Correlation Uncertainty, Heterogeneous Investors, and Asset Prices

Abstract

Under-diversification and limited participation, flight to quality, and asset comovement all involve uncertainty about the correlation structure, not only risk about asset payoffs. We present an equilibrium model that incorporates correlation uncertainty in which some investors (retail) are more ambiguity averse than other investors (institutional). When the correlation uncertainty dispersion is large, the institutional investors hold well-diversified portfolios, the retail investors hold under-diversified and even limited participation portfolios. Moreover, the retail investors demand less on the low-weighted volatility assets and more on the high-weighted volatility assets. All risky assets comove more under higher correlation uncertainty.

Keywords: asset pricing, correlation uncertainty, heterogeneous investors

JEL Classification Codes: G11, G12, G13, D52, and D90
Since the seminal work of Markowitz (1952) on portfolio choice and Ross (1976)’s arbitrage pricing theory, one of the fundamental research in finance has been the study of the correlated structure among assets. However, it is challenging to estimate the correlated structure precisely for several reasons. First, the correlated structure is difficult to estimate due to the limitation of the estimation methodology and data (Chan, Karceski, and Lakonishok, 1999), and the instability and complexity in the time-varying correlation process (Buraschi, Porchia, and Trojani, 2010; Engle, 2002). Second, the increasing interdependence of financial markets brings in additional estimation concerns (Forbes and Rigobon, 2002; Campbell, Forbes, Koedijk and Kofman, 2008). Moreover, from an economic perspective, investors could face uncertainty on the estimation of a correlated structure, and this uncertainty is not consistent with the standard expected utility preference (Ellsberg, 1961). The aim of our study is to examine the joint effects of the risk and the uncertainty of the correlated structure on optimal portfolios and equilibrium asset prices.

In this paper, we construct an equilibrium model with heterogeneous correlation uncertainty among investors. We demonstrate that the desire of investors to hedge against correlation uncertainty provides a uniform explanation for several well documented stylized facts in financial markets, which are often studied separably in literature. Specifically, we show that under-diversification and limited participation\(^1\), flight-to-quality and flight-to-safety\(^2\), and comovement and contagion\(^3\) can arise endogenously together in the presence of correlation uncertainty. Our equilibrium model also offers novel predictions for optimal portfolio choice and asset pricing.

When an investor evaluates an investment without knowing neither future realization of the payoff (risk) nor the probability of its occurring (ambiguity), the uncertainty is often

---

\(^1\)Mankiw and Zeldes (1991) presents evidence of limited participation of households in the capital market. Campbell (2006) finds that 20 percent of households has no public equity even when their retirement savings are considered. Vissing-Jorgensen (2002) suggests that accounting for limited asset market participation is important for the estimation of the elasticity of intertemporal substitution. Calvet, Campbell, and Sodini (2007) distinguishes underdiversification (“down”) from nonparticipation (“out”) in a European market. See also Calvet, Campbell, and Sodini (2009), and Dimmock, Kouwenberg, Mitchell, and Peijnenburg (2016).

\(^2\)Baur and Lucey (2009) finds that the flights from stock to bond and vice versa exist and are common features in many crisis episodes by using empirical data in eight developed counties. Baele, Bekaert, Inghelbrecht, and Wei (2013) identifies and characterizes flight to safety episodes for 23 countries and identifies major market crashes, such as October 1987, the Russia crisis in 1998 and the Lehman bankruptcy as flight to safety episodes.

\(^3\)Forbes and Rogibon (2002) documents strong evidence of high stock market co-movements during several financial crises and suggests it follows from a continuation of strong cross-market linkages. Pindyck and Rotemberg (1993) finds that the comovements of individual stock prices cannot be justified by fundamental variables. See also Barberis, Shleifer, and Wurgler (2005).
called ambiguity or Knightian uncertainty (Knight, 1921). In this paper, the investors’ preferences are presented by the “max-min” expected utility, following the axiomatization of aversion to ambiguity in Gilboa and Schmeidler (1989). In other words, investors propose a set of correlated structures among assets, and evaluate the outcome of an investment with respect to all correlated structures and then choose the one that leads to the lowest expected utility. Investors are heterogeneous in terms of ambiguity aversion, reflecting their various levels of sophistication in dealing with statistical data and estimation methodology. Some investors (retail) are more ambiguity averse than other investors (institutional). To concentrate on the role of correlation ambiguity, investors in our model have perfect knowledge of the marginal distributions of all assets, that is, they merely have concerns about the correlation structure.

In this paper, the equilibrium under correlation uncertainty is characterized by the simultaneous determination of both the optimal correlated structure and the optimal portfolio for each investor. There are two kinds of equilibriums in which the heterogeneity of investors’ correlation uncertainty plays a critical role. When the dispersion of correlation uncertainty among investors is small, each investor chooses the most correlated structure (associated with the highest correlation coefficient) in a full participation equilibrium. On the other hand, when the uncertainty dispersion is large, while the institutional investor still chooses the most correlated structure, the retail investor’s choice of the correlation structure is no longer relevant. A portfolio inertia occurs, resulting in a limited participation equilibrium.

We demonstrate how the correlation uncertainty affects asset prices, risk premiums, Sharpe ratios, and betas in different manners, depending on the characteristics of individual assets. In our model, we identify each asset by its weighted volatility, a product of size and return volatility, which is equivalent to a risk-adjusted size factor. We also use

---

4A growing body of research in asset pricing applies the multiple-priors framework of Gilboa and Schmeidler (1989) in which investors are heterogeneous in terms of ambiguity aversion. See, Cao, Wang, and Zhang (2005); Easley and O’Hara (2009, 2010); Epstein and Miao (2003). Moreover, there is both laboratory evidence and non-laboratory empirical evidence of ambiguity aversion heterogeneity. See Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010); Dimmock et al. (2016).

5By a copula theory (see McNeil, Frey, and Embrechts, 2005), the joint distribution for several random variables is characterized by the marginal distribution of each random variable, and a copula function that purely determines the correlation structure. In other words, the correlation structure can be independent of the marginal distributions.

6The risk-adjusted size factor or the weighted volatility captures a trade-off between asset quality versus firm size. The negative relation between size and quality has been empirically documented. In particular, small firms tend to be “junk”. See Ang, Hodrick, Xing, and Zhang (2006). Asness, et al. (2016) uses a wide variety of quality measures over different countries and industries and finds robust effect of the size factor after controlling for quality.
\( \eta \), the ratio of the weighted volatility over the total weighted volatility of all assets, to measure its relative risk contribution to the entire market. We show that for a low-\( \eta \) asset in the heterogeneous equilibrium, the higher the correlation uncertainty, the lower the price and the higher the Sharpe ratio. On the other hand, the price of a high-\( \eta \) asset increases with respect to the correlation uncertainty. Each asset’s beta and its correlation with the market portfolio also display different cross-sectional patterns depending on the change of the correlation uncertainty.

By focusing on the heterogeneity in correlation uncertainty, our equilibrium analysis has four implications. First, the under-diversification and limited participation arise endogenously in the equilibrium. We show that the retail investor chooses a more correlated structure than the institutional investor under all circumstances. As a result, the institutional investor always holds a well-diversified portfolio, whereas the retail investor holds an under-diversified portfolio. An increase in the correlation uncertainty induces a less diversified portfolio for the retail investor, and in some cases, the retail investor even has limited participation due to the portfolio inertia. Our model predicts that the retail investor’s portfolio is less risky because his higher correlation uncertainty yields higher implicit risk aversion, and the institutional investor’s portfolio achieves better performance than the retail investor.

It is well known that aversion to ambiguity helps to explain the limited participation puzzle but our results significantly differ from the previous studies. In particular, our approach is built on the methodology in Polkovnichenko (2005), Calvet, Campbell, and Sodini (2009), and Dimmock et al. (2016) by comparing the optimal portfolio with the market portfolio. To quantify the extent of under-diversification in a precise manner, we hinge upon a dispersion measure inspired by the portfolio selection literature (Ibragimov, Jaffee, and Walden, 2011). As shown explicitly in our model, the optimal portfolio is less diversified with a higher degree of correlation uncertainty, which are also consistent with the empirical

---

7 Calvet, Campbell and Sodini (2009) finds empirical evidence that better educated household tend to be better diversified. Dimmock et al. (2016) documents further empirical evidences of portfolio underdiversification with the increase of ambiguity.

8 Cao, Wang, and Zhang (2005), and Easley and O’Hara (2009) demonstrate the limited participation considering only two risky assets, but the limited participation in our model involve a large number of assets. Our results are consistent with the limited market participation in the aforementioned studies where \( N = 2 \).
findings in Dimmock et al.(2016). Our model suggests another important element that leads to the under-diversification and limited participation.\footnote{Cao, Wang, and Zhang (2005), Easley and O’Hara (2009), and Wang and Uppal (2003) focus on the situation in which the marginal distribution is ambiguous given that the correlation structure is known. In contrast, we assume the marginal distribution is known whereas the correlation structure is ambiguous.}

Second, our model generates flight-to-quality and flight-to-safety from a correlation uncertainty perspective.\footnote{Guerrieri and Shimer (2014) argues that adverse selection is another source of illiquidity. Vayanos (2004) constructs a balance sheet model in which the investor prefers more liquid and less risky assets when the balance sheet is tight during periods of market stress. Routledge and Zin (2014) shows that ambiguity can drastically increase the bid-ask spread and hence reduce liquidity.} When the dispersion among investors’ correlation estimation is large, the institutional investor acquires more low-eta assets whereas the retail investor demands more high-eta assets. Since large uncertainty dispersion among investors often comes in a stressed economy, the high-eta assets can be used to hedge against “catastrophic economic shock” or “macroeconomic uncertainty”. Therefore, our model provides an alternative interpretation of flight-to-safety or flight-to-quality episodes which results from retail investors’ large demand for high-eta assets. And the prices of high-eta assets move up enormously in a bad economic situation. Caballero and Krishnamurthy (2008) develops an information amplification mechanism with uncertainty averse investors. In the worst-case scenario, the investors choose to invest only on safe assets. Our model is similar to Caballero and Krishnamurthy (2008) in the Knightian uncertainty setting, but our focus is on the uncertainty from the correlation structure.

Third, our model helps in understanding asset comovement from an investment perspective. In our model, the relative Sharpe ratios of high-eta assets decrease whereas the relative Sharpe ratios of low-eta assets increase; thus the dispersion of all Sharpe ratios decreases endogenously with the increase of correlation uncertainty. Previous studies document that assets move closely together in a downside market and move apart in an upside market (Barberis, Shleifer, and Wurgler, 2005; Basak and Pavolva, 2013; Kyle and Xiong, 2001; and Veldkamp, 2006.).\footnote{Barberis, Shleifer, and Wurgler (2005) proposes three sources of frictions and suggest investor sentiments as an explanation for stock comovement. Basak and Pavolva (2013) develops models of asset class effect or community effect, in which comovement is implied by the correlated demand unrelated to fundamentals. Kyle and Xiong (2001) describes a wealth effect of convergence traders that create contagion, thus assets become more volatile and more correlated. Veldkamp (2006) argues that costly information generates excess covariance and comovements.} As a complementary analysis, our model suggests that risky assets are forced to comove more with larger correlation uncertainty due to similar investment trading opportunities in terms of the Sharpe ratios.
Last but not least, our model implies that high trading volume is associated with a large heterogeneity in correlation uncertainty among investors. When economic situation is very bad, the correlation uncertainty dispersion is large, the retail investor easily panics and overreacts to the market thereafter purchasing a significant amount of high-eta assets and selling low-eta assets. Our result is different from Easley and O’Hara (2010), in which they demonstrate market freeze when the investor has an incomplete preference in the presence of model uncertainty. Under Bewley (2002)’s incomplete preference, the investor evaluates the outcome in each model and determines the trading decision only when this decision leads to the highest expected utility in each plausible model. Therefore, when the uncertainty is high, the investors are almost unwilling to buy or sell at any posted asset price. By contrast, since the investor in our framework chooses the model that yields the least expected utility in the worst-case scenario, the investor’s desire to hedge against the uncertainty not only generates flight to quality, but also high trading volume. If we interpret the heterogeneity as disagreement, our result is consistent with empirical findings by Carlin, Longstaff, and Matoba (2014).

Our model draws from many important works of asset pricing under ambiguity literature. Boyle, Garlappi, Uppal, and Wang (2012), Cao, Wang, and Zhang (2005), Easley and O’Hara (2009), and Garlappi, Uppal, and Wang (2007) investigate expected return parameter uncertainty. Easley and O’Hara (2009) and Epstein and Ji (2013) discuss volatility parameter uncertainty. In an information ambiguity setting, Easley, O’Hara, and Yang (2015), Epstein and Schneider (2008), and Illeditsch (2011) address the conditional distribution ambiguity of the signals. In all these studies, however, the correlation structure is given as exogenous. Instead, we allow ambiguity to exist in the correlation structure while the marginal distribution is known. In this regard, our model creates a situation in which an ambiguity-averse investor views the overall market as highly ambiguous rather than made up of individual stocks, such as in Boyle, et al. (2012), and Uppal and Wang (2003). Therefore, correlation uncertainty can be viewed to some extent as systemic risk uncertainty because the ambiguity in the overall market is attributed to the macroeconomic uncertainty or the aggregate liquidity risk (Dicks and Fulghieri, 2015). The studies in asset pricing under uncertainty also include the works of Bossarts et al (2010), Hansen and Sargent (2001), and Routledge and Zin (2014).
The rest of the paper is organized as follows. Section 1 presents our correlation uncertainty model. In Section 2, we study the portfolio choice problem and characterize the equilibrium under heterogeneous correlation uncertainty. In Section 3 we discuss the joint effect of correlation uncertainty and asset characteristics on asset prices, risk premiums, correlations with the market portfolio and betas. Section 4 presents further implications of our model, including the under-diversification and limited participation, flight to quality and flight to safety, and asset comovement. We also use the 2007-2009 financial crisis as an example to illustrate these empirical implications. Section 5 presents the robustness of our results in a general setting. Section 6 concludes. Proofs and technical arguments are in the Appendices.

1 A Model of Correlation Uncertainty

There are $N$ risky assets and one risk-free asset in a two-period economy (date $t = 0$ and $t = 1$). The payoffs (or dividends) of these $N$ risky assets are $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N$, respectively, at time $t = 1$. The risk-free rate is assumed to be zero, alternatively speaking, the risk-free asset serves as a numeraire in the economy. The per capita endowment of risky asset $i$ is $\bar{x}_i, i = 1, \ldots, N$. Each risky asset can be viewed as an investment asset, an equity portfolio, an investment fund, or a market portfolio in an international market.

To focus entirely on the correlation uncertainty and its effect on asset pricing, we investigate the correlation structure instead of the joint distribution of $(\tilde{a}_1, \ldots, \tilde{a}_N)$. In our specification of the correlation matrix, we employ the dynamic equicorrelation (DECO) model of Engle and Kelly (2012), where any two distinct risky assets have a same correlation coefficient $\rho$, i.e., $\text{corr}(\tilde{a}_i, \tilde{a}_j) = \rho$ for each $i \neq j$.\footnote{This assumption on the correlation structure can be relaxed in a block equicorrelation model, in which all risky assets are grouped into several sectors and the assets within each sector have close pairwise correlation coefficients. Although the details are not presented here, we extend the presented setting using two block equicorrelation examples in Engle and Kelly (2012), and the main insights of correlation uncertainty on equilibrium are largely the same. The details are available upon request.} Engle and Kelly (2012) show that the (block) DECO estimation of U.S. stock return data can display a better fit for the data than a general dynamic conditional correlation (DCC in Engle, 2002) model. Consistent with Easley and O’Hara (2009, 2010), we assume that $(\tilde{a}_1, \ldots, \tilde{a}_N)$ has a multivariate Gaussian distribution. Therefore, investors are confident in the estimation of the expected mean $\bar{a}_i$ and the variance $\sigma_i^2$ for each risky asset $i = 1, \ldots, N$; but are severely concerned about the esti-
mation of $\rho$, which represents the ambiguity aversion to the correlated structure. Since the correlation coefficient between two payoffs is identical to the correlation coefficient between the returns, our assumption is equivalent to the ambiguity being purely on the correlation structure of the asset returns, but no ambiguity on the marginal distribution. In this paper we confine ourselves to a nonnegative correlated financial market.$^{13}$

In this economy, all investor have CARA-type risk preference with the same absolute risk aversion parameter $\gamma$. In most situations, investors are able to find the benchmark correlation coefficient through the calibration to a stochastic process with some estimation errors (Buraschi, Porchia, and Trojani, 2010; Chan, Karceski, and Lakonishok, 1999; Engle, 2002), and the more sophisticated the investors the smaller the estimation error. To highlight the heterogeneity of correlation estimation among investors, we consider two types of investors, institutional investors and retail investors, in our model. The percentage of institutional investors in the market is $\nu$ and the percentage of retail investors is $1 - \nu$.

