Analyzing Interactive Exercising Policies and Evaluations of Callable and (or) Convertible Bonds

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Abstract

Many embedded options are incorporated in bond provisions to meet various requirements from investors and issuing firms. Analyzing the policies of exercising these options and their relations to issuing firms’ creditworthiness, prevailing interest rate levels, and hence bond valuations have drawn much attention in academics and practitioner communities. This paper focuses on one particular example, a callable convertible bond (CCB), with its counterparts, a callable nonconvertible and a noncallable convertible, to investigate how the changes of aforementioned market conditions and the presence of one embedded option influence the policy of exercising another option. We provide theoretical analyses based on a structural model of credit risk. Compared with a CCB’s counterparts, our analyses suggest that a CCB issuer (holder) would precipitate its call (conversion) decision to maximize its value by destroying the option owned by its counter party. This precipitation becomes more salient as the counter party’s option value appreciates. The empirical implication can not only be visualized through our numerical pricing models but be supported by empirical tests carried out with twenty years bond data. We show that out-of-the-money calls (or early calls) of CCBs that are observed but are not well elucidated in past studies can be satisfactorily explained to pre-empt conversion rather than to pre-empt default. Similarly, subject to the early call policies, out-of-the-money conversions (early conversions) are triggered to pre-empt redemption.

Keywords: Callable Convertible Bond, Structural Model of Credit Risk, Out-of-the-Money Call Policy, Out-of-the-Money Conversion Policy

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1 Introduction

A corporate bond is becoming an important financial instrument for a firm to raise capital to fulfill expenses of operations and future business expansions. Securities Industry and Financial Markets Association (abbreviated as SIFMA) reports that the amount of new issuances in the US market grew from 626.4 billion in 2000 to 1,513.2 billion in 2014.1 Many outstanding bonds contain embedded options to meet various requirements from issuers and investors. For example, the statistical records in SIFMA says that about 60% of U.S. corporate bonds are callable bonds; callable convertible bonds that simultaneously grant bond issuing firms call options and bond holders options to convert (the bonds into the issuing firms’ stocks) are also widely traded in bond markets. The presence of an embedded option or even a complex embedded “game option” owned simultaneously by both the bond issuing firms and bond holders could significantly influence bond values. This paper analyzes how the presence of call and/or conversion options and prevailing market conditions like levels of interest rates influence the policies of exercising embedded options and thus bond evaluations. We extend theoretical analyses of Acharya and Carpenter (2002) that study the interaction effect between the call and default option embedded in a defaultable callable bond. They show that an issuing firm would delay its decisions to exercise its call or default option to avoid value destruction of the other option since both options are owned by the firm. On the other hand, we novelly analyze the policies of exercising game options for both the bond issuing firms and bond holders and show that both parties would precipitate their exercise decisions to destroy counter party’s option. Our analyses of precipitating exercise decision not only explain Bhattacharya (2012)’s insight on early call decisions but provide another tunnel to elucidate Jensen and Pedersen (2016)’s observation about the early conversion decision.2 Empirical tests verify our theoretical analyses of policies of exercising embedded options, their relations to issuers’ creditworthiness, prevailing interest rate levels, and hence bond valuations.

Theoretical works on pricing callable convertible bonds are well established, and the call policies implicit in pricing results are especially emphasized and empirically examined. Pioneered by the seminal works Ingersoll (1977a) and Brennan and Schwartz (1977) adopting the contingent claim analysis (see Black and Scholes, 1973; Merton, 1974), the optimal call policy is predicted to call the convertible bond once its conversion value—the market value of the common stocks the bond can be exchanged for—equals its call price. With this at-the-money call policy, the equity holder’s value can be maximized by minimizing the bond value when the capital market is perfect to accommodate the Modigliani-Miller theorem (see Modigliani and Miller, 1958). However, later empirical studies do not support such policy and thereby raise the puzzle: are the widely observed in-the-money calls (late calls) and out-of-the-money calls (early calls) of convertible bonds still rational to firms?

A variety of explanations have been offered for the first piece of the puzzle. Some argue that in-the-money calls are still rational if the capital market is imperfect. That includes the existence of information asymmetry (see Harris and Raviv, 1985), transaction costs (see Ingersoll, 1977b; Emery and Finnerty, 1989), and corporate taxes leading to tax shield benefits (see Asquith and Mullins, 1991; Campbell et al., 1991; Sarkar, 2003; Liao and Huang, 2006; Chen et al., 2013). Others refers to the rationale for mitigating agency conflict (see Billett et al., 2007; King and Mauer, 2014) or the existence of call notice periods (see Ingersoll, 1977b; Altintig and Butler, 2005) and call protection (see Asquith,

