th>
<th>Short ( \bar{R}_{pt+1} )</th>
<th>( \bar{R}_{pt+1} )</th>
<th>Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slice 1 - top +1/−1%</td>
<td>4.25</td>
<td>−8.74</td>
<td>12.99</td>
<td>1.61</td>
<td>−1.06</td>
<td>2.67</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>Slice 2 - top +2/−2%</td>
<td>3.45</td>
<td>−7.53</td>
<td>10.98</td>
<td>1.70</td>
<td>−0.55</td>
<td>2.25</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>Slice 3 - top +5/−5%</td>
<td>2.51</td>
<td>−5.97</td>
<td>8.47</td>
<td>1.65</td>
<td>−0.15</td>
<td>1.80</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>Slice 4</td>
<td>1.22</td>
<td>−3.64</td>
<td>4.86</td>
<td>1.49</td>
<td>0.33</td>
<td>1.16</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>Slice 5</td>
<td>0.79</td>
<td>−2.78</td>
<td>3.57</td>
<td>1.35</td>
<td>0.67</td>
<td>0.68</td>
<td>0.06</td>
<td></td>
</tr>
<tr>
<td>Slice 6</td>
<td>0.51</td>
<td>−2.23</td>
<td>2.75</td>
<td>1.31</td>
<td>0.78</td>
<td>0.53</td>
<td>0.03</td>
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<tr>
<td>Slice 7</td>
<td>0.30</td>
<td>−1.83</td>
<td>2.13</td>
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<td>0.84</td>
<td>0.39</td>
<td>−0.02</td>
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<tr>
<td>Slice 8</td>
<td>0.12</td>
<td>−1.51</td>
<td>1.63</td>
<td>1.22</td>
<td>0.97</td>
<td>0.25</td>
<td>−0.08</td>
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</tr>
<tr>
<td>Slice 9</td>
<td>−0.55</td>
<td>−1.25</td>
<td>1.21</td>
<td>1.20</td>
<td>1.00</td>
<td>0.20</td>
<td>−0.13</td>
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</tr>
<tr>
<td>Slice 10</td>
<td>−0.20</td>
<td>−1.03</td>
<td>0.83</td>
<td>1.12</td>
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<td>0.02</td>
<td>−0.27</td>
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<tr>
<td>Slice 11</td>
<td>−0.35</td>
<td>−0.84</td>
<td>0.49</td>
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<td>1.08</td>
<td>0.01</td>
<td>−0.33</td>
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<tr>
<td>Slice 12 - bottom +5/−5%</td>
<td>−0.50</td>
<td>−0.67</td>
<td>0.16</td>
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<td>1.08</td>
<td>0.08</td>
<td>−0.31</td>
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<tr>
<td>CRSP Index</td>
<td>0.89</td>
<td>0.10</td>
<td>(4.49)</td>
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