For simplicity, we assume that institutional investor knows the true correlation coefficient $\bar{\rho}$ without estimation error, being a Savage investor with perfect knowledge about the correlation structure among assets.$^{14}$ The retail investors are ambiguity-averse investors in the sense of Gilboa and Schmeidler (1989). Specifically, let $[\rho - \epsilon, \rho + \epsilon]$ be a set of correlation coefficients considered by the investor$^{15}$ and $E^\rho[\cdot]$ represents the expectation operator with respect to the belief generated by the correlation coefficient $\rho$. The retail investors do not have the requisite experience to form priors over the occurrence of particular states of the world, knowing only a set of correlation coefficients. Therefore, the ambiguous-averse investor evaluates outcomes under all possible correlated structures among assets, and chooses the one that leads to the lowest expected utility

$$\min_{\rho \in [\rho - \epsilon, \rho + \epsilon]} E^\rho[u(W)], u(W) = -e^{-\gamma W}.$$ \hspace{1cm} (1)

$^{13}$A positively correlated structure is driven by common shocks or factors in a financial market. Both the diversification benefits and synchronization are more critical in a positively correlated economy than in a negatively correlated environment. In fact, our results in this paper hold when all correlation coefficients are strictly larger than $-\frac{1}{N-1}$ from a technical point of view.

$^{14}$We extend the setting in Section 5 in which the institutional investor is also ambiguity-averse, but the retail investor has a larger ambiguity aversion than the institutional investor.

$^{15}$It is well known that the plausible linear correlation coefficient between any two variables $X$ and $Y$ is an interval, $[a, b]$. See McNeil, Frey, and Embrechts (2015). We change $[\rho - \epsilon, \rho + \epsilon]$ to be $[\rho - \epsilon, \rho + \epsilon] \cap [0, 1]$ in some special cases.
In this regard, our setting of correlation uncertainty is different from Ehling and Heyerdahl-Larsen (2016), which studies the correlation structure through the channel of heterogeneity in risk aversion.

The size of $[\rho - \epsilon, \rho + \epsilon]$ measures the degree of correlation uncertainty of the retail investor. The smaller $\epsilon$ the less ambiguous the investor. $\epsilon$ also measures the dispersion of correlation uncertainty among two types of investors.

## 2 Equilibrium

This section presents the characterization of the equilibrium under correlation uncertainty. We start with the portfolio choice problem of an ambiguity averse investor.

### 2.1 Optimal Portfolio Choices

Let $x_i$ be the number of shares on the risky asset $i$, $i = 1, \cdots, N$, and $W_0$ is the initial wealth of the investor. Then the final wealth at time $t = 1$ is

$$W = W_0 + \sum_{i=1}^{N} x_i (\tilde{a}_i - p_i),$$

where $p_i$ is the price of the risky asset $i$ at time $t = 0$. The optimal portfolio choice problem for the investor is

$$\max_{x \in \mathbb{R}^N} \min_{\rho \in [\rho - \epsilon, \rho + \epsilon]} \mathbb{E}^\rho \left[-e^{-\gamma W}\right]. \tag{2}$$

Under CARA preference and the multivariate Gaussian distribution assumption of the asset returns, it is standard to reduce this optimal portfolio choice problem to be

$$A \equiv \max_{x} \min_{\rho \in [\rho - \epsilon, \rho + \epsilon]} CE(x, \rho) \tag{3}$$

where $CE(x, \rho) = (\tilde{a} - p) \cdot x - \frac{1}{2} x^T \cdot D^T \cdot R(\rho) \cdot D \cdot x$ is the mean-variance utility of the investor when the demand vector on the risk assets is $x = (x_1, \cdots, x_N)^T$, $D = (d_{ij})$ is a

---

16 Previous literature measuring macroeconomic or aggregate uncertainty use proxies such as the VIX index, volatility of the CRSP value-weighted index, the Chicago Fed National Activity index (Bloom, 2009; Jurado, Ludvigson, and Ng, 2005). Consistent with these studies, we assume that the degree of correlation uncertainty is positively associated with these uncertainty indexes.
diagonal $N \times N$ matrix with entries $d_{ii} = \sigma_i$ for each $i = 1, \ldots, N$, and $R(\rho)$ is a correlation matrix with a common correlation coefficient $\rho$. We use $\cdot^T$ to denote the transpose operator of a matrix. Thus the certainty-equivalent of the ambiguity-averse investor is

$$CE(x) = \min_{\rho \in [\rho - \epsilon, \rho + \epsilon]} CE(x, \rho).$$

(4)

It is clear to obtain

$$CE(x) = \begin{cases} \quad CE(\rho + \epsilon, x), & \text{if } \sum_{i \neq j} \sigma_i x_i \sigma_j x_j > 0, \\ \quad CE(\rho - \epsilon, x), & \text{if } \sum_{i \neq j} \sigma_i x_i \sigma_j x_j < 0, \\ \quad \sum_{i=1}^N \left( (\bar{\pi}_i - p_i) x_i - \frac{\gamma}{2} \sigma_i^2 x_i^2 \right), & \text{if } \sum_{i \neq j} \sigma_i x_i \sigma_j x_j = 0. \end{cases}$$

(5)

The insight of Equation (5) is appealing and it can be illustrated by taking an example of $N = 2$. If $x_1 x_2 > 0$, the portfolio yields a synchronization strategy, then the highest correlation is the worst-case scenario for the ambiguity-averse investor. When $x_1 x_2 < 0$, the portfolio is a pair trading or a market-neutral strategy, the worst-case scenario for the least mean-variance utility should correspond to the lowest possible correlation coefficient. Finally, if $x_1 x_2 = 0$, then either $x_1 = 0$ or $x_2 = 0$ and the choice of correlation coefficient in computing $CE(x)$ is clearly irrelevant.

For a financial market with more than three risky assets, the intuition of Equation (5) is similar. If the holding positions are largely in the same direction, the worst-case scenario should be when the ambiguity-averse investor chooses the highest correlation coefficient among assets. By contrast, if the holding positions on the risky assets are opposite, the worst-case choice yielding the least expected utility should be the one when assets are the least correlated. The most interesting case is when the portfolio positions satisfy $\sum_{i \neq j} (\sigma_i x_i) (\sigma_j x_j) = 0$, and we call it a limited participation portfolio.\footnote{This equation deserves further comments. For $N \geq 3$, if each $x_i$ is non-negative, then $\sum_{i \neq j} (\sigma_i x_i) (\sigma_j x_j) = 0$ ensures that there is at most one non-zero position $x_i$; however, if short-selling is allowed, it is possible that each $x_i$ is non-zero in the equation $\sum_{i \neq j} (\sigma_i x_i) (\sigma_j x_j) = 0$. For instance, if $N = 3$, let $\sigma_1 = \sigma_2 = \sigma_3$, then $\sum_{i \neq j} (\sigma_i x_i) (\sigma_j x_j) = 0$ for $x_1 = 2, x_2 = 2$, and $x_3 = -1$.} For a limited participation portfolio, the choice of the correlation coefficient is irrelevant since each one leads to the same expected utility.
For several purposes of our subsequent discussions, we introduce a dispersion measure, \( \Omega(w) \), of a vector \( w = (w_1, \cdots, w_N) \) with \( \sum_{i=1}^{N} w_i \neq 0 \). Let

\[
\Omega(w) \equiv \sqrt{\frac{1}{N-1} \left( N \sum_{i=1}^{N} w_i^2 - \left( \sum_{i=1}^{N} w_i \right)^2 \right) - 1}.
\]  

(6)

Clearly, \( \Omega(w)^2 \) is up to a linear transformation of the Herfindahl index \( \sum_{i=1}^{N} w_i^2 \) for \( \sum_{i=1}^{N} w_i = 1 \). A formal justification of \( \Omega(\cdot) \) being a dispersion measure is presented in Appendix C.

By using the dispersion measure \( \Omega(\cdot) \), we reformulate the certainty-equivalent of the ambiguity-averse investor as

\[
CE(x) = \begin{cases}
CE(\bar{p} + \epsilon, x), & \text{if } \Omega(\sigma x) < 1, \\
CE(\bar{p} - \epsilon, x), & \text{if } \Omega(\sigma x) > 1, \\
\sum_{i=1}^{N} \left( (\bar{a}_i - p_i)x_i - \frac{\gamma}{2}\sigma_i^2 x_i^2 \right), & \text{if } \Omega(\sigma x) = 1.
\end{cases}
\]

(7)

Therefore, the optimal portfolio choice problem for the ambiguity-averse investor is to solve

\[
A = \max \left\{ \max_{\Omega(\sigma x) < 1} CE(\bar{p} + \epsilon, x), \max_{\Omega(\sigma x) > 1} CE(\bar{p} - \epsilon, x), \max_{\Omega(\sigma x) = 1} CE(\rho, x) \right\}.
\]

(8)

**Proposition 1 (Optimal Portfolio Choice)** Let \( s_i = (\bar{a}_i - p_i)/\sigma_i \) be the Sharpe ratio of asset \( i \) and \( \Omega(s) \) be the dispersion of the Sharpe ratios vector \( s = (s_1, \cdots, s_N) \) of all risky assets. \( \hat{\Omega}(s) \equiv \frac{1 - \Omega(s)}{1 + (N-1)\Omega(s)} \). \( x_\rho \equiv \frac{1}{\gamma} D^{-1} R(\rho)^{-1} s^T \) be the optimal portfolio in the absence of uncertainty when the correlation coefficient is \( \rho \). We assume that \( \sum_{i=1}^{N} s_i \neq 0 \).

1. If \( 0 < \epsilon < \bar{p} - \hat{\Omega}(s) \), the investor’s optimal demand is \( x_{\bar{p} - \epsilon} \); if \( 0 < \epsilon < \hat{\Omega}(s) - \bar{p} \), the investor’s optimal demand is \( x_{\bar{p} + \epsilon} \).

2. If \( \epsilon \geq |\bar{p} - \hat{\Omega}(s)| \), the investor’s optimal demand is \( x_{\hat{\Omega}(s)} \).

The first part of Proposition 1 concerns the situation in which the correlation uncertainty is small. If \( \epsilon \) is smaller than the absolute difference between \( \hat{\Omega}(s) \) and the true correlation

\footnote{For example, the dispersion of \( (1/3, 1/3, 1/3) \) is zero, the dispersion of \( (1, 0, 0) \) is 1, and the dispersion of \( (1/2, 1/3, 1/6) \) is \( \frac{1}{2\sqrt{3}} \). Evidently, the smaller the dispersion the more diversified these variables.}

\footnote{For any two \( 1 \times N \) vectors \( s \) and \( t \), \( st \) denotes \( (s_1 t_1, \cdots, s_N t_N) \), and its dispersion is written as \( \Omega(st) \). \( \sigma = (\sigma_1, \cdots, \sigma_N) \).}
coefficient, the optimal demand of the retail investor is \( x_{\bar{\rho} - \epsilon} \) or \( x_{\bar{\rho} + \epsilon} \), depending on whether \( \hat{\Omega}(s) > \bar{\rho} \) or \( \hat{\Omega}(s) < \bar{\rho} \). The intuition is straightforward. First, let us assume that all risky assets offer similar investment opportunities, the worst-case scenario for the ambiguity-averse investor is associated with the highest correlation. In this case, \( \Omega(s) \) is close to zero, thus \( \hat{\Omega}(s) \) is close to one and all plausible correlation coefficients are smaller than \( \hat{\Omega}(s) \). Therefore, the optimal demand of the ambiguity-adverse investor is determined by the correlation coefficient \( \bar{\rho} + \epsilon \) as characterized by Proposition 1 (1).

Next, we consider a situation in which risky assets display significantly different investment opportunities thus \( \Omega(s) \) is large. For example, all risky assets’ returns are close to the risk-free rate except only one asset. The worst-case scenario for the retail investor is naturally related to the least correlated structure among the risky assets for diversification benefits. Hence, the optimal demand is determined as \( x_{\bar{\rho} - \epsilon} \). Indeed, a large \( \Omega(s) \) yields a small \( \hat{\Omega}(s) \), thus all plausible correlation coefficients are larger than \( \hat{\Omega}(s) \). By Proposition 1 (1), the optimal demand of the retail investor is associated with \( \bar{\rho} - \epsilon \).

The second part of Proposition 1 shows that when the correlation uncertainty is large such as \( \epsilon \geq \left| \bar{\rho} - \hat{\Omega}(s) \right| \), the optimal demand for the retail investor displays a completely different pattern. The intuition is as follows. If the correlation uncertainty is large, investor will decide not to diversify, but invest on the risky asset with a higher Sharpe ratio for \( N = 2 \), as shown in Proposition 1 (2). For \( N \geq 3 \), the optimal positions are

\[
x_i = \frac{1}{\gamma \sigma_i} \frac{1 + (N - 1)\Omega(s)}{N \Omega(s)} \left( s_i - \frac{1 - \Omega(s)}{N} S \right)
\]

where \( S = \sum_{i=1}^{N} s_i \) is the total sum of all Sharpe ratios. Since the optimal demand \( x_i \) does not depend on the correlated structure at all, a portfolio inertia occurs.\(^{20}\)

For \( N = 2 \), this portfolio inertia in the correlated structure generates a limited market portfolio, \( x_1 x_2 = 0 \), which is the same as a classical nonparticipation portfolio. For \( N \geq 3 \), however, it is significantly different from the anti-diversification portfolio in Goldman (1979),

\[^{20}\]It has been well documented that high ambiguity may result in portfolio inertia since Cao, at el. (2005), Easley and O’Hara (2009), Epstein and Schneider (2008), and Illeditsch (2011). In particular, investors’ holdings are constant in a range of prices when the uncertainty on the marginal distribution is high. We identify another type of portfolio inertia in a set of correlated structures when the correlation uncertainty is high.
in which at most one risky asset participates in the portfolio\footnote{If can be shown that any limited participation portfolio \((x_1, \cdots, x_N)\) under our definition can be constructed as \(x_1 = \frac{c}{\sigma_i}(y_1 - \frac{1-\hat{\Omega}(s)}{N}Y), i = 1, \cdots, N\) for an arbitrary \(N \times 1\) vector \((y_1, \cdots, y_N)\), where \(Y = \sum_{i=1}^N y_i\) and \(c\) is an arbitrary non-zero constant.}. The difference between the limited market portfolio and anti-diversification in three risky assets is displayed in Figure 1 (a).

Since the portfolio inertia feature in the correlated structure is driven by the correlation uncertainty, we display the portfolio inertia region in \((s_1 \geq 0, s_2 \geq 0)\) in Figure 1 (b). This portfolio inertia region, \(|\Omega(s) - \overline{\rho}| \leq \epsilon\), consists of two components, one is bounded by \(s_2 = (\overline{\rho} + \epsilon)s_1\) and \(s_2 = (\overline{\rho} - \epsilon)s_1\), the other is bounded by \(s_2 = \frac{1}{\overline{\rho} + \epsilon}s_1\). The larger the correlation uncertainty the more likely the portfolio inertia will occur.

The portfolio inertia property can be also explained by the retail investor’s optimal demand. Assuming \(\overline{\rho} < \hat{\Omega}(s)\) and the retail investor has a small correlation uncertainty, he chooses the correlation structure with the highest correlation coefficient \(\overline{\rho} + \epsilon\), yielding the optimal demand \(x_{\overline{\rho} + \epsilon}\) accordingly. If the correlation uncertainty is larger than \(\hat{\Omega}(s) - \overline{\rho}\), the optimal demand is determined as if the correlation coefficient is \(\hat{\Omega}(s)\), thus, the optimal demand is independent of the correlation uncertainty. See Figure 1 (c) for a description of the portfolio inertia feature in terms of optimal demand. Figure 1 (d) displays another example of portfolio inertia when \(\overline{\rho} > \hat{\Omega}(s)\).

2.2 Characterization of Equilibrium

We start with the characterization of the equilibrium first in a baseline model with retail investors only \((\nu = 0)\).

Proposition 2 (Homogeneous Equilibrium) Assume the set of plausible correlation coefficients is \([\overline{\rho} - \epsilon, \overline{\rho} + \epsilon]\) in a homogeneous environment. There exists a unique uncertainty equilibrium in which the endogenous pairwise correlation coefficient among assets is \(\overline{\rho} + \epsilon\), and the price of the risky asset \(i\) is given by

\[
p_i = \overline{a}_i - \gamma \sigma_i (1 - \overline{\rho} - \epsilon) \sigma_i x_i - \gamma \sigma_i (\overline{\rho} + \epsilon) \left( \sum_{n=1}^{N} \overline{\sigma}_n \overline{x}_n \right),
\]

(10)
Proposition 2 follows easily from Proposition 1. Since the optimal portfolio must be the market portfolio \( \sum_{i=1}^{N} \bar{x}_i \bar{a}_i \) in equilibrium, there is no short position in the optimal portfolio of the representative investor, thus \( \Omega(\sigma x^*) = \Omega(\sigma \bar{x}) < 1 \). According to Proposition 1, the worst-case scenario for the ambiguity-averse investor corresponds to the highest possible correlation coefficient, thus the endogenous pairwise correlation coefficient is \( \bar{\rho} + \epsilon \).