2Jensen and Pedersen (2016) address that bond holders convert their convertible bonds early when facing market frictions, such as short-sale costs, transaction costs, or funding costs. However, their bond data set is consisted of convertible bonds with call provisions.
However, recent research present the empirical evidence that shows in-the-money calls would disappear once convertible bonds are dividend-protected. Grundy and Verwijmeren (2016) argue that such anti-dilution provision would considerably increase the issuing firm’s incentive to call because the conversion value of its convertible bond is protected to immunize against its dividend payouts. The aforementioned rationales for in-the-money calls become trivial thereafter, and the link between near-zero call delay and dividend-protected convertible bonds are established to thereby preserve the importance of the at-the-money call policy.\(^3\)

Diminishing in-the-money calls allow us to shift the focus on the discussion about out-of-the-money calls of convertible bonds that are observable (see Cowan et al., 1993; Grundy and Verwijmeren, 2012; King and Mauer, 2014; Bechmann et al., 2014) but still excluded from Ingersoll (1977a) and Brennan and Schwartz (1977)’s predictions of call policy. The literatures on out-of-the-money calls are limited. Among the works trying to elucidate this phenomenon, both of Sarkar (2003) and Chen et al. (2013) propose that firms trigger out-of-the-money calls to pre-empt default. However, this explanation contradicts Bhattacharya (2012)’s arguments that firms may trigger out-of-the-money calls when the levels of interest rates fall or when they want to pre-empt conversion to avoid equity dilution. In particular, the former argument preserves the rationale for redeeming callable nonconvertible bonds as in King and Mauer (2000), whereas the latter are captured in Bechmann et al. (2014) as they observe “avoid-dilution arguments” when out-of-the-money calls are announced.\(^4\) To bridge the gap between the aforementioned two streams of works on demonstrating out-of-the-money calls, we propose the interaction effect that reveals the conflict of interest between the bond issuing firm and bond holder by adopting the Kifer (2000)’s game option model under Acharya and Carpenter (2002, hereafter AC)’s framework.

AC analyze a firm’s policies to call or default its non-perpetual bond when the interest rate and the firm’s asset value are stochastic. Within their framework, a callable defaultable bond is equivalent to the bond holder’s portfolio that comprises longing an otherwise identical default-free bond but selling the issuing firm a combined option to call or default. When the firm exercises its call option to redeem the bond prematurely, the value of its default option is destroyed. Similarly, when the firm exercises its default option to trigger bankruptcy, the value of its call option is destroyed. Thus, AC argue that potential call delays default and potential default delays call if both of the two options belong to the firm behaving to maximize its equity holder’s value. The issuing firm’s tradeoff further implies that the more valuable the default or call option the more significant the delays in calling or defaulting. Rather than documenting the interaction that reveals the firm’s tradeoff, Sirbu et al. (2004) and Sirbu and Shreve (2006) establish another framework to explicitly recognize a zero-sum two-person game for the circumstances that the callable defaultable bond includes the bond holder’s conversion option. Though adopting constant interest rates, their framework suggests the possibility of out-of-the-money calls but do not provide explanations for this firm behavior. Extending Sirbu et al. (2004) to nonzero-sum game, Chen et al. (2013) argue that out-of-the-money calls are triggered only for pre-empting default.

To the best of our knowledge, this is the first paper that investigates the policy of calling and

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\(^3\)According to Grundy and Verwijmeren (2016), more than 82% of the convertible bond issued between 2003 and 2006 are dividend-protected, and Finnerty (2015) confirms that the predominance has continued through 2013.

\(^4\)Bechmann et al. (2014) state “Several out-of-the-money call announcements explicitly mention the ‘avoid-dilution arguments’ as the main reason for the call. For example, on November 5, 1997, BancTec Inc. made an out-of-the-money convertible bond call. In the call announcement, the firm said “the call will be funded with internal capital and existing lines of credit and should allow the company to avoid dilution of 1.5 million shares,” which should be compared to the 21.1 million shares outstanding. Besides, they also find in their sample that relatively more out-of-the-money calls occur in 2002 and 2003 because the levels of interest rates decrease in the period.”
converting a non-perpetual bond when both of the interest rate and the issuing firm’s asset value are stochastic.\textsuperscript{5} We apply AC’s framework and treat a callable convertible bond as the bond holder’s portfolio that comprises longing an otherwise identical default-free bond and a game option (or Israeli option) granting both of the holder and the bond issuing firm the right to terminate the bond prematurely. Specifically, the pricing framework for a game option developed by Kifer (2000) and Bielecki et al. (2008) is exploited and adequately adapted for the analyses as in AC. In a perfect capital market accommodating the Modigliani-Miller theorem, the equity holder’s value can be maximized when the bond holder’s value is minimized, and vice versa. Thus, we theoretically document that, as a callable bond without a call notice period, call of a callable convertible bond will be triggered earlier than that of an otherwise identical callable nonconvertible bond, because potential conversion at the bond holder’s discretion precipitates call to maximize the equity holder’s value. This precipitation would become salient further as the conversion option becomes valuable. On the other hand, as a convertible bond, conversion of a callable convertible bond will be triggered earlier than that of an otherwise identical noncallable convertible bond, because potential call at the equity holder’s discretion precipitates conversion to maximize the bond holder’s value. This precipitation would again become salient as the call option becomes valuable. The intuition following this interaction suggests that the firm would call early to destroy the bond holder’s conversion option at expense of its equity holder’s default option when the former option is valuable but the latter is invaluable. That is, to pre-empt conversion (to avoid too much equity dilution), the firm has more incentives to redeem the convertible and thus trigger an out-of-the-money call or a forced in- or out-of-the-money conversion. On the other hand, to pre-empt redemption (to avoid too much loss of conversion value), the bond holder has more incentives to convert the callable early. This precipitation then makes the conversion policy insensitive to the levels of interest rates, because the conversion would be triggered either voluntarily or compulsively as levels of interest rates are relatively high or low. Instead of the pre-empt default explanation for out-of-the-money calls, we argue that both of the firm’s call and bond holder’s conversion can simultaneously pre-empt potential default on the callable convertible. Our framework can not only capture Bhattacharya (2012)’s insight into the early call decision but provide another tunnel to elucidate Jensen and Pedersen (2016)’s observation about the early conversion decision. The interaction between a bond issuing firm’s call and default decision is empirically consolidated by Jacoby and Shiller (2010), and we in this paper provide empirical confirmation for the interaction between the firm’s call decision and the bond holder’s conversion decision as the complement to the literature on callable convertible bonds.