The risk premium \( \bar{a}_i - p_i \) can be written as a sum of two components:

\[
\bar{a}_i - p_i = \gamma (1 - \bar{\rho}) \sigma_i^2 \bar{x}_i + \gamma \sigma_i \bar{\rho} \sum_{n=1}^{N} \sigma_n \bar{x}_n \\
+ \epsilon \gamma \left( \sigma_i \sum_{j \neq i} \sigma_j \bar{x}_j \right)
\]

(11)

where the first component represents the risk premium in the absence of correlation uncertainty (for institutional investor), and the second one is the correlation-uncertainty premium. Equation (11) is useful to explain the equity premium puzzle arising from a positive uncertainty premium, in addition to the uncertainty on the expected return or the volatility (Cao, et al., 2005; Epstein and Ji, 2013).

Since the uncertainty equity premium has been well studied in literature, our goal is to characterize the general equilibrium and demonstrate the asset pricing implications in a heterogeneous correlation uncertainty setting.

Define three auxiliary functions which capture the parameters in the heterogeneous setting. Let

\[
m(x, y) \equiv \frac{\nu}{1-x} + \frac{1-\nu}{1-y},
\]

(12)

\[
n(x, y) \equiv \frac{\nu x}{(1-x)(1+(N-1)x)} + \frac{(1-\nu)y}{(1-y)(1+(N-1)y)},
\]

(13)

and

\[
K(\rho) \equiv \frac{1}{1-\rho} \frac{\Omega(\sigma \bar{x})}{1+(N-1)\rho} + \frac{(1-\rho)(1-\Omega(\sigma \bar{x}))}{\nu} \frac{1}{1-\rho} + \frac{(N-1)\Omega(\sigma \bar{x})}{1+(N-1)\rho}.
\]

(14)

Proposition 3 (Heterogenous Equilibrium)
1. Full Participation: If

\[ \bar{p} \geq \frac{1}{N-1} \left\{ \nu \frac{\Omega(\sigma \bar{x})}{1-\nu} N - 1 \right\} , \]  

(15)

or if \( \epsilon < K(\bar{p}) - \bar{p} \), there exists a unique equilibrium in which

\[ p_i = \bar{a}_i - \frac{\gamma \sigma_i}{m(\bar{p}, \bar{p} + \epsilon)} \left( \sigma_i \bar{x}_i + \frac{n(\bar{p}, \bar{p} + \epsilon)}{m(\bar{p}, \bar{p} + \epsilon)} - N n(\bar{p}, \bar{p} + \epsilon) \sum_{j=1}^{N} \sigma_j \bar{x}_j \right) . \]

(16)

2. Limited Participation: If \( \bar{p} \) does not satisfy Equation (15) and \( \epsilon \geq K(\bar{p}) - \bar{p} \), there exists a unique equilibrium in which

\[ p_i = \bar{a}_i - \frac{\gamma \sigma_i}{m(\bar{p}, K(\bar{p}))} \left( \sigma_i \bar{x}_i + \frac{n(\bar{p}, K(\bar{p}))}{m(\bar{p}, K(\bar{p}))} - N n(\bar{p}, K(\bar{p})) \sum_{j=1}^{N} \sigma_j \bar{x}_j \right) . \]

(17)

3. In both equilibriums, each risky asset is priced at discount in equilibrium. That is, \( p_i < \bar{a}_i \) and \( s_i > 0 \) for each \( i = 1, \ldots, N \).

Since the institutional investor knows the true correlation coefficient, her optimal demand is determined by \( \bar{p} \) in equilibrium. while for the retail investor, since he is ambiguity averse, he makes decisions under the worst-case scenario which leads to the least expected utility, thus the highest correlation coefficient is chosen under most circumstances. Following the terminology in Cao, et al. (2005), and Easley and O’Hara (2009), we name a full participation equilibrium in which the retail investor’s optimal demand corresponds to \( \bar{p} + \epsilon \). The intuition of the full equilibrium is straightforward. If investors are relatively homogeneous, either \( \nu \) or \( \epsilon \) is small, the market is close to the homogeneous equilibrium considered in Proposition 2. Then the highest possible correlation coefficient characterizes the worst-cast scenario for the ambiguity-averse retail investor.

However, as we have explained in the optimal portfolio choice problem, a portfolio inertia may occur when the uncertainty of the retail investor is high, thus, the correlation coefficient choice to his optimal demand may not necessarily be the corner solution. When portfolio inertia occurs, the retail investor will have a limited participation portfolio in our definition above, so we name it a limited participation equilibrium in Proposition 3 (2). Specifically, if there is a large degree of heterogeneity among investors’ correlation uncertainty such that \( \bar{p} \)
is reasonable small but $\epsilon$ is large, any choice of the correlation coefficient in the estimation range is feasible but irrelevant for the retail investor. Therefore, a limited participation equilibrium is generated in which the retail investor’s optimal demand is $x_{Ω(s)}$ by Proposition 1. Moreover, in the limited participation equilibrium, the dispersion of all endogenous Sharpe ratios is $Ω(s) = \frac{1-\frac{K(\rho)}{1+(N-1)K(\rho)}}{1+\frac{K(\rho)}{1+(N-1)K(\rho)}}$.

3 Equilibrium Analysis

In this section we conduct a detailed analysis examining how the correlation uncertainty affects the equilibrium with respect to different asset characteristics.

Let $\hat{\sigma}_i$ be the return volatility of asset $i$. Then the payoff volatility $\sigma_i = \hat{\sigma}_i p_i$ and thus $\sigma_i x_i = \hat{\sigma}_i (p_i x_i)$. Since $p_i x_i$ is the market capitalization of asset $i$, $w_i = \frac{p_i x_i}{\sum_{i=1}^{N} p_i x_i}$ represents the “size factor” of asset $i$. $\sigma_i x_i$ is proportional to $\hat{\sigma}_i w_i$, a product of the volatility and the size factor and we call it a risk-adjusted size factor or weighted volatility. In contrast to a simple risk factor, $\hat{\sigma}_i w_i$ is large only when both the size and the volatility are large or at least one factor is extremely large; and $\hat{\sigma}_i w_i$ is small if both factors are small or at least one is very small. Hence, the risk-adjusted size factor is able to capture the trade-off between risk and size.

Furthermore, we introduce $\eta_i \equiv \frac{\hat{\sigma}_i w_i}{\sum_{n=1}^{N} \hat{\sigma}_n w_n}$, a proxy to represent the individual asset’s risk contribution to the market. We demonstrate in Appendix A that the set of $\eta_i$, the correlation coefficient $corr(\tilde{R}_i, \tilde{R}_m)$, and the weighted beta $\hat{\beta}_i w_i$, are mutually determined by each other in an equicorrelation model. It is worth noting that the eta concept is relative. By a low eta we mean its weighted volatility is small when comparing with the weighted volatilities of all other risky assets in the market.

---

22 As documented in Moskowitz (2003), the firm size is a significant factor for predicting future covariation. Ang, Hodrick, Xing, and Zhang (2006) argue that a small firm asset tends to have high quality.

23 More precisely, $\eta_i \leq \alpha$, if and only if its weighted volatility, $\hat{\sigma}_i w_i \leq \frac{\alpha}{1-\alpha} \sum_{j \neq i} \hat{\sigma}_j w_j$. In particular, $\eta_i < \frac{1}{N}$ if and only if $\hat{\sigma}_i w_i \leq \frac{1}{1-\alpha} \sum_{j \neq i} \hat{\sigma}_j w_j$. 

---

16
3.1 Risk Premium and Asset Price

The effects of correlation uncertainty and the asset characteristics on the risk premium and the asset price are given by the next proposition.

**Proposition 4**

1. The Sharpe ratio \(s_i \geq s_j\), if and only if \(\eta_i \geq \eta_j\). Moreover, \(s_i\) is larger than the average Sharpe ratio, \(\frac{S}{N}\), if and only if \(\eta_i \geq \frac{1}{N}\).

2. For asset \(i\) with \(\eta_i < \frac{1}{N}\), the higher the correlation uncertainty, the larger its Sharpe ratio and the smaller its price; the effect of correlation uncertainty on the Sharpe ratio and the price is opposite if \(\eta_i\) is large.

Proposition 4 (1) displays the symmetric property between the Sharpe ratio and the eta of an individual asset. It states that a larger Sharpe ratio always relates to a higher eta among all risky assets. Thus, an asset with higher-eta is more attractive than the one with smaller-eta from an investment perspective. By the same reason, a risky asset’s Sharpe ratio is above the average Sharpe ratio only when its eta is above the average level \(\frac{1}{N}\). To study the joint effects of the correlation uncertainty and the asset characteristics, we divide assets into high-eta assets with relatively high \(\eta_i\), and low-eta assets with smaller \(\eta\) in subsequent discussions.

Proposition 4 (2) demonstrates the effect of the correlation uncertainty on the risk premium. We decompose the Sharpe ratio into two components:

\[
s_i = \frac{S}{N} + \frac{\gamma L}{m(\overline{p}, \overline{p} + \epsilon)} \left( \eta_i - \frac{1}{N} \right)
\]

where \(L = \sum_{i=1}^{N} \sigma_i x_i\) is the aggregate market volatility if all assets are perfectly correlated. In the right side of Equation (18), the first component is the average Sharpe ratio, and the second represents how much it differs from the average Sharpe ratio. We call the second component a specific Sharpe ratio. The specific Sharpe ratio of asset \(i\) is proportional to the difference between its eta and the average level, \(\eta_i - \frac{1}{N}\).

The correlation uncertainty affects the Sharpe ratios of high and low-eta assets very differently. For a low-eta asset (say, its eta is smaller than the average), the correlation
uncertainty contributes to the increase of the Sharpe ratio through two distinct channels: the increase on the average Sharpe ratio and the increase on the specific Sharpe ratio.

However, for an asset whose eta is larger than the average, the effect of correlation uncertainty is not straightforward because of the opposing effects of the average Sharpe ratio and the specific Sharpe ratio. As the uncertainty increases, the average Sharpe ratio always increases, but the specific Sharpe ratio decreases. When the eta is large enough under certain circumstances, the negative effect of the specific Sharpe ratio dominates the positive effect of the average Sharpe ratio, thus reaching an overall negative effect on the risk premium and the Sharpe ratio, and the asset price increases. Figure 2 (a1) \( (\nu = 0.3) \) demonstrates how the Sharpe ratios are affected by the correlation uncertainty for two low-eta assets with \( \eta_1 = 0.1, \eta_2 = 0.3 \) and one high-eta asset with \( \eta_3 = 0.6 \). Since \( \nu = 0.3, \bar{\rho} = 0.5 \), Equation (15) holds, and Figure 2 (a1) displays a full participation equilibrium. Given the same parameters except that \( \nu = 0.5 \), we obtain a limited participation equilibrium for \( \epsilon \geq K(\bar{\rho}) - \bar{\rho} = 0.3476 \). The sensitivity of the Sharpe ratio with respect to the correlation uncertainty for \( \nu = 0.5 \) is presented in Figure 2 (b1).

3.2 Correlation and Beta

We explain next how the asset characteristics and the correlation uncertainty jointly influence an asset’s correlation with the market and its beta.

**Proposition 5** Let \( \tilde{R}_i = (\tilde{a}_i - p_i)/p_i \) be the asset i’s return, and \( \tilde{R}_m \) be the market portfolio return, \( \tilde{R}_m = \sum_{i=1}^{N} w_i \tilde{R}_i \).

1. The correlation between asset return and the market portfolio is positively associated with the asset eta. Specifically,

\[
\text{corr}(\tilde{R}_i, \tilde{R}_m) \geq \text{corr}(\tilde{R}_j, \tilde{R}_m) \text{ if and only if } \eta_i \geq \eta_j, \forall i, j = 1, \ldots, N. \quad (19)
\]

Moreover, when the etas of risky assets display in a reasonable range such that \( \eta_1 \geq \cdots \geq \eta_N \) and \( \frac{2\eta_N}{1-\eta_N} \geq \frac{\eta}{1-\eta} \), \text{corr}(\tilde{R}_i, \tilde{R}_m) \) is increasing with respect to the correlation uncertainty.

2. The weighted beta is positively associated with the asset eta. For an asset with a very large or small eta, its beta is positively associated with the correlation uncertainty.
But for an asset with an intermediate level of eta, its beta is negatively related to the correlation uncertainty.

From Proposition 4, we demonstrate that the higher the eta, the higher the Sharpe ratio and the better the investment opportunities. Given the fact that a better investment opportunity is associated with a higher correlation with the market and a higher beta, we can obtain that a higher quality asset has a higher correlation with the market and vice versa.

Similar to the Sharpe ratio, an asset’s correlation with the market portfolio varies with respect to the correlation uncertainty. For example, let us assume all etas are within a reasonable range, that is, all risky assets offer fairly similar investment opportunities. Since the pairwise correlation coefficient increases with the correlation uncertainty (according to Proposition 3), the correlation with the market for each asset must be positively related to the correlation uncertainty. In general, if there are significant differences among asset characteristics thus $\Omega(\eta)$ is high, the effect of the uncertainty on the correlation $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ are also determined by the asset characteristics in a more complicated manner. To illustrate it graphically, we draw the correlation with the market portfolio in Figure 2 (a3) ($\nu = 0.3$) and Figure 2 (b3) ($\nu = 0.5$). The correlation with the market return for (low-eta) asset 1 and asset 2 increase with respect to the correlation uncertainty, $\epsilon$. But a high correlation uncertainty reduces the correlation with the market portfolio for (high-eta) asset 3.

In a similar fashion, we also study the joint effects of correlation uncertainty and asset characteristics on the weighted beta. Proposition 5 (2) follows from the formula of asset’s beta (See Appendix A, Proposition 11):

$$\beta_i = \frac{\eta_i}{w_i} \cdot \frac{\rho + \eta_i(1 - \rho)}{\rho + (1 - \rho)\sum_{i=1}^{n} \eta_i^2}. \tag{20}$$

An asset’s beta is determined not only by the correlation uncertainty but also the asset characteristics, as stated in Proposition 5 (2). We can observe this feature in Figure 2 (a4) and Figure 2 (b4), in which the weighted beta for asset 3 (asset 1 and asset 2) is decreasing (increasing) with respect to the correlation uncertainty.
### 3.3 Effects of Institutional Investor and Risk Distribution

In addition to the degree of correlation uncertainty, other parameters including the proportion of the institutional investors $\nu$ and the risk distribution $\Omega(\eta)$ play important roles in the equilibrium analysis. In this subsection we use $K(\rho, \nu, \Omega(\eta))$ in equation (14) to highlight the impact of $\nu$ and $\Omega(\eta)$.

First of all, the number of institutional investors in the market is a critical factor determining which kind of equilibrium it will be. Note that $K(\rho, \nu, \Omega(\eta))$ is a decreasing function of $\nu$. When $\epsilon$ is small such that $\bar{\rho} + \epsilon$ is smaller than $\lim_{\nu \to 1} K(\bar{\rho}, \nu, \Omega(\eta))$, a full participation equilibrium occurs regardless of the percentage $\nu$. Let us assume the retail investor has a reasonable large uncertainty about the market such that $\rho + \epsilon = \lim_{\nu \to 1} K(\bar{\rho}, \nu, \Omega(\eta))$ and $\nu^*$ satisfies $\bar{\rho} + \epsilon = K(\bar{\rho}, \nu^*, \Omega(\eta))$. For a small number of institutional investors, $\nu \leq \nu^*$, $\bar{\rho} + \epsilon$ is within $K(\bar{\rho}, \nu, \Omega(\eta))$, a full participation equilibrium is generated. But for more institutional investors with $\nu > \nu^*$, a limited participation equilibrium prevails. Since more institutional investor reduces the endogenous correlation coefficient and thus excess covariance in equilibrium, each investor’s maxmin expected utility increases and the entire market becomes more stable.\[24\]

Second, the proportion of institutional investors has significant effects on individual assets. Consider a low-eta asset with $\eta_i < \frac{1}{N}$, it is easy to see

$$\frac{\partial}{\partial \nu} (s_i) = \frac{\partial}{\partial \nu} \left( \frac{S}{N} \right) + \frac{\partial}{\partial \nu} \left( \frac{\gamma L}{m (\bar{\rho} + \rho + \epsilon)} \right) \left( \eta_i - \frac{1}{N} \right) < 0. \tag{21}$$

Therefore, the risk premium of low-eta assets drops with an increase of institutional investors. Similarly, the risk premium of high-eta assets increases with $\nu$. In this regard, Equation (21) is closely related to the findings in Gompers and Metrick (2001), which empirically document that the small-company stock premium drops due to increasing demand from institutional investors. By the same reasoning, the institutional investors’ demand for a high-eta firm would increase the premium. Equation (21) asserts that the low-eta firm premium decreases and the high-eta firm premium increases with the number of the institutional investors present.

\[24\] By enhancing investor’s education and information transparency, the uncertainty on assets’ correlated structure can be reduced. Easley and O’Hara (2009), Easley, O’Hara and Yang (2015) present similar insights from different perspectives.
Lastly, the eta distribution $\Omega(\eta)$ also has great impact on the heterogeneous equilibrium. If each asset has similar weighted volatility risk (eta) in the market, a small $\Omega(\eta)$ yielding equation (15), then a full participation equilibrium emerges. On the other hand, when assets offer a skewed eta distribution such that $\Omega(\eta)$ is close to one, a limited participation equilibrium is obtained according to Proposition 3. The intuition is straightforward. When $\Omega(\eta)$ approaches to 1, $K(\overline{\rho}, \nu, \Omega(\eta))$ is arbitrarily close to $\rho$. Therefore, $K(\overline{\rho}, \nu, \Omega(\eta))$ is smaller than $\overline{\rho} + \epsilon$ for any ambiguity-averse retail investor, thus the retail investor must hold a limited portfolio for an sufficiently skewed eta distribution. Our result demonstrates that a large risk dispersion among asset characteristics could be another channel for limited participation phenomena.

To summarize, Table 1 reports the conditions on model parameters $\{\overline{\rho}, \epsilon, \nu, \Omega(\eta)\}$ under which either a full equilibrium or a limited equilibrium is endogenously generated. There are three panels in Table 1. In Panel A, we characterize the equilibrium with varying levels of correlation uncertainties, while $\nu$ and $\Omega(\eta)$ are fixed. In our model, a limited participation occurs if and only if $\overline{\rho}$ is small and $\overline{\rho} + \epsilon \geq K(\overline{\rho})$. Otherwise, a full participation equilibrium prevails. Panel B presents the impact of the institutional investors’ proportion, $\nu$, when other parameters, $\overline{\rho}, \epsilon$ and $\Omega(\eta)$ are fixed. Clearly, if there are many retail investors such that $\frac{\nu}{1-\nu} \leq \frac{1+(N-1)\overline{\rho} (1-\Omega(\eta))}{N \Omega(\eta)}$, in particular for $\nu = 0$ with only retail investors, there is a full participation equilibrium. If there are many institutional investors and the correlation uncertainty is small, the retail investors behave similarly as the institutional investors, yielding a full participation equilibrium. When the correlation uncertainty $\epsilon$ is very high, whether it will be a full equilibrium or limited equilibrium really depends on whether there are a large number of institutional investors. Similarly, we characterize the equilibrium under conditions of the risk distribution $\Omega(\eta)$ in Panel C, when other parameters $\overline{\rho}, \epsilon$ and $\nu$ are fixed. $\Omega^*$ satisfies the equation $K(\overline{\rho}, \nu, \Omega^*) = \overline{\rho} + \epsilon$. If $\Omega(\eta) \geq \Omega^*$, we obtain a limited participation; otherwise, a full participation equilibrium occurs.

4 Implications

We present several asset pricing implications and testable properties on the optimal portfolios in this section. The heterogeneity of correlation uncertainty is shown to be one fundamental channel to explain the following stylized facts in the financial market.
4.1 Limited Participation and Under-diversification

We start with the limited participation and under-diversification puzzle by comparing each investor’s optimal portfolio with the market portfolio.

Proposition 6

1. (Under-diversification and well-diversification) Compared with the market portfolio \( \sum_{i=1}^{N} \bar{x}_i \tilde{a}_i \), the retail investor has an under-diversified portfolio while the institutional investor has a well-diversified portfolio. As a consequence, the institutional investor always holds a better diversified portfolio than the retail investor.

2. (Comparative Analysis) An increasing perceived level of correlation uncertainty induces a less diversified optimal portfolio for retail investor but a more diversified optimal portfolio of the institutional investor in a full participation equilibrium.

3. (Portfolio Risk) The institutional investor holds a riskier portfolio than the retail investor.

4. (Portfolio Performance) The institutional investor has a better portfolio performance than the retail investor.

In Proposition 6, we make use of the dispersion measure \( \Omega(\cdot) \) to measure the diversification extent of each investor’s optimal portfolio. The market portfolio \( \sum_{i=1}^{N} \bar{x}_i \tilde{a}_i \) serves as the benchmark to compare with each investor’s optimal portfolio.

Let \( x^{(s)} \) and \( x^{(r)} \) denote the optimal demand vector for institutional investor and retail investor respectively. Since \( \Omega(\sigma x^{(s)}) \) is smaller than \( \Omega(\sigma \bar{x}) \), the institutional investor has a better diversified portfolio than the market portfolio. By contrast, the retail investor’s optimal portfolio is less diversified because \( \Omega(\sigma x^{(r)}) \) is always larger than \( \Omega(\sigma \bar{x}) \). As a consequence, the institutional investor holds a well-diversified optimal portfolio and the retail investor’s optimal portfolio is under-diversified. In our model, the nature of the under-diversification of the retail investor’s optimal portfolio, and simultaneously, the well-diversified optimal portfolio of the institutional investor, is essentially driven from the heterogeneity of correlation uncertainty.

If the retail investor’s correlation uncertainty is large enough, the model shows that the retail investor holds a limited participation portfolio. In general, the retail investor’s optimal
portfolio satisfies $\Omega \left( \sigma x^{(r)} \right) = 1$, and it can be reduced to $x_1 x_2 = 0$ for $N = 2$. By quantifying the difference between the investor’s optimal portfolio and the market portfolio, Proposition 6 (1) helps in understanding the under-diversification and limited participation puzzle.

Our approach is novel compared with the extant theoretical studies that posit under-diversification from perspectives such as model misspecification (Easley and O’Hara, 2009; Uppal and Wang, 2003), heterogeneous beliefs (Mitton and Vorkink, 2008), and costly information (Van Nieuwerburgh and Veldkamp, 2010). We demonstrate that under-diversification can be generated endogenously from the dispersion of correlation uncertainty. Furthermore, a better diversified portfolio is associated with a better estimation of the correlated structure or a smaller degree of correlation uncertainty. Otherwise, under-diversified or even a limited participation portfolio emerges in equilibrium.

Proposition 6 (2) is related to several empirical studies of household portfolio choice. Calvet, Campbell, and Sodini (2009) presents evidence suggesting that the position of equity held in individual stocks on top of well-diversified portfolio (mutual fund or a market portfolio) is a reasonable proxy for portfolio under-diversification. By using the under-diversification measure proposed in Calvet et al. (2009), Dimmock, et al. (2016) further examines ambiguity-averse investors who view the overall market as a highly ambiguous under-diversified portfolio. They find that a one standard deviation increase in ambiguity aversion leads to a 38.9 percentage point increase in the proportion of equity allocated to individual stocks for those who view the overall market as highly uncertain. However, institutional investors with good knowledge about the market allocate little to individual stocks. In our approach, we also use the market portfolio as a benchmark and the overall market uncertainty is interpreted as the correlation uncertainty. Proposition 6 (2) states that the higher the correlation uncertainty, the less diversified the optimal portfolio of the retail investor. Moreover, when the retail investor is more ambiguity averse, the institutional investor is relatively more sophisticated about the estimation of the correlation structure, thus her optimal portfolio is better diversified. Our Proposition 6 (2) is consistent with these recent empirical findings.

Figure 3 (a1)-(a4) show the optimal demand for both the institutional investor and the retail investor, as well as the dispersion of their optimal portfolio when the correlation uncertainty changes in a full participation equilibrium. As drawn in Figure 3, the dispersion of the institutional investor’s optimal portfolio in graph (a2) is always smaller than the corre-
sponding dispersion of the retail investor in graph (a4), given the same degree of correlation uncertainty. Moreover, when $\epsilon$ goes up, we observe that the institutional investor’s portfolio dispersion decreases, and the retail investor’s portfolio dispersion increases. Figure 3 (b1)-(b4) explain our results robustly in a limited participation equilibrium.

Proposition 6 (3) states that the institutional investor is willing to choose a riskier portfolio than the retail investor due to the dispersion of correlation uncertainty among investors. The intuition is simple. Since ambiguity aversion leads to risk aversion, the retail investor behaves more risk-averse; hence, he takes smaller risk in his optimal portfolio. Proposition 6 (3) demonstrates that a higher correlation uncertainty yields a higher risk aversion, then the corresponding optimal portfolio is less risky. Specifically, the variance of $\sum_i \tilde{a}_i x_i^{(s)}$ is strictly larger than the variance of $\sum_i \tilde{a}_i x_i^{(r)}$. Finally, as shown in Proposition 6 (4), the institutional investor holds a better performed optimal portfolio in terms of Sharpe ratios. Moreover, the institutional investor has a higher maxmin expected utility than the retail investor.

To conclude, our comparative portfolio analysis demonstrates that a robust limited participation and under-diversification phenomenon resulting from the correlation uncertainty dispersion among investors.

4.2 Flight-to-Quality and Flight-to-Safety

After examining the optimal portfolio as a whole, we study the trading positions as well as the trading volume of each individual asset in the optimal portfolio. By investigating how the correlation uncertainty affects the trading position and trading volume, our analysis helps explain the flight-to-quality and flight-to-safety phenomenon.

To study precisely how the degree of correlation uncertainty affects risk-sharing among investors, we assume that each investor initially holds a market portfolio (without the correlation uncertainty).

Proposition 7 1. (Portfolio Position) For a low-eta asset $i$ with $\eta_i < \frac{1}{N}$, the holding of the institutional investor increases and the holding of the retail investor decreases as the correlation uncertainty increases. For the holding of a high-eta asset, the effect of correlation uncertainty on both types of investors is opposite.
2. (Trading Pattern) Put

\[
J(\epsilon, \nu) = \frac{1}{1 + (N - 1)\rho + (N - 1)\nu}\epsilon.
\]

The institutional investor always sells high-eta assets satisfying \(\eta > J(\epsilon, \nu)\) and purchases low-eta assets with \(\eta < J(\epsilon, \nu)\); The retail investor always purchases high-eta assets with \(\eta > J(\epsilon, \nu)\) and sells low-eta assets with \(\eta < J(\epsilon, \nu)\).

3. (Trading Volume) The higher the correlation uncertainty, the larger the trading volume for the institutional investor, \(\lvert x_i^{(r)} - \bar{x}_i \rvert\), and the retail investor, \(\lvert x_i^{(s)} - \bar{x}_i \rvert\), for both high-eta assets and low-eta assets.

As demonstrated in Proposition 7(1), the correlation uncertainty affects each investor’s position differently. When the retail investor’s perceived degree of uncertainty increases, the institutional investor holds larger positions on low-eta assets and smaller positions on high-eta assets. On the contrary, the retail investor holds smaller positions on low-eta assets, and purchases more shares of high-eta assets. The property of the portfolio position under correlation uncertainty is displayed in Figure 3 (a1) and (a3), Figure 3 (b1) and (b3).

We investigate next the trading pattern and trading volume by assuming that each investor initially holds the market portfolio. As shown in Proposition 7(2), the retail investor sells low-eta assets and purchases high-eta assets. Correspondingly, the institutional investor buys low-eta assets and sells high-eta assets. Moreover, Propositions 7(3) states that the trading volume for each investor increases on almost all assets regardless low-eta or high-eta. Put differently, the more uncertain the retail investor is on the correlated structure, the more trading or overreaction occurs in the market.

Propositions 7(1)-(3) together describe a flight-to-safety or flight-to-quality episode under correlation uncertainty. When investors have different beliefs and the correlation uncertainty is high, investors’ trading activities influence the asset prices significantly - a price decline with a fire sale of one asset class is associated with an increase in price and trading volume of another asset class during the same time period. Caballero and Krishnamurthy (2008), Guerrieri and Shimer (2014), and Vayanos (2004) characterize the flight-to-quality in contexts of model uncertainty, adverse selection, and liquidity risk, respectively. Our model
complements these previous studies to demonstrate that correlation uncertainty could be another factor to generate flight-to-quality endogenously.

Proposition 7 (3) has another interpretation when we view $\epsilon$ as one form of disagreement between the investors. Under this interpretation, Proposition 7 (3) shows that a larger trading volume is associated with a larger disagreement between the institutional investor and the retail investor. Our finding is consistent with the empirical evidence in Carlin, Longstaff, and Matoba (2014). In this paper, they argue that high volatility itself does not lead to higher trading volume, rather it is only when disagreement arises in the market that higher uncertainty is associated with more trading. Proposition 7 (3) presents a theoretical explanation of their findings through the correlation uncertainty mechanism. Because the correlation uncertainty is positively associated with the aggregate market volatility, the model is also consistent with the market microstructure literature on the positive relation between the aggregate market volatility and trading volumes.

4.3 Asset Comovement

Finally, we examine how the correlation uncertainty impacts the asset comovement through the Sharpe ratios from an investment perspective.

**Proposition 8**

1. The relative Sharpe ratio $\frac{s_i}{S}$ always decreases with respect to the correlation uncertainty and increases with more institutional investors when $\eta_i > \frac{1}{N}$; and displays an opposite monotonic feature when $\eta_i < \frac{1}{N}$;

2. $\Omega(s)$ depends negatively on the correlation uncertainty, but it increases with respect to the number of institutional investors.

Proposition 8 provides a comparative analysis on the relative Sharpe ratio $\frac{s_i}{S}$ and the dispersion of Sharpe ratios $\Omega(s)$, assuming the correlation uncertainty changes or the percentage of institutional investors varies. Proposition 8 (1) follows from the decomposition of

---

25 According to the construction of disagreement index in Carlin, Longstaff, and Matoba (2014), the disagreement largely depends on the dispersion of investors’ forecast, which is also often used to measure the ambiguous level.
the relative Sharpe ratio \( s_i^\frac{s}{S} \): \[
\frac{s_i}{S} - \frac{1}{N} = \frac{\Omega(s)}{\Omega(\eta)} \left( \eta_i - \frac{1}{N} \right). \tag{22}
\]

Equation (22) reveals how the relative Sharpe ratio departing from \( \frac{1}{N} \) is proportional to the distance between its \( \eta \) and \( \frac{1}{N} \). The sensitivity of the relative Sharpe ratio with respect to the degree of correlation uncertainty relies on how large its \( \eta \) is, that is, the sign of \( \eta_i - \frac{1}{N} \). This sensitivity is negative for a high-\( \eta \) asset and positive for assets with low-\( \eta \).

Proposition 8 (1) is important for examining the effect of correlation uncertainty on different assets. For a low-\( \eta \) asset, its relative Sharpe ratio increases as the perceived correlation uncertainty grows; however, for a high-\( \eta \) asset, the relative Sharpe ratio decreases. In other words, high correlation uncertainty makes a high-\( \eta \) asset less attractive, and at the same time, the low-\( \eta \) asset becomes relatively more attractive. In the end, all assets comove under high correlation uncertainty, as presented in Proposition 8 (2). As drawn in Figure 2 (a2) and Figure 2 (b2), the dispersion of all Sharpe ratios with respect to the degrees of uncertainty decreases, thus, all risky assets are forced to comove closer.

Equation (22) is also useful in understanding the effect of correlation uncertainty on the comovement (contagion) pattern in financial market. Fixing the risk of each asset, the higher \( \frac{\Omega(\eta)}{\Omega(s)} \), the smaller the dispersion of the Sharpe ratio; thus, the higher the likelihood the assets move together from an investment perspective. Conversely, fixing the Sharpe ratio of each asset, the higher \( \frac{\Omega(\eta)}{\Omega(s)} \), the larger the dispersion of the individual risks. Therefore, we argue

\[\text{By a multiplicative version of Equation (18), we obtain } \frac{s_i}{S} - \frac{1}{N} = \kappa \left( \eta_i - \frac{1}{N} \right), \text{ where } \kappa \text{ is one number that is independent of the asset characteristics. Let } y_i = \kappa \left( \eta_i - \frac{1}{N} \right). \text{ Then } \sum_{i=1}^{N} y_i = 0, \text{ and}
\]

\[
\Omega(s)^2 = \Omega \left( \frac{s}{S} \right)^2 = \frac{1}{N-1} \left( \frac{N \sum_{i=1}^{N} (y_i + \frac{1}{N})^2}{\left( \sum_{i=1}^{N} (y_i + \frac{1}{N}) \right)^2} - 1 \right)
\]

\[= \frac{1}{N-1} \sum_{i=1}^{N} y_i^2 = \kappa^2 \frac{N}{N-1} \left( \sum_{i=1}^{N} \eta_i^2 - \frac{1}{N} \right) = \kappa^2 \Omega(\eta)^2. \]

Then \( \kappa = \frac{\Omega(s)}{\Omega(\eta)} \).
that \( \frac{\Omega(\eta)}{\Omega(s)} \) can measure the contagion of the market.\(^{27}\) Remarkably, \( \frac{\Omega(\eta)}{\Omega(s)} \) depends only on each investor’s endogenous correlation, and is independent of each asset’s marginal distribution.

### 4.4 Empirical Implications

Our model offers several important empirical implications to the financial markets. As we have explained above, the model provides a new approach to explain under-diversification or limited participation, flight to quality, and asset coovement. It also generates some testable cross-sectional predictions on assets and portfolios with different characteristics.

Table 2 summarizes our model implications when the correlation uncertainty increases in three different categories. Panel A presents the effect of the correlation uncertainty at the market level. We consider three popular measures of asset comovement: the aggregative market volatility, the pairwise market correlation and the dispersion of Sharpe ratios. In Panel B, we present the cross-sectional effect on the individual asset given different asset characteristics, including the asset prices, the risk premiums, Sharpe ratios, the relative Sharpe ratios, the correlations with the market portfolio and the weighted betas. The high-eta and low-eta asset is largely opposite on each element. Finally in Panel C, we compare the institutional investor and retail investor, in terms of their optimal portfolio, holding position, and trading volume on individual assets. The under-diversification and limited participation puzzle as well as the flight to quality phenomenon are shown.