The remainder of this paper proceeds as follows. Section 2 lays out our baseline assumptions for our model, including the dynamics of interest rates and the issuing firm’s asset value. Following this, we describe as in AC the values of the bond with the embedded option only granting the firm additional right, the call and default options. In Section 3, we evaluate the bond with the game option granting both of the bond holder and the firm additional right, the conversion options and the combined options to call or default. For comparison purpose, we first evaluate the bond only with the conversion option. Then the corresponding optimal conversion policy will be emphasized to shed light on the difference.

\textsuperscript{5}Ingersoll (1977a) and Brennan and Schwartz (1977) analyze the optimal call policy for the issuing firm of a non-perpetual convertible bond considering constant interest rates. Some evaluate a non-perpetual convertible bond with an exogenous specifying call policy. For example, Yagi and Sawaki (2005) considers constant interest rates, whereas Brennan and Schwartz (1980) and Finnerty (2015) consider stochastic interest rates. Sarkar (2003), Liao and Huang (2006) and Chen et al. (2013) analyze the optimal call policy of a perpetual convertible bond considering constant interest rates. Some provide pricing framework only. For example, Hung and Wang (2002), Gapeev and Künn (2005), Chambers and Lu (2007), Bielecki et al. (2008) and many others (see Batten et al., 2014; Kwok, 2014).
in call, default, and conversion decision once the call and default options are incorporated in the bond. Finally, we summarize our theoretical results as theorems and list the empirical implications based on them. Section 4 reports the results of our empirical tests with callable nonconvertible, noncallable convertible and callable convertible corporate bond data over a twenty-year period. Section 5 concludes this paper. All the material reviews in favor of our analyses and technical issues arising in the text are deferred to the Appendix.

2 Baseline Assumptions

In light of Grundy and Verwijmeren (2016)’s empirical findings that convertible bond pricing models assuming perfect capital market, like Ingersoll (1977a) and Brennan and Schwartz (1977), can fairly predict call policies after the prevalence of anti-dilution provisions since about 2003, we adopt this perfect capital market setting as AC and Finnerty (2015) do. Under this setting, all investors (including equity and bond holders) have equal access to market information and trade continuously without arbitrage opportunities. The Modigliani-Miller theorem (see Modigliani and Miller, 1958) holds due to absence of corporate taxes and bankruptcy costs. All contingent claims values and exercise policies of their embedded options are assessed under the risk neutral valuation method. In this method, an unique risk-neutral probability measure \( \tilde{P} \) that is equivalent to the physical measure \( P \) is constructed to make discounted price processes of all assets under \( \tilde{P} \) be martingale processes (see Harrison and Kreps, 1979).

We follow AC to model the dynamics of the stochastic interest rate and the bond issuing firm’s asset value under the probability measure \( \tilde{P} \) described as follows.