While a complete empirical test of our model is beyond the scope of this paper, we take the 2007-2009 financial crisis as one example to illustrate our model implications.\(^{28}\)

In the period of the 2007-2009 financial crisis, the investor was very uncertain about the entire financial market. Following Bloom (2009), and Baele et al. (2013), we use the Chicago Board Options Exchange Volatility Index (VIX) to measure ambiguity in the overall

\(^{27}\)It is easy to derive

\[
\frac{\Omega(\eta)}{\Omega(s)} = \frac{\nu}{1 - \rho_2} + \frac{1 - \nu}{1 - \rho_2},
\]

where \( \rho_2 = \rho + \epsilon \) or \( K(\rho) \) in a full participation equilibrium or a limited participation equilibrium, respectively. In particular, \( \frac{\Omega(\eta)}{\Omega(s)} = \frac{1 + (N-1)\rho}{1 - \rho} \) in a homogeneous equilibrium increases with the correlation coefficient \( \rho \). It displays similar property of the contagion measure in Forbes and Rigobon (2002).

\(^{28}\)Given the results in this paper, our model can also be used to discuss other recent flight-to-quality phenomenon such as Black Monday 1987, the Russian debt default and Long-term capital management (LTCM) in 1998, the September 11 in 2001, among many others.
financial market. Figure 7(a) displays the VIX as well as S&P 500 index from 2006 to 2016. As shown clearly, the VIX is extremely high for all of 2008, representing a very high degree of uncertainty in the market. We also observe that the VIX index and S&P index move in opposite directions consistently over the entire period from 2006 to 2016.

To conduct a general analysis in the heterogeneous environment, we consider two types of asset classes. We treat the entire stock market as one asset class and the fixed income market (in particular, the Treasury market) as another asset class. Although the volatility of the stock market is larger than the volatility of the fixed income market, the volume of the fixed income market is much larger. The stock market can be seen as a low-eta asset and the fixed income market as a high-eta asset. To illustrate, we follow McKinsey Global Institute research (www.mckinsey.com/mgi) and report, in Figure 7(b), the global stock market and the fixed-income market (including public debt, financial bonds, corporate bonds, securitized loan, and unsecuritized loans outstanding) between 2005 to 2014. The total volume of the fixed-income market is about four times larger than that of the stock market. Given that the volatility of the stock market is around three times that of the fixed-income market according to historical data (Reilly, Wright, and Chan, 2000), the weighted volatility of the stock market is about 75 percent of the weighted volatility of the fixed income market. Therefore, the stock market can be viewed as a low-eta asset class while the fixed income market, as another asset class, can be viewed as a high-eta asset class.

The asset price movement during the period of the 2007-2009 financial crisis is consistent with Proposition 4. During the financial crisis time period model, the prices of low-eta assets (such as the stock market) drop significantly; at the same time, the prices of high-eta assets (such as Treasury bonds) increase.

The flight-to-quality and flight-to-safety episodes during the period of the 2007-2009 financial crisis can be also explained in our model. Proposition 7 states that the institutional investor holds more on the stock market since she has perfect knowledge of the overall market, and the price decline of the stock market virtually follows from the retail investor’s overselling on the stock market. Moreover, the more uncertain the retail investor is about the entire market, the fewer positions he holds on the stock market; in turn, the institutional investor holds more on the stock market.

---

29During financial crisis periods, the stock market typically displays a significant decline and the Treasury yield rallies in a short-term period. Baele et al. (2013) empirically characterize flight-to-quality episode using equity index and the Treasury bond. We follow Batsky (1989), and Baele et al. (2013) to compare the equity market and the Treasury market in our explanations.
investor holds more equity positions. Similarly, for high-eta assets (for instance, government bonds), which are used to hedge against “economic catastrophe risk” (a safe haven), the retail investor holds more and more positions.

During the 2007-2009 financial crisis time period, when the retail investor had a very high perceived degree of ambiguity aversion for the entire financial market, dramatic trading activities and extreme price declines took place on the stock market and a substantial price increase pattern emerged for government bonds, especially Treasury bonds, in a short time period. Moreover, Proposition 7 (3) explains the huge volume of trading during this time period due to high correlation uncertainty.

5 Extension

In this section we extend the model in which the institutional investor also has some ambiguity on the correlation estimation. Everything else is the same as in Section 2, we assume that for each institutional investor, the plausible correlation coefficient range is \([\rho - \epsilon_1, \rho + \epsilon_1]\). The set of correlation coefficients for the retail investor is \([\rho - \epsilon_2, \rho + \epsilon_2]\), and we assume \(0 \leq \epsilon_1 < \epsilon_2\) to reflect the fact that the estimation on the correlated structure of the institutional investor is more accurate than the retail investor.

We first characterize the equilibrium.

Proposition 9

1. Full Participation: If

\[
\bar{p} + \epsilon_1 \geq \frac{1}{N - 1} \left\{ \nu \frac{\Omega(\sigma x)}{1 - \nu} (\sigma x) \right\} N - 1, \tag{23}
\]

or if \(\epsilon_2 < K(p + \epsilon_1) - p\), there exists a unique equilibrium in which

\[
p_i = \frac{\gamma \sigma_i}{m (\bar{p} + \epsilon_1, \bar{p} + \epsilon_2)} \left( \sigma_i x_i + \frac{n (\bar{p} + \epsilon_1, \bar{p} + \epsilon_2)}{m (\bar{p} + \epsilon_1, \bar{p} + \epsilon_2) - N n (\bar{p} + \epsilon_1, \bar{p} + \epsilon_2) \sum_{j=1}^{N} \sigma_j x_j} \right). \tag{24}
\]

\[30\]Our results hold in a more general setting in which \([\rho_a, \rho_b] \subseteq [\rho_a', \rho_b']\), \([\rho_a', \rho_b']\) is a set of plausible correlation coefficient for the institutional investor, and \([\rho_a', \rho_b']\) is a set of plausible correlation coefficient for the retail investor. In this setting, the institutional investor can have different benchmark coefficient from the retail investor. \(\rho_b - \rho_a\) measures the correlation uncertainty for each agent. Alternatively, by adopting Cao et al. (2005), we can also assume that there is a continuum of investors, say, \([\rho - \epsilon, \bar{p} + \epsilon]\), each type of investor’s correlation uncertainty is captured by the parameter \(\epsilon\) while \(\epsilon\) is uniformly distributed among investors on \([\epsilon - \delta, \epsilon + \delta]\) with a density of \(1/(2\delta)\). Our main results are fairly robust in these alternative settings.
2. Limited Participation: If Equation (23) does not hold and \( \epsilon_2 \geq K(\bar{\rho} + \epsilon_1) - \bar{\rho} \), there exists a unique equilibrium in which

\[
p_i = \bar{a}_i - \frac{\gamma \sigma_i}{m(\bar{\rho} + \epsilon_1, K(\bar{\rho} + \epsilon_1))} \left( \sigma_i \bar{x}_i + \frac{n(\bar{\rho} + \epsilon_1, K(\bar{\rho} + \epsilon_1))}{m(\bar{\rho} + \epsilon_1, K(\bar{\rho})) - Nn(\bar{\rho} + \epsilon_1, K(\bar{\rho} + \epsilon_1))} \sum_{j=1}^{N} \sigma_j \bar{x}_j \right).
\]

(25)

3. In both equilibriums, each risky asset is priced at discount in equilibrium. That is, \( p_i < \bar{a}_i \) and \( s_i > 0 \) for each \( i = 1, \cdots, N \).

If the dispersion of correlation uncertainty among investors is small such that \( \epsilon_1 < \epsilon_2 < K(\bar{\rho} + \epsilon_1) - \bar{\rho} \), extending the homogeneous environment (for \( \epsilon_2 = \epsilon_1 \)) in Proposition 2, a full participation equilibrium is obtained. There are two special cases in which Equation (23) holds. First, \( \bar{\rho} \geq \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma \bar{x})}{1-\Omega(\sigma \bar{x})} N - 1 \right\} \), which often applies to a certain asset class in which each pair of assets displays a high correlation by nature, eg. stocks in one sector, or bonds with different maturities. Second, the benchmark correlation coefficient \( \bar{\rho} \) is small, but each investor has high correlation uncertainty. This condition is often true in a stressed economy. As long as Equation (23) is satisfied, each investor chooses the most correlated structure among the set of plausible correlated structure in a full participation equilibrium regardless of the uncertainty dispersion between investors.

In contrast to the full participation equilibrium, a limited participation equilibrium is generated when there is a large amount of heterogeneity of correlation estimation among investors. Specifically, if \( \epsilon_1 \) is smaller than \( \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma \bar{x})}{1-\Omega(\sigma \bar{x})} N - 1 \right\} - \bar{\rho} \), and \( \epsilon_2 \) is larger than \( K(\bar{\rho} + \epsilon_1) - \bar{\rho} \), the retail investor holds a limited participation portfolio. As discussed in Section 3.3, when the retail investor is highly ambiguity averse to the correlation uncertainty, he holds a limited participation portfolio in some market situations. For instance, assuming many institutional investors and a high risk dispersion such that a total sum of \( \nu \) and \( \Omega(\sigma \bar{x}) \) is greater than 1 (Equation (23) does not satisfied), if the retail investor is very uncertain about the correlated structure, any choice of the correlation coefficient is feasible and irrelevant for the retail investor in the limited participation equilibrium.

Interestingly, regardless of the retail investor’s decision in the equilibrium, the worst-case scenario for the institutional investor is always associated with \( \bar{\rho} + \epsilon_1 \). The intuition is as
follows. The institutional investor is ambiguity averse and she is the most sophisticated 
agent in the market, her hedging strategy to the worst-case scenario can be implemented 
in equilibrium. By the same reason, the retail investor also wants to hedge the worst-case 
scenario in equilibrium, then the correlation coefficient $\rho + \epsilon_2$ is the one that leads to the 
least expected utility in most situations. Only if the dispersion of correlation uncertainty is 
very large, a portfolio inertia occurs and the retail investor chooses a limited participation 
portfolio. In either full participation or limited participation equilibrium, each risky asset is 
price at discount because of investor’s risk aversion and ambiguity aversion.

If the retail investor chooses a limited participation portfolio in equilibrium, his correla-
tion uncertainty does not affect the endogenous asset prices at all due to the portfolio inertia 
feature in the correlated structure. Therefore, the equilibrium asset prices and the Sharpe 
ratios depend only on the institutional investor’s correlation uncertainty $\epsilon_1$. By the solution 
of the portfolio choice problem in Proposition 1 the optimal portfolio for the retail investor 
is $x^{\hat{\Omega}(s)}$. Precisely,

$$
x_i^{(r)} = \frac{1}{\gamma \sigma_i} \frac{1 + (N - 1)\Omega(s)}{N\Omega(s)} \left( s_i - \frac{1 - \Omega(s)}{N} S \right), \ i = 1, \cdots, N,
$$

(26)

where $\Omega(s) = \frac{1 - K(\rho + \epsilon_1)}{1 + (N - 1)K(\rho + \epsilon_1)}$ is the dispersion of all Sharpe ratios in equilibrium.

By the above equilibrium analysis, the worst-case scenario of the institutional investor 
is $\rho + \epsilon_1$, which is strictly smaller than $\rho + \epsilon_2$ in a full participation equilibrium, or $\hat{\Omega}(s)$ 
in a limited participation equilibrium and $\hat{\Omega}(s) = K(\rho + \epsilon_1) > \rho + \epsilon_1$. Therefore, the asset 
pricing implications of our results can be extended to the general setting, given in the next 
proposition.

**Proposition 10** 1. (Under-diversification and well-diversification) **Compared with the** 
market portfolio $\sum_{i=1}^{N} \bar{x}_i \bar{a}_i$, the retail investor has an under-diversified portfolio while 
the institutional investor has a well-diversified portfolio.

2. (Flight to quality) **For a low-eta asset, the holding of the institutional investor increases** 
and the holding of the retail investor decreases as the correlation uncertainty increases. 
**For the holding of a high-eta asset, the effect of correlation uncertainty on both types** 
of investors is opposite.
3. (Asset comovement) The relative Sharpe ratio $\hat{\gamma}_i$ always decreases with respect to the correlation uncertainty for high-eta asset, and increase for low-eta assets. Moreover, $\Omega(s)$ always depends negatively on the correlation uncertainty.

To explain the main insights in Proposition 10, we draw several three-dimensional graphs. Figure 4 (a) - (b) depict the dispersions of the institutional and retail investor’s optimal portfolio with respect to the correlation uncertainty in our extended model. We see that the institutional investor’s optimal portfolio is always smaller than $\Omega(\sigma \bar{x})$, the dispersion of the marker portfolio. On the other hand, since the dispersion of the retail investor’s optimal portfolio is larger than $\Omega(\sigma \bar{x})$, the retail investor has an under-diversified portfolio. Similarly, we can derive the limited participation portfolio for the retail investor when the dispersion of the correlation uncertainty is high. Therefore, under-diversification and limited participation can be explained quantitatively by using the dispersion measure to each investor’s optimal portfolio.

Proposition 10 (2) is explained in Figure 5. As seen in the upper panel of Figure 5, the retail investor’s demand on the low-eta asset is decreasing with respect to $\epsilon_1$ and $\epsilon_2$. Simultaneously, the retail investor has more and more demand on the high-eta asset when the degree of the correlation uncertainty goes up. The demand from the institution investor is just opposite on both low-eta and high-eta asset. We can further observe high trading volumes, $|x_i^{(s)} - \bar{x}_i|, |x_i^{(r)} - \bar{x}_i|$, if $\epsilon_1$ and $\epsilon_2$ increases.

Finally, Figure 6 explains the asset comovement phenomena in the extended model. Consistent with Proposition 10 (3), Figure 6 (a)- (b) show that high-eta asset’s relative Sharpe ratio decreases and low-eta asset’s relative Sharpe ratio increases when the degree of correlation uncertainty increases. Moreover, Figure 6 (c) reports that the dispersion of all Sharpe ratios increases, thus all assets comove closer with increasing of correlation uncertainty.

6 Conclusion

This paper develops an equilibrium model in the presence of correlation uncertainty to investigate the complicated correlation structure among asset classes and the well-documented stylized facts on the correlated structure. We find that those correlation-related phenomena...
can be inherently connected through the disagreement among investors (institutional and retail investors) on the correlation structure and the asset characteristics, when the marginal distribution of each risky asset is perfectly known. Our model demonstrates that correlation uncertainty is an essential factor in studying asset prices, volatilities, and correlations, which cannot be fully explained by fundamentals.

Specifically, the institutional investor always holds a diversified portfolio versus the retail investor, who is under-diversified. The optimal portfolio becomes less diversified when the perceived level of the correlation uncertainty increases. When the correlation uncertainty is high, retail (institutional) investor demands more (less) position on high-eta assets and sells (purchases) low-eta assets. As a consequence, low-eta assets’ prices decline and high-eta assets’ prices increase because of retail investors’ overreact activity and high trading volumes. Moreover, all assets comove together if the correlation uncertainty is high. This equilibrium model is helpful for explaining several empirical puzzles heretofore presented concerning correlation, including under-diversification, flight-to-quality, and asset comovement.
Appendix A: Asset Characteristics

In this appendix we explain the economic insights of the asset-characteristic parameter $\eta_i$ in an equicorrelation model. We show that the set of $\{\eta_i\}$, $\{\text{corr}(\tilde{R}_i, \tilde{R}_m)\}$, and the weighted beta $\{\beta_i w_i\}$, are determined by each other. Our result does not depend on any distribution assumption of the asset return $\tilde{R}_i$.

Proposition 11 In an equicorrelation model with $\text{corr}(\tilde{R}_i, \tilde{R}_j) = \rho, \forall i \neq j$, the market portfolio return $\tilde{R}_m = \sum_{i=1}^{N} w_i \tilde{R}_i$.

1. Given $\eta_i, i = 1, \ldots, N$, the correlation coefficient between individual asset return with the market portfolio return, $\text{corr}(\tilde{R}_i, \tilde{R}_m)$, is given by the following equation

$$\text{corr}(\tilde{R}_i, \tilde{R}_m) = \frac{\rho + \eta_i (1 - \rho)}{\sqrt{\rho + \sum_{i=1}^{N} \eta_i^2 (1 - \rho)}}. \quad (A-1)$$

Conversely, let $\alpha_i = \text{corr}(\tilde{R}_i, \tilde{R}_m), i = 1, \ldots, N$ with $\sum_{i=1}^{N} \alpha_i \neq 0$, then

$$\eta_i = \frac{\alpha_i}{\sum_{i=1}^{N} \alpha_i} \cdot \frac{1 + (N-1)\rho}{1 - \rho} - \frac{\rho}{1 - \rho}. \quad (A-2)$$

2. The weighted beta for asset $i$ is

$$\beta_i w_i = \frac{\eta_i (\rho + \eta_i (1 - \rho))}{\rho + \sum_{i=1}^{N} \eta_i^2 (1 - \rho)}. \quad (A-3)$$

Conversely, given a set of weighted beta, $\{\beta_i w_i, i = 1, \ldots, N\}$, we obtain

$$\eta_i = -\rho + \sqrt{\rho^2 + 4(1 - \rho) V \beta_i w_i} \quad \frac{2(1 - \rho)}{2(1 - \rho)}, i = 1, \ldots, N \quad (A-4)$$

where $V$ is solved by the following equation

$$\sum_{i=1}^{N} \sqrt{\rho^2 + 4V (1 - \rho) \beta_i w_i} = 2(1 - \rho) + N\rho. \quad (A-5)$$
Proposition 11 (1) determines explicitly the correlation coefficient $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ in terms of its eta and the dispersion of etas, and vice versa. Proposition 11 (2), demonstrates that the weighted beta $\beta_i w_i$ is determined by the eta, and vice versa.