\[
 dr_t = \mu(r_t, t) dt + \sigma(r_t, t)d\tilde{Z}_t,
\]

where \( \tilde{Z}_t \) denotes a standard Brownian motion. \( \mu \) and \( \sigma \) are continuous functions satisfying Lipschitz and linear growth condition:

\[
|\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq L|x - y|,
\]
\[
|\mu(x, t)| + |\sigma(x, t)| \leq L(1 + |x|),
\]

where \( L \) denotes a given constant, and \( x, y, \) and \( t \) can be arbitrary positive real numbers. The bond issuing firm’s asset value at time \( t \), \( V_t \), is assumed to follow a log-normal diffusion process

\[
 \frac{dV_t}{V_t} = (r_t - \gamma_t)dt + \phi_t d\tilde{W}_t,
\]

where \( \gamma_t \) denotes the payout ratio for serving contractually-obligated debt payments and dividends at time \( t \), \( \phi_t \) represents the firm value volatility, and \( \tilde{W}_t \) refers to a standard Brownian motion under \( \tilde{P} \). Note that \( \gamma_t \geq 0, \phi_t > 0, \) and \( \rho_t \in [-1, 1] \) are deterministic functions of time. For ease of analyses, we follow AC by considering a firm with one outstanding non-perpetual bond. Protective bond covenants included in the bond deter the firm from arbitrarily adjusting both the payout policy \( \gamma_t \) and the investment policy proxied by the firm value volatility \( \phi_t \). Under these restrictions, the available cash payout \( \gamma_t V_t \) is first used to fulfill interest payments, and the rest (if any) is then distributed to equity holders as dividends. On the other hand, if the payout fails to meet interest payments, we follow Chen (2010) by assuming that the firm will issue new equity to cover the shortfall. In addition, default
decisions are viewed as options owned by the firm as defined in AC. In other words, bond provisions that force the firm to default when its asset value $V_t$ or cash flow $\gamma_tV_t$ falls below certain thresholds, like minimum net worth covenants (see Leland, 1994) or net cash flow covenants (see Fan and Sundaresan, 2000), are assumed to be absent in our analyses.

Values of contingent claims can be evaluated as conditional expectations of discounted payoffs, and claim holders will exercise their own options optimally to maximize their benefits. The discount factor from time $\tau$ back to $t$ can be expressed as

$$\beta_{t,\tau} \equiv e^{-\int_{\tau}^{t} r_s ds}.$$ (3)

A defaultable and/or callable bond that grants the bond issuing firm the right to call back its bond or to default is analyzed by AC as a host bond, a default-free straight bond with unit face value, plus an embedded option. By assuming that this host bond pays a fixed continuous coupon $c$ and matures at time $T$, the value of the host bond at time $t \in [0, T]$ can be evaluated as

$$P_t = \tilde{E} \left[ c \int_{t}^{T} \beta_{t,s} ds + 1 \cdot \beta_{t,T} \mid \mathcal{F}_t \right],$$ (4)

where $\tilde{E}[\cdot]$ represents the expectation under $\tilde{P}$, and $\{\mathcal{F}_t\}$ be a filtration generated by the evolutions of interest rate and the firm’s asset value defined in Equations (1) and (2). The value of an embedded option $f_X$ can be interpreted as a function of the host bond price $p$, the firm’s asset value $v$, and the current time $t$. AC then express the value of a bond with the embedded option as

$$p_X = p - f_X(p, v, t),$$ (5)

where the subscript $X$ is $C$ for a callable default-free (i.e., pure callable) bond, $D$ for a noncallable defaultable (i.e., pure defaultable) bond, and $CD$ for a callable defaultable bond. In a perfect capital market, Modigliani-Miller theorem asserts that the optimal option execution policy for the bond issuing firm to maximize its equity value is equivalent to minimizing the bond value; in other words, the firm would optimize its exercise policy by maximizing the value of the embedded option $f_X(p, v, t)$ as

$$f_X(p, v, t) = \sup_{t \leq \tau \leq T} \tilde{E} \left[ \beta_{t,\tau} (P_{\tau} - \kappa(V_{\tau}, \tau))^+ \mid \mathcal{F}_t \right],$$ (6)

where the strike price $\kappa(V_{\tau}, \tau) = k_{\tau}, V_{\tau}$, and $k_{\tau} \wedge V_{\tau}$ for $X = C$, $D$, and $CD$, respectively. The stopping time $\tau_X$ of the filtration $\{\mathcal{F}_t\}$ denotes the optimal option exercise time that satisfies the condition

$$\tau_X = \inf\{t \in [0, T] : f_X(p, v, t) = (p - \kappa(v, t))^+\}.$$

Specifically, the embedded option value $f_X(p, v, t)$ is larger than or equal to the exercise value $(p - \kappa(v, t))$, and the option is exercised once the option value is exact the exercise value.

AC theoretically document the interaction effect between call and default decisions made by a bond issuing firm as described in Appendix A. They show that the presence of the firm’s default option would delay its call decision, since its call announcement would simultaneously destroy the value of its default option. Thus the timing for redeeming a callable defaultable bond tends to be later than the timing for redeeming an otherwise identical pure callable bond. This delay would be more salient as the firm’s creditworthiness deteriorates due to appreciation of the default option. Similarly, the
presence of the firm’s call option would delay its default decision, since its default announcement would simultaneously destroy the value of its call option. Thus the timing for defaulting a callable defaultable bond tends to be later than the timing for defaulting an otherwise identical pure defaultable bond. This delay would be more significant as the firm’s creditworthiness improves due to appreciation of the call option. This paper would focus on exercise policies of various embedded options simultaneously owned by bond issuing firms and bond holders and provide theoretical analyses for game options with empirical confirmations.

3 Evaluating Callable Convertible Bonds

Instead of the call and default options that decrease the bond value for granting the bond issuing firm additional right, a conversion option increases the bond value for granting the bond holder the right to convert the bond into the firm’s common stocks. Following AC’s framework, we analyze a callable convertible bond, a bond simultaneously granting the firm and bond holder additional right, by treating it as an otherwise identical host bond plus a game option. Similar to the combined option to call or default that has an interaction revealing the firm’s tradeoff, the firm’s option to call and the bond holder’s option to convert also has an interaction though the conversion right has priority. In this section, we first characterize the properties of the conversion option by analyzing a bond only with a conversion option. For comparison purpose, the corresponding optimal conversion policy will then be emphasized to shed light on the difference in call, default, and conversion decisions once the call and default options are included in the bond.

3.1 A Bond Only with a Conversion Option

3.1.1 Option and Bond Valuation

A bond coupling with a conversion option allows the bond holder to convert the bond into the bond issuing firm’s common stocks worth a fraction \( z \) of the firm’s asset value.\(^6\) Intuitively, a bond only with a conversion option (i.e., a pure convertible bond) can be regarded as the bond holder’s portfolio that comprises longing an otherwise identical host bond and an option to exchange the host bond for the common stocks at any time prior to the bond maturity \( T \). Thus, its price at time \( t \), \( t \in [0, T] \), can be expressed as

\[
p_{CB} = p + f_{CB}(p,v,t),
\]

where the subscript “CB” is the abbreviation of “Convertible”. Similar to that in AC, \( f_{CB}(p,v,t) \) represents the optimal option value at current time \( t \) associated with the conversion policy to maximize the current bond holder’s value \( p_{CB} \). That is,

\[
f_{CB}(p,v,t) = \sup_{t \leq \tau_{CB} \leq T} \mathbb{E} \left[ \beta_{t,\tau_{CB}} (zV_{\tau_{CB}} - P_{\tau_{CB}})^+ \bigg| \mathcal{F}_t \right],
\]

\(^6\)The setting of the constant conversion fraction follows Sarkar (2003), Liao and Huang (2006), Sirbu and Shreve (2006) and Chen et al. (2013). This simplification implies that the bond holder is protected by the anti-dilution covenant once the bond issuing firm raises additional shares to repay its interest payments. If the firm’s payout at time \( t \), \( \gamma_t V_t \), is less than the bond coupon payment, \( c \), the issuing firm needs to raise new shares to cover the shortfall. The new shares dilute the value of each existing share, but the bond with the conversion value worth constant fraction of the firm’s asset value is indifferent to this effect.
where $zV_{\tau CB}$ is the conversion value at time $\tau_{CB}$ with $0 < z < 1$ and the corresponding optimal stopping time of the filtration $\{F_t\}$ is defined as

$$\tau_{CB} = \inf\{t \in [0, T] : f_{CB}(p, v, t) = (zv - p)^+\},$$

ensuring the condition

$$f_{CB}(p, v, t) \geq (zv - p)^+$$

for $(p, v, t) \in R^+ \times R^+ \times [0, T]$. The moneyness of the conversion option at time $t$ is determined through the sign of the value $zv - p$. In the following theorem, we summarize the properties of $f_{CB}(p, v, t)$.

**Theorem 1** Denote two different host bond prices at time $t$ by $p^{(1)}$ and $p^{(2)}$ and two different firm’s asset values at time $t$ by $v^{(1)}$ and $v^{(2)}$. The following properties hold for the $f_{CB}(p, v, t)$.

1. $p^{(1)} > p^{(2)} \Rightarrow f_{CB}(p^{(1)}, v, t) < f_{CB}(p^{(2)}, v, t)$.
2. $v^{(1)} < v^{(2)} \Rightarrow f_{CB}(p, v^{(1)}, t) < f_{CB}(p, v^{(2)}, t)$.
3. $p^{(1)} \neq p^{(2)} \Rightarrow -1 \leq \frac{f_{CB}(p^{(2)}, v, t) - f_{CB}(p^{(1)}, v, t)}{p^{(2)} - p^{(1)}} < 0$. (Put delta inequality)
4. $v^{(1)} \neq v^{(2)} \Rightarrow 0 < \frac{f_{CB}(p, v^{(1)}, t) - f_{CB}(p, v^{(2)}, t)}{v^{(1)} - v^{(2)}} < z$. (Call delta inequality)

According to Equation (8), the embedded conversion option can be viewed as the put option on the host bond price, and the put delta inequality entails that the value of the embedded option decreases with the underlying host bond price at a slower rate. Similarly, the conversion option can be regarded as the call option on the conversion value measured in the bond issuing firm’s asset value, and the call delta inequality entails that the value of the embedded option increases with the asset value at a slower rate. With all of the items and Equation (7), this theorem summarizes that the value of $p_{CB}$ increases with the host bond price and firm’s asset value but at slower rate. The proof of all parts of this theorem displayed in Appendix A.1 are based on AC’s corollaries and lemma listed in Appendix A and our extended corollaries listed in Appendix B.