**Proof:** (1) Since $\text{corr}(\tilde{R}_i, \tilde{R}_j) = \rho, \forall i \neq j$, we have

$$\text{Cov} \left( \tilde{R}_i, \tilde{R}_m \right) = \text{Cov} \left( \tilde{R}_i, \sum_j w_j \tilde{R}_j \right) = \left( \sum_j w_j \tilde{\sigma}_j \right) \tilde{\sigma}_i \rho + \tilde{\sigma}_i^2 (1 - \rho). \quad (A-6)$$

It follows that

$$\text{corr}(\tilde{R}_i, \tilde{R}_m) = \frac{\left( \sum_j w_j \tilde{\sigma}_j \right) \rho + w_i \tilde{\sigma}_i (1 - \rho)}{\tilde{\sigma}_m}, \quad (A-7)$$

where $\tilde{\sigma}_m$ is the volatility of the market portfolio. Equation (A-6) yields

$$\tilde{\sigma}_m^2 = \left( \sum_j w_j \tilde{\sigma}_j \right)^2 \rho + \sum_j w_j^2 \tilde{\sigma}_j^2 (1 - \rho). \quad (A-8)$$

By using the dispersion $\Omega(\eta) = \Omega(w\tilde{\sigma})$, we obtain

$$\tilde{\sigma}_m = \left( \sum_{j=1}^N w_j \tilde{\sigma}_j \right) \sqrt{\rho + \frac{(N - 1)\Omega(\eta)^2 + 1}{N} (1 - \rho)}. \quad (A-9)$$

By plugging Equation (A-9) into Equation (A-7), we derive Equation (A-1) as desired.

Conversely, let $x$ denote the denominator in Equation (A-7), then by Equation (A-1) again with $\alpha_i = \text{corr}(\tilde{R}_i, \tilde{R}_m)$, we obtain

$$\rho + \eta_i (1 - \rho) = \alpha_i x. \quad (A-10)$$

By using $\sum_{i=1}^N \eta_i = 1$, it follows that

$$\sum_{i=1}^N \alpha_i x = N\rho + 1 - \rho, \quad (A-11)$$

yielding

$$x = \frac{1 + (N - 1)\rho}{\sum_{i=1}^N \alpha_i}. \quad (A-12)$$

36
Equation (A-2) follows from Equation (A-10).

(2) By definition, the asset beta is given by

$$\beta_i = \frac{\text{Cov}(\tilde{R}_i, \tilde{R}_m)}{\text{Var}(\tilde{R}_m)} = \frac{\text{corr}(\tilde{R}_i, \tilde{R}_m)\tilde{\sigma}_i}{\tilde{\sigma}_m},$$

then

$$\beta_i w_i = \text{corr}(\tilde{R}_i, \tilde{R}_m) \frac{\tilde{\sigma}_i w_i}{\tilde{\sigma}_m}.$$ 

By plugging equation (A-8) into the last equation and using equation (A-1), we obtain equation (A-3). Conversely, given a set of weighted beta, and employing equation (A-1), we have

$$\eta_i = \frac{-\rho + \sqrt{\rho^2 + 4V(1 - \rho)\beta_i w_i}}{2(1 - \rho)}, \quad i = 1, \cdots, N \quad (A-13)$$

where $V = \rho + \sum_{i=1}^{N} \eta_i^2(1 - \rho)$. It suffices to derive the constant $V$ by using the weighted betas. In fact, by squaring both sides of equation (A-4), we have

$$\eta_i^2 = \frac{2\rho^2 + 4(1 - \rho)V\beta_i w_i - 2\rho \sqrt{\rho^2 + 4(1 - \rho)V\beta_i w_i}}{4(1 - \rho)^2},$$

then

$$\sum_{i=1}^{N} \eta_i^2 = \frac{2N\rho^2 + 4(1 - \rho)V - 2\rho \sum_{i=1}^{N} \sqrt{\rho^2 + 4(1 - \rho)V}\beta_i w_i}{4(1 - \rho)^2}. $$

Since $V = \rho + \sum_{i=1}^{N} \eta_i^2(1 - \rho)$, replacing $\sum_{i=1}^{N} \eta_i^2$ by $\frac{V - \rho}{1 - \rho}$ in the last equation, we derive the equation of $V$ as desired. \[\square\]
Appendix B: Proofs of Propositions 1 - 10

To simplicity the notations, let \( \tau \) be a linear fractional transformation: \( \tau(t) \equiv \frac{1-t}{1+(N-1)t} \) for any real number \( t \neq -\frac{1}{N-1} \). We start with the Sherman-Morrison formula in linear algebra.

**Lemma 1** Suppose \( A \) is an invertible \( N \times N \) matrix, and \( u, v \) are \( N \times 1 \) vectors. Suppose further that \( 1 + v^T A^{-1} u \neq 0 \). Then the matrix \( A + uv^T \) is invertible and

\[
(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}.
\]

(B-1)

The next lemma computes the dispersion of optimal demand in terms of the dispersion of the Sharpe ratios.

**Lemma 2** Let \( \rho \neq 1, \rho \neq -\frac{1}{N-1}, x_\rho \) is defined in Proposition 1. Then

\[
\Omega(\sigma x_\rho) = \Omega(s) \frac{1 + (N - 1)\rho}{1 - \rho}.
\]

(B-2)

**Proof.** Recall that \( \sigma x_\rho = \frac{1}{\gamma} R(\rho)^{-1} s \). By Lemma 1,

\[
R(\rho)^{-1} = \frac{1}{1 - \rho} I_N - \frac{1}{1 + (N - 1)\rho} \frac{\rho}{1 - \rho} ee^T,
\]

where \( e = (1, 1, \ldots, 1) \). Then \( \Omega(\sigma x_\rho) = \Omega(t) \), where \( t_i = s_i - \frac{\rho}{1 + (N - 1)\rho} S \). Then \( \sum_{i=1}^N t_i = \frac{1-\rho}{1+(N-1)\rho} S \), and

\[
\sum_{i=1}^N t_i^2 = \sum_{i=1}^N \left( s_i^2 - 2s_i S \frac{\rho}{1 + (N - 1)S} + \left( \frac{\rho}{1 + (N - 1)S} \right)^2 S^2 \right)
\]

\[
= \sum_{i=1}^N s_i^2 - \frac{(N - 2)\rho^2 + 2\rho}{(1 + (N - 1)\rho)^2} S^2.
\]

By straightforward computation, we obtain

\[
\Omega(t) = \Omega(s) \frac{1 + (N - 1)\rho}{1 - \rho}.
\]

(B-4)
The next two lemmas follow from straightforward calculations and their proofs are simply omitted.

**Lemma 3** Let

\[
G(\rho) = \frac{S^2}{N} \left( \frac{(N-1)\Omega(s)^2}{1 - \rho} + \frac{1}{1 + (N-1)\rho} \right).
\]

Then argmin\(\rho \in [\rho_a, \rho_b] G(\rho)\) is given by, when \(\Omega(s) \neq \frac{1}{N-1}\),

\[
\rho^* = \begin{cases} 
\rho_a, & \text{if } \rho_a > \tau(\Omega(s)), \\
\rho_b, & \text{if } \rho_b < \tau(\Omega(s)), \\
\tau(\Omega(s)), & \text{if } \tau(\Omega(s)) \in [\rho_a, \rho_b].
\end{cases}
\] (B-5)

If \(\Omega(s) = \frac{1}{N-1}\), then \(\rho^*\) is given similarly in which \(\tau(\Omega(s))\) is replaced by \(\frac{N-2}{2(N-1)}\).

**Lemma 4** Assume that \(\kappa = \frac{\nu a + (1-\nu)b}{\nu c + (1-\nu)d}\) with \(a, b, c, d > 0\) and \(\nu \in (0, 1)\). Then

\[
\min \left\{ \frac{a}{c}, \frac{b}{d} \right\} \leq \kappa \leq \max \left\{ \frac{a}{c}, \frac{b}{d} \right\}.
\] (B-6)

The both inequalities are strictly if \(\frac{a}{c} \neq \frac{b}{d}\).

**Proof of Proposition 1**

We prove the proposition when the set of correlation coefficient is \([\rho_a, \rho_b]\).

By Sion’s theorem (1958),

\[
A = \max_{x \in \mathbb{R}^N} \min_{\rho \in [\rho_a, \rho_b]} \left\{ (\bar{a}_i - p_i)x_i - \frac{\gamma}{2} \sum_{i,j=1}^{N} x_ix_j\sigma_i\sigma_j R(\rho)_{ij} \right\}
\]

\[
= \min_{\rho \in [\rho_a, \rho_b]} \max_{x \in \mathbb{R}^N} \left\{ (\bar{a}_i - p_i)x_i - \frac{\gamma}{2} \sum_{i,j=1}^{N} x_ix_j\sigma_i\sigma_j R(\rho)_{ij} \right\}
\]

39
It is well known that \( \max_{x \in \mathbb{R}^N} \left\{ (\pi_i - p_i)x_i - \frac{\gamma}{2} \sum_{i,j=1}^N x_i x_j \sigma_i \sigma_j R(\rho)_{ij} \right\} \) is given by \( \frac{1}{2\gamma} G(\rho) \), where

\[
G(\rho) = s^T R(\rho)^{-1} s = \frac{N \sum_{n=1}^N s_n^2 - (\sum_{n=1}^N s_n)^2}{N(1 - \rho)} + \frac{(\sum_{n=1}^N s_n)^2}{N(1 + (N - 1)\rho)}
\]

Therefore, \( A = \min_{\rho \in [\rho_a, \rho_b]} \frac{1}{2\gamma} G(\rho) \).

By Lemma 3, we obtain

\[
A = \frac{1}{2\gamma} G(\rho^*) = CE(\rho^*, x_{\rho^*}). \tag{B-7}
\]

(1). If \( \rho_a > \tau(\Omega(s)) \), then by Lemma 3, \( \rho^* = \rho_a \). Moreover, by Lemma 2, \( \Omega(\sigma x_{\rho_a}) > 1 \) and thus \( \max_{\Omega(\sigma x) > 1} CE(\rho_a, x) = CE(\rho_a, x_{\rho_a}) \). Since Equation (B-7) ensures that \( A = CE(\rho_a, x_{\rho_a}) \), and by using Equation (8), the solution of the problem (2) is given by \( \rho^* = \rho_a \), and \( x^* = x_{\rho_a} \).

(2). If \( \rho_b < \tau(\Omega(s)) \), then Lemma 3, \( \rho^* = \rho_b \). By Lemma 2, \( \Omega(\sigma x_{\rho_b}) < 1 \) and \( \max_{\Omega(\sigma x) < 1} CE(\rho_b, x) = CE(\rho_b, x_{\rho_b}) \). By Equation (B-7), \( A = CE(\rho_b, x_{\rho_b}) = \max_{\Omega(\sigma x) < 1} CE(\rho_b, x) \). Therefore, by using Equation (8), \( \rho^* = \rho_b, x^* = x_{\rho_b} \) is the solution of the portfolio choice problem (2).

(3). Assume that \( \rho_a \leq \tau(\Omega(s)) \leq \rho_b \). Lemma 3 ensures that \( \rho^* = \tau(\Omega(s)) \), and by Equation (B-7), \( A = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))}) \). Moreover, by Lemma 2, \( \Omega(x_{\tau(\Omega(s))}) = 1 \). By straightforward calculation, we have

\[
A = \frac{\left( \sum_i s_i \right)^2}{2\gamma} \left( \frac{1 + (N - 1)\Omega(s)}{N} \right)^2.
\]

For each \( x^* \) with \( \Omega(\sigma x^*) = 1 \) and \( CE(\tau(\Omega(s)), x^*) = \max_{\Omega(\sigma x) = 1} CE(\tau(\Omega(s)), x^*) \), we have \( CE(\tau(\Omega(s)), x^*) = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))}) \), and because of the uniqueness \( x_\rho \) for maximizing \( CE(\rho, x) \), \( x^* = x_{\tau(\Omega(s))} \). Therefore, the unique demand for the ambiguity-averse investor is \( x_{\tau(\Omega(s))} \), but the investor is irrelevant to choosing any correlation coefficient \( \rho \in [\rho_a, \rho_b] \) since \( CE(\rho, x_{\tau(\Omega(s))}) = A \) for each \( \rho \in [\rho_a, \rho_b] \).
The proof of Proposition 1 is completed. □

Proof of Proposition 2.

The optimal demand $x^*$ is presented by Proposition 1. By the market-clearing condition, $x^* = \bar{x}$ in equilibrium. Then $\Omega(\sigma x^*) = \Omega(\sigma \bar{x})$. Since $\Omega(\sigma \bar{x}) < 1$, the optimal demand in equilibrium satisfies $\Omega(\sigma x^*) < 1$. Then, by Proposition 1 again and Equation (8), the optimal correlation coefficient is $\rho^* = \bar{\rho} + \epsilon$, the highest possible correlation coefficient. □

Proof of Proposition 3.

By Proposition 1, the optimal demand of the retail investor is $x_{\rho_2}$ for $\rho_2 \in \left\{ \rho - \epsilon, \rho + \epsilon, \hat{\Omega}(s) \right\}$. Since the institutional investor knows the true correlation coefficient, her optimal demand is $x_{\rho_1}$ for $\rho_1 = \bar{\rho}$. Let $R_j = R(\rho_j)$ for $j = 1, 2$. Notice that in a portfolio inertia situation, $\rho_2 = \tau(\Omega(s))$ which is determined endogenously since the optimal demand is $x_{\tau(\Omega(s))}$. To simplify notations we also use $x^{(j)}$ to represent $x_{\rho_j}$, $j = 1, 2$.

By the market clearing condition, we have $\nu x^{(1)} + (1 - \nu) x^{(2)} = \bar{x}$. Then

$$\frac{1}{\gamma}(\nu R_1^{-1} + (1 - \nu) R_2^{-1}) \cdot s = \sigma \bar{x}. \quad (B-8)$$

Let $X = \nu R_1^{-1} + (1 - \nu) R_2^{-1}$, $m \equiv m(\rho_1, \rho_2) \equiv \frac{\nu}{1 - \rho_1} + \frac{1 - \nu}{1 - \rho_2}$, $n \equiv n(\rho_1, \rho_2) = \frac{\nu \rho_1}{(1 - \rho_1)(1 + (N - 1)\rho_1)} + \frac{(1 - \nu) \rho_2}{(1 - \rho_2)(1 + (N - 1)\rho_2)}$ and notice that $m - N n = \frac{\nu}{1 + (N - 1)\rho_1} + \frac{1 - \nu}{1 + (N - 1)\rho_2} > 0$. Then by Lemma 1, $X$ is invertible and its inverse matrix is

$$X^{-1} = \frac{1}{m} \left( I_N + \frac{n}{\kappa m} ee^T \right) \quad (B-9)$$

where

$$\kappa \equiv \kappa(\rho_1, \rho_2) = \frac{m(\rho_1, \rho_2) - N n(\rho_1, \rho_2)}{m(\rho_1, \rho_2)} = \frac{\nu}{1 + (N - 1)\rho_1} + \frac{1 - \nu}{1 - \rho_2}, \quad (B-10)$$

Therefore, by Equation (B-8),

$$s = \gamma X^{-1} \cdot (\sigma \bar{x}). \quad (B-11)$$

By straightforward calculation on Equation (B-11), we obtain the following fundamental relation between the dispersion of Sharpe ratios and the dispersion of risks:

$$\Omega(s) = \kappa \Omega(\sigma \bar{x}). \quad (B-12)$$
Assume first that $\Omega(\sigma \bar{x}) = 0$, then each $\sigma_i x_i = c$ and Equation (B-12) ensures that $\Omega(s) = 0$ and $\tau(\Omega(s)) = 1$. Therefore, by Proposition 1, $\rho_2 = \bar{\rho} + \epsilon$, and all Sharpe ratios are the same. By using Equation (B-11), all Sharpe ratios equal to

$$s_i = \frac{c}{m} \left(1 + \frac{nN}{m - nN}\right) = \frac{c}{m - nN}.$$ (B-13)

We next assume that $\Omega(\sigma \bar{x}) \in (0, 1)$ and characterize the equilibrium.

By using Proposition 1, there are three different cases regarding the choice of $\rho_2$ in equilibrium.

**Case 1.** We first assume $\tau(\Omega(s)) < \bar{\rho} - \epsilon$, which ensures that $\Omega(s) > \tau(\bar{\rho} - \epsilon)$.

By Proposition 1, $\rho_2 = \bar{\rho} - \epsilon$. By Equation (B-12),

$$\kappa = \kappa(\rho_1, \rho_2) = \frac{\Omega(s)}{\Omega(\sigma \bar{x})} > \Omega(s) > \tau(\rho_1), \tau(\rho_2),$$ (B-14)

which is impossible by Lemma 4.

**Case 2.** We assume that $\bar{\rho} - \epsilon \leq \tau(\Omega(s)) \leq \bar{\rho} + \epsilon$. By Proposition 1, we can choose $\rho_2 = \tau(\Omega(s))$, and the optimal holding for the retail investor is $x_{\tau(\Omega(s))}$. Moreover,

$$\tau(\bar{\rho} + \epsilon) \leq \Omega(s) < \tau(\bar{\rho} - \epsilon).$$ (B-15)

By Equation (B-12) and direct computation, we have

$$\kappa = \kappa(\rho_1, \tau(\Omega(s))) = \frac{\Omega(s)}{\Omega(\sigma \bar{x})} = \frac{\nu}{1 + \frac{(N-1)\rho_1}{\nu} + \frac{1 - \nu}{N}(1 + (N-1)\Omega(s)) - \frac{1 - \nu}{\Omega(s)}(1 - \Omega(\sigma \bar{x}))}. \quad \kappa$$ (B-16)

By solving the last equation in $\Omega(s)$, we obtain

$$\Omega(s) = \frac{\nu}{1 + \frac{(N-1)\rho_1}{\nu} + \frac{1 - \nu}{N}(1 - \Omega(\sigma \bar{x}))} \cdot (1 - \Omega(\sigma \bar{x}))^{-1}. \quad \Omega(s) \geq 0$$ (B-17)

Moreover, by simple algebra, $\tau(\Omega(s)) = K(\rho_1)$ where $K(\cdot)$ is defined in Equation (14). $\Omega(s) \geq 0$ ensures that

$$\rho_1 \leq \frac{1}{N-1} \left\{ \frac{1 - \nu}{\Omega(\sigma \bar{x})} - \frac{\nu}{1 - \nu} \right\} N - 1.$$ (B-18)
Moreover, the left side of Equation (B-15) is translated as $\bar{\rho} + \epsilon \geq K(\bar{\rho})$, and the right side of Equation (B-15) holds always because of $K(\rho) \geq \rho, \forall 0 \leq \rho \leq 1$. Then there exists a unique equilibrium in this case, a limited participation equilibrium, under conditions presented in Proposition 3.

**Case 3.** We characterize the equilibrium in which $\tau(\Omega(s)) > \bar{\rho} + \epsilon$.

By Proposition 1, $\rho_2 = \bar{\rho} + \epsilon$, and $\Omega(s) < \tau(\bar{\rho} + \epsilon)$. By Equation (B-12), we obtain $\kappa(\rho_1, \rho_2)\Omega(\sigma x) < \frac{1 - \rho_2}{1 + (N - 1)\rho_2}$. By straightforward computation, the last inequality is the same as $\rho_2 < K(\rho_1)$.

To the end, we note that when $\bar{\rho}$ is large enough such that

\[
\frac{\Omega(\sigma x)N}{(1 - \nu)(1 - \Omega(\sigma x))} \leq 1 + (N - 1)\rho_1, \tag{B-19}
\]

or equivalently,

\[
\rho_1 \geq \frac{1}{N - 1} \left\{ \frac{\nu \Omega(\sigma x)}{1 - \nu} \right\} \left( N - 1 \right),
\]

then $K(\rho_1) \geq 1$. Therefore $\rho_2 < K(\rho_1)$ holds naturally since $\rho_2 < 1$. Then we have characterized the equilibrium in Proposition 3.

Finally, by using this equilibrium characterization, we see that each $s_i > 0$. Therefore, each risky asset is priced at discount in equilibrium. □

**Proof of Proposition 4.**

By the characterization of the Sharpe ratios in Proposition 3, we obtain

\[
s_i = \frac{S}{N} + \frac{\gamma}{m(\rho_1, \rho_2)} \left( \sigma_i \bar{x} - \frac{L}{N} \right). \tag{B-20}
\]

Both (1) and (2) follow easily from Equation (B-20) and $\frac{\partial}{\partial \epsilon} m(\bar{\rho}, \bar{\rho} + \epsilon) > 0$. □

**Proof of Proposition 5.**

(1) The first part follows from the expression of $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ in terms of $\eta_i$ in Appendix A, Proposition 11. Let $a \equiv \sum_{i=1}^{N} \eta_i^2 = \frac{(N-1)\Omega(\eta)^2 + 1}{N}$. By Equation (A-1), it suffices to show
that
\[
\frac{\partial}{\partial \rho} \left( \frac{(\rho + \eta_i(1 - \rho))^2}{\rho + a(1 - \rho)} \right) > 0. \tag{B-21}
\]

By straightforward calculation, we see that this partial derivative is a positive number times
\[
2a(1 - \eta_i) - (1 - a)\eta_i + (1 - a)(1 - \eta_i)\rho. \tag{B-22}
\]
Under conditions on \( \eta_i, i = 1, \ldots, N \), we obtain
\[
\frac{2a}{1 - a} = \frac{2 \sum_{i=1}^{N} \eta_i^2}{\sum_{i=1}^{N} (\eta_i - \eta_i^2)} \geq \frac{2\eta_N}{1 - \eta_N} \geq \frac{\eta_i}{1 - \eta_i}, i = 1, \ldots, N.
\]
Thus, \( 2a(1 - \eta_i) - (1 - a)\eta_i + (1 - a)(1 - \eta_i)\rho > 0 \), and the sensitivity of the correlation with
the market portfolio with respect to \( \epsilon \) is positive.

(2) By using the expression of the weighted beta in Proposition 11 (2), the positive effect
of the correlation coefficient on the weighted beta follows from the fact that
\[
\frac{w_i \partial \beta_i}{\eta_i \partial \rho} = \frac{\partial}{\partial \rho} \left( \frac{\rho + \eta_i(1 - \rho)}{\rho + \sum_{i=1}^{N} \eta_i^2(1 - \rho)} \right)
= \frac{\sum_{j \neq i} \eta_j^2 + \eta_i^2 - \eta_i}{\left( \rho + \sum_{i=1}^{N} \eta_i^2(1 - \rho) \right)^2}.
\]
The numerator \( \sum_{j \neq i} \eta_j^2 + \eta_i^2 - \eta_i \) is positive if and only of when \( \eta_i \) is small or large, and in this
case, its beta is positively associated with the endogenous pairwise correlation coefficient.
For a moderate level of eta, the numerator is negative, thus, its beta is negatively associated
with the correlation coefficient.

\[
\text{Proof of Proposition 6.}
\]

(1). We compare the investors’ optimal portfolios with the market portfolio in terms
of the dispersion measure \( \Omega(\cdot) \). By Proposition 3 and Equation (B-12), we obtain
\[
\Omega(s) = \kappa(\rho_1, \rho_2)\Omega(\sigma \bar{x}).
\]
In the full participation equilibrium, Lemma 2 implies that
\[
\Omega(\sigma x^{(s)}) = \Omega(\sigma \bar{x})\kappa(\bar{\rho}, \bar{\rho} + \epsilon) \frac{1 + (N - 1)\bar{\rho}}{1 - \bar{\rho}}
\]
and
\[
\Omega(\sigma x^{(r)}) = \Omega(\sigma \bar{x})\kappa(\bar{\rho}, \bar{\rho} + \epsilon) \frac{1 + (N - 1)(\bar{\rho} + \epsilon)}{1 - \bar{\rho} - \epsilon}.
\]
Then, by Lemma 4 we have
\[ \Omega(\sigma x^{(s)}) < \Omega(\sigma \overline{x}) < \Omega(\sigma x^{(r)}). \] (B-23)

The proof in the limited participation equilibrium is similar. Since \( \rho \leq K(\rho) = \tau(\Omega(s)) = \rho_2 \), then \( \kappa(\rho, \tau(\Omega(s))) \frac{1 + (N-1)\rho}{1-\rho} < 1 \). Lemmas 2 and 4 together imply that \( \Omega(\sigma x^{(s)}) = \Omega(\sigma \overline{x}) \kappa(\rho, \tau(\Omega(s))) \frac{1 + (N-1)\rho}{1-\rho} < \Omega(\sigma \overline{x}) \). Moreover \( \Omega(\sigma \overline{x}) = \Omega(\sigma x^{(r)}) = 1 = \Omega(\sigma x^{(r)}) \).

(2). When the dispersion of correlation uncertainty \( \epsilon \) increases, it is easy to see that \( \kappa(\rho, \rho + \epsilon) \) decreases, and \( \kappa(\rho, \rho + \epsilon) \frac{1 + (N-1)\rho}{1-\rho} \) increases. Therefore,
\[ \frac{\partial \Omega(\sigma x^{(s)})}{\partial \epsilon} < 0, \quad \frac{\partial \Omega(\sigma x^{(r)})}{\partial \epsilon} > 0. \] (B-24)

(3). Let \( X^{(s)} = \sum_i \tilde{a}_i x_i^{(s)} \) be the optimal portfolio of the institutional investor and \( X^{(r)} \) the optimal portfolio of the retail investor. We have \( Var(X^{(s)}) = \frac{1}{\gamma} s^T R(\rho_1)^{-1} s \) where the correlation coefficient is \( \rho_1 \), and \( Var(X^{(r)}) = \frac{1}{\gamma^2} s^T R(\rho_2)^{-1} s \) with the correlation coefficient \( \rho_2 \). By using the same notation in Lemma 3, we have
\[ Var(X^{(s)}) - Var(X^{(r)}) = \frac{1}{\gamma^2} \left\{ G(\rho_1) - G(\rho_2) \right\}. \] (B-25)

Applying Proposition 3 since the optimal portfolio choice is under the worst-case correlation uncertainty, \( G(\rho_1) > G(\rho_2) \) holds. Therefore, \( Var(X^{(s)}) > Var(X^{(r)}) \).

(4). \( \mathbb{E}[X^{(s)}] = \sum_{i=1}^N x_i^{(s)} (\overline{p}_i - p_i) = (\sigma x^{(s)})^T \gamma s = \frac{1}{\gamma} s^T R(\rho_1)^{-1} s \). By (3), the variance of \( X^{(s)} \) is \( \frac{1}{\gamma^2} s^T R(\rho_1)^{-1} s \). Then the Sharpe ratio of the portfolio \( X^{(s)} \) is \( SR(X^{(s)}) = \sqrt{s^T R(\rho_1)^{-1} s} \). Therefore, \( SR(X^{(s)}) > SR(X^{(r)}) \) follows from \( G(\rho_1) > G(\rho_2) \). Moreover, by the proof of Proposition 4 the retail investor’s maxmin expected utility is \( \frac{1}{Z} G(\rho_2) \). Again, the fact that \( G(\rho_1) > G(\rho_2) \) ensures that the institutional investor has a higher maxmin expected utility than the retail investor.

\[ \square \]

Proof of Proposition 7.

(1). By using the characterization of Sharpe ratios in Proposition 3 we obtain
\[ \frac{s_i}{S} - \frac{1}{N} = \frac{m(\overline{p}, \rho + \epsilon) - Nn(\overline{p}, \rho + \epsilon)}{m(\overline{p}, \rho + \epsilon)} \left( \eta_i - \frac{1}{N} \right). \] (B-26)
We can easily show that $\frac{d}{d\epsilon} S$ is increasing with respect to $\epsilon$ when $\eta_i < \frac{1}{N}$. Since the average Sharpe ratio $S$ is always increasing with respect to the level of correlation uncertainty, and

$$\gamma \sigma_i x_i^{(s)} = \frac{1}{1 - \rho} S \left( \frac{s_i}{S} - \frac{\rho}{1 + (N - 1)\rho} \right), \quad (B-27)$$

thus $\frac{\partial}{\partial \epsilon} \left( x_i^{(s)} \right) > 0$. By using the market-clearing equation, $\nu x_i^{(s)} + (1 - \nu) x_i^{(r)} = \bar{x}_i$, we have $\frac{\partial}{\partial \epsilon} \left( x_i^{(r)} \right) < 0$.

If $\eta_i$ is large, the level of correlation uncertainty to $s_i$ is negative, by Proposition 4, but $S$ is positively relates to $\epsilon$. Since Equation (B-27) implies that

$$\gamma \sigma_i x_i^{(s)} = \frac{1}{1 - \rho} \left( s_i - \frac{\rho}{1 + (N - 1)\rho} S \right),$$

$\frac{\partial}{\partial \epsilon} \left( x_i^{(s)} \right) < 0$ and $\frac{\partial}{\partial \epsilon} \left( x_i^{(r)} \right) > 0$.

(2). By using the expression of $x_i^{(s)}$ in Equation (B-27) and the expression of $s_i, S$, it is easy to check that $\gamma \sigma_i x_i^{(s)} < \gamma \sigma_i \bar{x}_i$ is equivalent to $\eta_i > J(\epsilon, \nu)$. Because of the market-clearing condition, $x_i^{(r)} < \bar{x}_i$ if and only if $\eta_i < J(\epsilon, \nu)$.

(3). First, the retail investor buys the high-eta asset with $\eta > J(\epsilon, \nu)$, so the trading volume is $|x_i^{(r)} - \bar{x}_i| = x_i^{(r)} - \bar{x}_i$. Then, by Proposition 7 (1),

$$\frac{\partial}{\partial \epsilon} \left( x_i^{(r)} - \bar{x}_i \right) = \frac{\partial}{\partial \epsilon} \left( x_i^{(r)} \right) > 0.$$

However, for the low-eta asset, the naive trading volume is $\bar{x}_i - x_i^{(r)}$ since he needs to sell the initial position, thus, by Proposition 7 (1), we obtain

$$\frac{\partial}{\partial \epsilon} \left( \bar{x}_i - x_i^{(r)} \right) = -\frac{\partial}{\partial \epsilon} \left( x_i^{(r)} \right) > 0.$$

By the similar argument, we can show that, for the institutional investor,

$$\frac{\partial}{\partial \epsilon} \left| \bar{x}_i - x_i^{(s)} \right| > 0.$$
Proof of Proposition 8

(1). By Equation (B-26), the result follows from the fact that $\frac{\partial \kappa(\rho, \rho + \epsilon)}{\partial \epsilon} < 0$, and $\frac{\partial \kappa(\rho, \rho + \epsilon)}{\partial \nu} > 0$.

(2). By Equation (B-12), $\Omega(s) = \kappa(\rho, \rho + \epsilon)\Omega(\eta)$. Then the result also follows from $\frac{\partial \kappa(\rho, \rho + \epsilon)}{\partial \epsilon} < 0$, and $\frac{\partial \kappa(\rho, \rho + \epsilon)}{\partial \nu} > 0$. □

Proof of Proposition 9

We apply the same idea of proving Proposition 3. By Proposition 1, the institutional investor’s optimal demand is $x(1) = x_{\rho_1}$ where $\rho_1 \in \{\bar{\rho} - \epsilon_1, \bar{\rho} + \epsilon_1, \tau(\Omega(s))\}$. Similarly, the retail investor’s optimal demand is $x(2) = x_{\rho_2}$ where $\rho_2 \in \{\bar{\rho} - \epsilon_2, \bar{\rho} + \epsilon_2, \tau(\Omega(s))\}$. Notice that $\Omega(s)$ is determined in equilibrium. Then, by the market clearing condition, we have $\nu x(1) + (1 - \nu)x(2) = \bar{x}$. We employ the same derivations and notations as in Proposition 3.

By using Proposition 1, there are five different cases to characterize the equilibrium.

Case 1. We assume $\tau(\Omega(s)) \leq \bar{\rho} - \epsilon_2 < \bar{\rho} - \epsilon_1$, which ensures that $\Omega(s) \geq \tau(\bar{\rho} - \epsilon_2) > \tau(\bar{\rho} - \epsilon_1)$.

By Proposition 1, $\rho_i = \bar{\rho} - \epsilon_i, i = 1, 2$. By Equation (B-12),

$$\kappa = \kappa(\bar{\rho} - \epsilon_1, \bar{\rho} - \epsilon_2) = \frac{\Omega(s)}{\Omega(\sigma \bar{x})} > \Omega(s) > \tau(\bar{\rho} - \epsilon_1), \tau(\bar{\rho} - \epsilon_2), \quad (B-28)$$

which is impossible by Lemma 4.

Case 2. We assume that $\tau(\bar{\rho} - \epsilon_2) > \Omega(s) \geq \tau(\bar{\rho} - \epsilon_1)$.