### 3.1.2 Optimal Conversion Policy

Once we characterize the properties of $f_{CB}(p, v, t)$, the optimal conversion policy for the bond holder can then be theoretically portrayed. In Theorem 2, we first establish the existence of the critical boundary level to trigger conversion in terms of the host bond price and the bond issuing firm’s asset value, respectively. With the existence of these boundaries, we then in Theorem 3 describe their shapes to characterize the optimal conversion policy for the bond holder. The proof of the two theorems is based on Theorem 1 and is displayed in Appendix C.1.

**Theorem 2** Let $t \in [0, T]$.

1. Consider the firm’s asset value at time $t$ is $v > 0$. If there is any host bond price at time $t$, $p > 0$, such that it is optimal to exercise the conversion option given that the state at time $t$ is the pair of the host bond price and firm’s asset value $(p, v)$, then there exists a critical host bond price $b_{CB}(v, t) < zv$ such that it is optimal to convert the pure convertible bond at time $t$ if and only if $p \leq b_{CB}(v, t)$.
2. Consider the host bond price at time $t$ is $p > 0$. If there is any firm’s asset value at time $t$, $v > 0$, ...
such that it is optimal to exercise the conversion option given that the state at time $t$ is $(p, v)$, then there exists a critical firm’s asset value $v_{CB}(p, t)$, $zv_{CB}(p, t) > p$, such that it is optimal to convert the pure convertible bond at time $t$ if and only if $v \geq v_{CB}(p, t)$.

This theorem points out the orientation that the bond holder tends to convert the bond when the level of interest rate is relatively high or when the bond issuing firm is relatively healthy. Besides, it implies that out-of-the-money conversion is always not the optimal strategy for the bond holder because $b_{CB}(v, t) < zv$ and $zu_{CB}(p, t) > p$.

**Theorem 3** For each $t \in [0, T)$

1. $v^{(1)} < v^{(2)} \Rightarrow b_{CB}(v^{(1)}, t) \leq b_{CB}(v^{(2)}, t)$.
2. $p^{(1)} > p^{(2)} \Rightarrow v_{CB}(p^{(1)}, t) \geq v_{CB}(p^{(2)}, t)$.

This theorem indicates that the critical host bond price increases with the firm’s asset value. This implies that it takes higher interest rate to trigger conversion when the issuing firm is unhealthy. Similarly, the critical firm’s asset value increases with the host bond price. This on the other hand implies that it takes higher firm’s asset value to trigger conversion when the level of interest rate is low.

### 3.2 A Bond with a Game Option

#### 3.2.1 Option and Bond Valuation

A bond simultaneously coupling with an option to convert and a combined option to call or default can be viewed as the bond holder’s portfolio that comprises longing an otherwise identical host bond and a game option granting both the bond issuing firm and bond holder the right to terminate the bond prematurely. Both parties will behave to maximize the values of their holdings. That is, to minimize the current bond value, the firm choose to call or default the bond at any time before the bond maturity $T$ or the bond holder’s voluntary conversion. However, to maximize the bond value subject to this strategy, the bond holder chooses to convert the bond before or once the firm announces call. To capture the interaction revealing this conflict of interest, we express the price of this callable convertible bond at current time $t$, $t \in [0, T]$, as

$$p_{C^0B^0C^0D^0} = p + f_{C^0B^0C^0D^0}(p, v, t),$$

where the subscript “$C^0B^0C^0D^0$” is the abbreviation of “Convertible Callable Defaultable”, and the value of the embedded option is expressed as

$$f_{C^0B^0C^0D^0}(p, v, t) = \mathsf{E} \left[ \beta_t \mathbb{I}_{\tilde{\tau}_{CB^*} > \tilde{\tau}_{CD^*}} G(\tilde{\tau}_{CB^*}, \tilde{\tau}_{CD^*}) \bigg| \mathcal{F}_t \right].$$

The function $G(\tilde{\tau}_{CB^*}, \tilde{\tau}_{CD^*})$ is the payoff as the option is exercised and

$$G(\tilde{\tau}_{CB^*}, \tilde{\tau}_{CD^*}) = \left(zV_{\tilde{\tau}_{CB^*}} - P_{\tilde{\tau}_{CB^*}}\right) \mathbb{I}_{\tilde{\tau}_{CB^*} \leq \tilde{\tau}_{CD^*}} - \left(P_{\tilde{\tau}_{CD^*}} - \kappa \left(V_{\tilde{\tau}_{CD^*}}, \tilde{\tau}_{CD^*}\right)\right) \mathbb{I}_{\tilde{\tau}_{CD^*} < \tilde{\tau}_{CB^*}}$$

(10)
with the bond holder’s optimal conversion time $\tilde{\tau}_{CB^*}$ subject to the firm’s optimal call or default strategy characterized as the stopping time $\tilde{\tau}_{CD^*}$, and

$$\tilde{\tau}_{CB^*} = \inf\{t \in [0, T] : f_{CBCD}(p, v, t) = (zv - p)\},$$
$$\tilde{\tau}_{CD^*} = \inf\{t \in [0, T] : f_{CBCD}(p, v, t) = -(p - \kappa(v, t))\}.$$ 

Notice that call announcement may force conversion because the conversion right has priority over the redemption right. Thus to clarify, the state $\tilde{\tau}_{CB^*} < \tilde{\tau}_{CD^*}$ in Equation (10) refers to voluntary conversion, and $\tilde{\tau}_{CB^*} = \tilde{\tau}_{CD^*}$ refers to forced conversion.\(^7\) Besides, this payoff function also allows negative execution payoffs: $(zV_{\tilde{\tau}_{CB^*}} - P_{\tilde{\tau}_{CB^*}})$ and $(P_{\tilde{\tau}_{CD^*}} - \kappa(V_{\tilde{\tau}_{CD^*}}, \tilde{\tau}_{CD^*}))$ to capture an out-of-the-money conversion and call of the bond as found in empirical studies.

To examine whether the option holders’ exercise policies are influenced by the existence of other embedded options, we first compare the values of the embedded options discussed above. For convenience, let

$$f_{C\text{BC}}(p, v, t) \equiv -f_{C\text{DCB}}(p, v, t).$$

Intuitively,

$$f_{C\text{BCD}}(p, v, t) \leq f_{C\text{B}}(p, v, t),$$
$$f_{C\text{DCB}}(p, v, t) \leq f_{C\text{D}}(p, v, t),$$

because the game option grants each side of the counterparties additional right to terminate the bond prematurely at expense of the other. Other than the comparison, another two types of embedded game options are considered for clarification purpose. First, following the aforementioned discussion, the value of an otherwise identical default-free callable convertible bond at time $t$ can be regarded as bond holder’s portfolio decomposed as

$$p_{C\text{BC}} = p + f_{C\text{BC}}(p, v, t),$$

where the firm’s optimal call time subject to the bond holder’s optimal conversion time $\tilde{\tau}_{CB^*}$ is denoted by $\tilde{\tau}_{C^*}$ and the corresponding exercise price $\kappa(V_{\tilde{\tau}_{C^*}}, \tilde{\tau}_{C^*})$ in Equation (10) is redefined as $k_{\tilde{\tau}_{C^*}}$ due to the absence of the default risk. Second, the value of an otherwise identical defaultable noncallable convertible bond is

$$p_{C\text{BD}} = p + f_{C\text{BD}}(p, v, t),$$

where the firm’s optimal default time subject to bond holder’s optimal conversion time $\tilde{\tau}_{CB^*}$ is denoted by $\tilde{\tau}_{D^*}$ and the exercise price $\kappa(V_{\tilde{\tau}_{D^*}}, \tilde{\tau}_{D^*})$ is redefined as $V_{\tilde{\tau}_{D^*}}$ due to the absence of the call risk. We especially note that the conversion fraction $0 < z < 1$ excludes the status $\tilde{\tau}_{CB^*} = \tilde{\tau}_{D^*}$, because conversion once default is always the suboptimal policy for the bond holder. The values of the four

\(^7\)However, given the conversion fraction $0 < z < 1$, it is always optimal for the bond holder to keep the bond rather than to convert it once the bond issuing firm announces default. That is because the bond holder will receive $v$ and $v > zv$ once the callable convertible is defaulted at time $t$. Thus, default never forces conversion.
options then relate as follows:

\[ f_{CBC}(p, v, t) \vee f_{CBD}(p, v, t) \leq f_{CB}(p, v, t), \]  
\[ f_{CBC}(p, v, t) \wedge f_{CBD}(p, v, t) \geq f_{CBD}(p, v, t). \]

Similar to the relation illustrated as Inequality (12), Inequality (16) entails the intuition that the value of a simple option to convert is always greater than that of an otherwise identical game option granting the bond issuing firm additional right. More than this, Inequality (17) further argues that the more option right the bond holder grants the firm, the less value the game option is. This argument can be rigorously proved as follows.