In this case, by Proposition 1, $\rho_1 = \bar{\rho} - \epsilon_1$ and we can choose $\rho_2 = \tau(\Omega(s))$. We obtain

$$\kappa = \kappa(\bar{\rho} - \epsilon_1, \tau(\Omega(s))) > \kappa\Omega(\sigma \bar{x}) = \Omega(s) \geq \frac{1 - \bar{\rho} - \epsilon_1}{1 + (\bar{\rho} - \epsilon_1)(N - 1)}, \kappa > \Omega(s) = \frac{1 - \tau(\Omega(s))}{1 + (N - 1)\tau(\Omega(s))}. \quad (B-29)$$

Hence, it is impossible by using Lemma 4 again.

Case 3. In this case, $\tau(\Omega(s)) \in [\bar{\rho} - \epsilon_1, \bar{\rho} + \epsilon_1]$, and we prove this is impossible in equilibrium.

47
In fact, if there exists such an equilibrium, the optimal holding of each investor is \( x_{\tau(\Omega(s))} \) by Proposition 1. Then the market-clearing condition yields that \( x_{\tau(\Omega(s))} = \overline{\rho} \). However, by Lemma 2, \( \Omega(\sigma x_{\tau(\Omega(s))}) = 1 \) but \( \Omega(\sigma \overline{\rho}) < 1 \). Therefore, Case 3 is not possible in equilibrium.

**Case 4.** We characterize the equilibrium in which \( \rho + \epsilon < \tau(\Omega(s)) \leq \rho + \epsilon_2 \).

In this case, by Proposition 1 \( \rho_1 = \rho + \epsilon_1 \), we can choose \( \rho_2 = \tau(\Omega(s)) \), and the optimal holding for the retail investor is \( x_{\tau(\Omega(s))} \). Moreover,

\[
\tau(\rho + \epsilon_2) \leq \Omega(s) < \tau(\rho + \epsilon_1). \tag{B-30}
\]

By Equation (B-12) and direct computation, we have

\[
\kappa = \kappa(\rho + \epsilon_1, \tau(\Omega(s))) = \frac{\Omega(s)}{\Omega(\sigma \overline{\rho})} = \frac{\nu}{1+\nu(\rho + \epsilon_1)} + \frac{\nu}{1-\rho - \epsilon_1} + \frac{1}{\Omega(s)}(1 + (N-1)\Omega(s)). \tag{B-31}
\]

By solving the last equation in \( \Omega(s) \), we obtain

\[
\Omega(s) = \frac{\nu}{1+\nu(\rho + \epsilon_1)} \Omega(\sigma \overline{\rho}) - \frac{1}{N} \nu (1 - \Omega(\sigma \overline{\rho})). \tag{B-32}
\]

By simple algebra, we obtain \( \tau(\Omega(s)) = K(\rho + \epsilon_1) \). Moreover, \( \Omega(s) \geq 0 \) ensures that

\[
\rho + \epsilon_1 \leq \frac{1}{N-1} \left\{ \frac{\nu}{1-\rho - \epsilon_1} \Omega(\sigma \overline{\rho}) - \frac{1}{N-1} \right\}. \tag{B-33}
\]

By the same derivation as in Case 2 of Proposition 3, we see that Equation (B-30) is equivalent to \( \rho + \epsilon_2 \geq K(\rho + \epsilon_1) \), and under this condition, a limited participation equilibrium is generated.

**Case 5.** We characterize the equilibrium in which \( \tau(\Omega(s)) > \rho + \epsilon_2 \).

By Proposition 1 \( \rho_1 = \rho + \epsilon_1, \rho_2 = \rho + \epsilon_2 \). In this case, \( \Omega(s) < \tau(\rho + \epsilon_2) \) is the same as \( \kappa(\rho + \epsilon_1, \rho + \epsilon_2) \Omega(\sigma \overline{\rho}) < \tau(\rho + \epsilon_2) \). By straightforward computation, this condition equals to \( \overline{\rho} + \epsilon_2 < K(\rho + \epsilon_1) \).

Moreover, if

\[
\rho + \epsilon_1 \geq \frac{1}{N-1} \left\{ \frac{\nu}{1-\rho - \epsilon_1} \Omega(\sigma \overline{\rho}) - \frac{1}{N-1} \right\}. \tag{B-34}
\]
then $K(\bar{\rho} + \epsilon_1) \geq 1$; thus $\bar{\rho} + \epsilon_2 < K(\bar{\rho} + \epsilon_1)$ holds. The characterization of equilibrium is completed. The property of each risky asset priced at discount in equilibrium follows from the characterization of the equilibrium. □

Proof of Proposition 10. We note that $\rho_1 < \rho_2$ in equilibrium. Then the proofs of Proposition 10 (1)-(3) are similar to the proof of Proposition 6 - Proposition 8 □
Appendix C: A Dispersion Measure

A function \( f : (X_1, \cdots, X_N) \in \mathbb{R}^N \rightarrow [0, 1] \) is a dispersion measure if it satisfies the following three properties:

1. (Positively homogeneous property) Given any \( \lambda > 0 \), \( f(\lambda X_1, \cdots, \lambda X_N) = f(X_1, \cdots, X_N) \);

2. (Symmetric property) Given any translation \( \sigma : \{1, \cdots, N\} \rightarrow \{1, \cdots, N\} \), \( f(X_1, \cdots, X_N) = f(X_{\sigma(1)}, \cdots, X_{\sigma(N)}) \);

3. (Majorization property) Assuming \((X_1, \cdots, X_N)\) weakly dominates \((Y_1, \cdots, Y_N)\), then \( f(X_1, \cdots, X_N) \geq g(Y_1, \cdots, Y_N) \).

By a vector \( a = (a_1, \cdots, a_N) \) weakly dominates \( b = (b_1, \cdots, b_N) \) we mean that

\[
\sum_{i=1}^{k} a_i^* \geq \sum_{i=1}^{k} b_i^*, \quad k = 1, \cdots, N
\]

where \( a_i^* \) is the element of \( a \) stored in decreasing order. When \( f(Y) \leq f(X) \) for a dispersion measure we call \( Y \) is more dispersed than \( X \) under the measure \( f \). One example is the portfolio weight in a financial market, so the dispersion measure captures how one portfolio is more dispersed than another.

This paper concerns one example of a dispersion measure.

**Lemma 5** Given a non-zero vector \( X = (X_1, \cdots, X_N) \),

\[
f(X_1, \cdots, X_N) = \sqrt{\frac{1}{N-1} \left( N \frac{\sum X_i^2}{(\sum_i X_i)^2} - 1 \right)}
\]

is a dispersion measure.

**Proof:** Both the positive homogeneous property and the symmetric property are obviously satisfied. To prove the majorization property, we first assume that, because of the positively homogeneous property, \( \sum X_i = \sum Y_i = 1 \). Let \( g(x_1, \cdots, x_N) = N(\sum_i x_i^2) - (\sum_i x_i)^2 \). Notice that \( g(\cdot) \) is a Schur convex function in the sense that

\[
(x_i - x_j) \left( \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_j} \right) \geq 0, \forall i, j = 1, \cdots, N.
\]
Then by the majorization theorem, (Marshall and Olkin, 1979), $g(X) \geq g(Y)$. Then $f(X) \geq f(Y)$. □
References


Goldman, M. Barry, 1979, Anti-Diversification or Optimal Programmes for Infrequently Revised Portfolio, *Journal of Finance* 34, 505-516.


Figure 1: Limited participation portfolio and Portfolio inertia

This figure presents the portfolio inertia feature under correlation uncertainty. (a) displays the limited participation portfolios \(x_1, x_2, x_3\) for which \(\sigma_1 x_1 \sigma_2 x_2 + \sigma_1 x_1 \sigma_3 x_3 + \sigma_2 x_2 \sigma_3 x_3 = 0\). The anti-diversification portfolio consists of each \(x_i\)-axis only. (b) displays the region of \((s_1, s_2)\) in which the portfolio inertia in the correlated structure occurs. The true correlation coefficient \(\bar{\rho} = 0.5\) and the uncertainty \(\epsilon = 0.1\). By Proposition 1, the portfolio inertia is generated when \(0.4s_1 \leq s_2 \leq 0.6s_1\) and \(\frac{1}{0.6}s_1 \leq s_2 \leq \frac{1}{0.4}s_1\). (c) and (d) draw the retail investor’s optimal demand \((x_1, x_2, x_3)\) with respect to the correlation uncertainty \(\epsilon\) when \(\bar{\rho} = 0.5, 0.8\) respectively. Parameters: \((s_1, s_2, s_3) = (0.7, 0.8, 1.2), (\sigma_1, \sigma_2, \sigma_3) = (0.2, 0.3, 0.5)\) and \(\gamma = 1\). By calculation, \(\Omega(s) = 0.1697\) and \(\Omega(s) = 0.6199\).
This figure displays the equilibrium analysis when the correlation uncertainty changes. We report the sensitivity of Sharpe ratios, the correlation with the market portfolio, and weighted betas in a full equilibrium (Graph (a)), and a limited equilibrium (Graph (b)). Parameters: $\sigma = [0.09, 0.1, 0.12], x = [1, 5, 10.5], \gamma = 1, \nu = 0.3$, and $\bar{p} = 0.5$. In Graph (a), $\nu = 0.3$ and in Graph (b), $\nu = 0.5$. A limited equilibrium for any $\epsilon \geq 0.3476$. 

(a) A full participation equilibrium

(b) A limited participation equilibrium
Figure 3: Institutional vs Retail Investor under Correlation Uncertainty

This figure reports the optimal holdings of institutional and retail investor when the correlation uncertainty changes. It also displays the dispersions of institutional (retail) investor’s optimal portfolio with respect to the correlation uncertainty. Parameters are the same as in parameters in Figure 2.

(a) A full participation equilibrium

(b) A limited participation equilibrium
This figure explains precisely that the institutional investor holds a well-diversified portfolio while the retail investor’s portfolio is under-diversified. It demonstrates the dispersion of the optimal portfolio for the institutional and retail investor respectively in a full participation equilibrium with respect to the level of correlation uncertainty. Therefore, the figure also explains that an increasing perceived level of correlation uncertainty induces a less diversified optimal portfolio for each investor. The dispersion of the institutional investor’s optimal portfolio is smaller than the corresponding dispersion of the retail investor’s optimal portfolio. Moreover, the institutional investor’s optimal portfolio has a smaller dispersion than that of the market portfolio, whereas the retail investor’s optimal portfolio has a larger dispersion than that of the market portfolio. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Note that the dispersion of the market portfolio is $\Omega(\sigma \bar{x}) = 0.56$, and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma \bar{x})}{1-\Omega(\sigma \bar{x})} N - 1 \right\} = 0.3$. 

![Dispersion of Institutional Investor](image1)

![Dispersion of Retail Investor](image2)
Figure 5: Optimal holdings of investors in an extension model

This figure explains the flight-to-quality episode from the correlation uncertainty perspective. It reports the optimal holdings of the institutional and retail investor on low-eta asset 1 and high-eta asset 3 respectively when the level of uncertainty changes. This figure demonstrates how the trading positions and trading volumes are affected by the level of uncertainty for different asset. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma \bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma \bar{x})}{1-\Omega(\sigma \bar{x})} N - 1 \right\} = 0.3$. $\eta_1 = 0.048 < \frac{1}{N}$, $\eta_3 = 0.68 > \frac{1}{N}$. 

Institutional Investor Holdings on Low Eta Asset  
Retail Investor Holdings on Low Eta Asset

Institutional Investor Holdings on High Eta Asset  
Retail Investor Holdings on High Eta Asset
This figure demonstrates the asset comovements when the correlation uncertainty changes in an extension model. Graph (a) displays the relative Sharpe ratio of a high-eta asset. Graph (b) displays the relative Sharpe ratio of a low-eta asset. Graph (c) presents the dispersion of all Sharpe ratios. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \nu \frac{\Omega(\sigma\bar{x})}{1-\nu} \right\}_N = 0.3$. 

(a) Relative Sharpe ratios of high-eta assets

(b) Relative Sharpe ratios of low-eta assets

(c) Dispersion of Sharpe ratios
Graph (a) displays the VIX and S&P 500 index from Jan. 2006 to Jan. 2016. As documented in Bloom (2009), VIX measure the macroeconomic uncertainty as well as the ambiguity on the entire financial market. The high level of VIX in 2007-2009 reflects to higher level of Knightian uncertainty. It is clear that the stock market largely moves in opposite direction with the VIX, in particular, in 2007-2009. Source: Chicago Board Option Exchange. Graph (b) shows that the stock market is a low-eta asset class compared to the fixed income market which is a high-eta asset class. In the graph, we report the weight of the equity market compared to the fixed income market between 2005 to 2014. “Fixed Income Securities” includes “Public Debt market”, “Financial Bonds”, “Corporate Bonds”, “Securitized Loan Market”, and “Unsecuritized Loans Outstanding Market”. As shown, the fixed income market is about four time size large of the stock market in term of market capitalization. Since the stock volatility is around three time of the fixed income volatility (Reilly, Wright and Chan, Journal of Portfolio Management, 2000), the eta of the stock marker is about 75% of the eta of the fixed income market. Source: McKinsey Global Institute research.

(a) VIX vs S&P 500 index

(b) Comparison between equity market and debt/loan markets
Table 1: Limited and Full Equilibrium

This table reports the conditions under which the full equilibrium and the limited equilibrium prevails. For simplicity we assume that the institutional investor knows the perfect correlation $\rho$ and the retail investor's degree of correlation uncertainty is $\epsilon$. In Panel A, we characterize the equilibrium under conditions of the correlation uncertainty while $\nu$ and $\Omega(\eta)$ are fixed. Panel B discusses the equilibrium cases under conditions of $\nu$, given other parameters, $\bar{\rho}$, $\epsilon$ and $\Omega(\eta)$, are fixed. $\nu^*$ is defined by the equation $K(\bar{\rho}, \nu^*, \Omega(\eta)) = \bar{\rho} + \epsilon$. In Panel C, we characterize the equilibrium under the conditions of the risk distribution $\Omega(\eta)$ when parameters $\bar{\rho}$, $\epsilon$ and $\nu$ are fixed. $\Omega^*$ satisfies the equation $K(\bar{\rho}, \nu, \Omega^*) = \bar{\rho} + \epsilon$.

**Panel A: Conditions on $\epsilon$**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Full Participation</th>
<th>Limited Participation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\rho} + \epsilon \geq K(\bar{\rho})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{\rho} + \epsilon &lt; K(\bar{\rho})$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Panel B: Conditions on $\nu$, where $\bar{\rho} + \epsilon = K(\bar{\rho}, \nu^*, \Omega)$**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Full Participation</th>
<th>Full Participation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho + \epsilon \leq \lim_{\nu \to 1} K(\bar{\rho}, \nu, \Omega(\eta))$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{\rho} + \epsilon &gt; \lim_{\nu \to 1} K(\bar{\rho}, \nu, \Omega(\eta))$:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu &lt; \nu^*$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu \geq \nu^*$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Panel C: Conditions on $\Omega(\eta)$, where $\bar{\rho} + \epsilon = K(\bar{\rho}, \nu, \Omega^*)$**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Full Participation</th>
<th>Full Participation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega(\eta) &lt; \Omega^*$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega(\eta) \geq \Omega^*$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2: Model Implication

This table summarizes our model implications. We report the model implications in three different categories when the correlation uncertainty increases. Panel A presents the effect of the correlation uncertainty at an overall market level. We consider three popular measures of asset comovement: the aggregative market volatility, the pairwise market correlation and the dispersion of Sharpe ratios. In Panel B, we present the cross-sectional effect on the individual asset given different asset characteristics, including the asset prices, the risk premiums, Sharpe ratios, the relative Sharpe ratios, the correlations with the market portfolio and the weighted betas. In Panel C, we compare the institutional investor and retail investor, in terms of their optimal portfolio, holding position, and trading volume on individual assets, which explains the under-diversification or limited participation, and flight to quality in our model. For simplicity, we assume that institutional investor knows the exact correlation coefficient $\rho$ while the degree of correlation uncertainty for the retail investor is $\epsilon$.

### Panel A: Effect on the overall market level when $\epsilon$ increases

| Aggregate market volatility: $\sum_{i=1}^{N} \sigma_i \bar{x}_i$ | increase |
| Pairwise market correlation: $\rho$ | increase |
| Dispersion of Sharpe ratios: $\Omega(s)$ | increase |

### Panel B: Effect on the individual asset when $\epsilon$ increases

| Asset price: $p_i$ | High-eta asset | Low-eta asset |
| Risk premium: $\bar{a}_i - p_i$ | decrease | increase |
| Sharpe ratio: $s_i$ | decrease | increase |
| Relative Sharpe ratio: $\frac{s_i}{\bar{S}_i}$ | decrease | increase |
| Correlation with market: $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ | increases or mixed | decrease or mixed |
| (Weighted) Beta: $\beta_i \omega_i$ | increases or mixed | increases or mixed |

### Panel C: Effect on portfolio holdings and volume when $\epsilon$ increases

| Optimal portfolio: $\Omega(\sigma x)$ | Institutional investor | Retail investor |
| High-eta (holding) | decrease | increase |
| Low-eta (holding) | increase | decrease |
| High-eta (trading volume) | increase | increase |
| Low-eta (trading volume) | increases | increase |