**Proof.** To confirm the Inequality (17), we need to prove \( f_{CBC}(p, v, t) - f_{CBD}(p, v, t) \geq 0 \) and \( f_{CBD}(p, v, t) - f_{CBD}(p, v, t) \geq 0 \). Here we display the technique to prove the first inequality. Similar technique is applied to confirm the second inequality and the proof will be listed in Appendix C.2. In the beginning of the proof, we note that \( f_{CBC}(p, v, t) \) is the value of the game option at time \( t \) associated with the bond holder’s optimal conversion time \( \hat{\tau}_{CBC} \) subject to the bond issuing firm’s optimal call time \( \hat{\tau}_{CB^*} \), and \( \hat{\tau}_{CB^*}, \hat{\tau}_{C^*} \in [t, T] \). For clarification, we on the other hand note that \( f_{CBD}(p, v, t) \) is the value of another game option associated with the bond holder’s optimal conversion time \( \hat{\tau}_{CB} \) subject to the firm’s optimal call or default time \( \hat{\tau}_{CD^*} \), and \( \hat{\tau}_{CB}, \hat{\tau}_{CD^*} \in [t, T] \). Then given that the state at time \( t \) is the pair of the host bond price and issuing firm’s asset value \((p, v)\), let \( \tau_{C^*}, \tau_{CB^*} \in [t, T] \), be the optimal time to call the default-free callable convertible bond subject to the strategy to convert the bond \( \tau_{CB^*}, \tau_{CB} \in [t, T] \), which is a feasible but not optimal stopping time at this state. However, given the same state, the \( \tau_{CB^*} \) is the optimal time to convert the otherwise identical defaultable callable convertible bond subject to the strategy to call the bond \( \tau_{CD^*} \), \( \tau_{CD^*} = \tau_{C^*} \), which is on the other hand a feasible but not optimal stopping time at this state. Thus, we have

\[
\begin{align*}
&f_{CBC}(p, v, t) - f_{CBD}(p, v, t) \\
&\geq \underbrace{\mathbb{E}\left[ \beta_{t, \tau_{CB^*}} (z V_{\tau_{CB^*}} - P_{\tau_{CB^*}}) I_{(\tau_{CB^*} \leq \tau_{C^*})} - \beta_{t, \tau_{C^*}} (P_{\tau_{C^*}} - k_{\tau_{C^*}}) I_{(\tau_{C^*} < \tau_{CB^*})}\right]|_{\mathcal{F}_t}}_{A} \\
&\quad - \underbrace{\mathbb{E}\left[ \beta_{t, \tau_{CB}} (z V_{\tau_{CB}} - P_{\tau_{CB}}) I_{(\tau_{CB} \leq \tau_{C^*})} - \beta_{t, \tau_{C^*}} (P_{\tau_{C^*}} - \kappa (V_{\tau_{C^*}}, \tau_{C^*})) I_{(\tau_{C^*} < \tau_{CB})}\right]|_{\mathcal{F}_t}}_{B} \\
&= \mathbb{E}\left[ (\beta_{t, \tau_{CB^*}} (P_{\tau_{CB}} - \kappa (V_{\tau_{C^*}}, \tau_{C^*})) - \beta_{t, \tau_{C^*}} (P_{\tau_{C^*}} - k_{\tau_{C^*}})) I_{(\tau_{C^*} < \tau_{CB^*})}\right]|_{\mathcal{F}_t} \\
&\geq 0.
\end{align*}
\]

Part \( A \) of Inequality (18) is less than \( f_{CBC}(p, v, t) \) because, rather than the optimal conversion strategy \( \hat{\tau}_{CB^*} \) subject to firm’s optimal call strategy \( \hat{\tau}_{C^*}, \tau_{CB^*} \) is the suboptimal strategy subject to the firm’s optimal strategy \( \tau_{C^*} \) to call the default-free callable convertible given that the state at time \( t \) is \((p, v)\). However, part \( B \) is greater than \( f_{CBD}(p, v, t) \) because, rather than the optimal conversion strategy \( \hat{\tau}_{CB} \) subject to firm’s optimal call or default strategy \( \hat{\tau}_{CD^*} \), \( \tau_{CB^*} \) is the optimal strategy subject to the firm’s suboptimal strategy to call the defaultable callable convertible given the same state at time \( t \). Thus, Inequality (18) is established. Equation (19) is obtained after rearranging Inequality (18). The value \( \beta_{t, \tau_{C^*}} (k_{\tau_{C^*}} - \kappa (V_{\tau_{C^*}}, \tau_{C^*})) \) in Equation (19) is nonnegative almost surely because \( \kappa (V_{\tau_{C^*}}, \tau_{C^*}) \) is the minimum of \( V_{\tau_{C^*}} \) and \( k_{\tau_{C^*}} \), and that makes the expectation value in Equation (20) nonnegative. This thus ensures \( f_{CBC}(p, v, t) - f_{CBD}(p, v, t) \geq 0 \).
Following the relation illustrated as Inequality (17), we then summarize the properties of \( f_{\text{CBCD}}(p, v, t) \), \( f_{\text{CBC}}(p, v, t) \), and \( f_{\text{CBD}}(p, v, t) \) through the following theorem to further clarify their difference.

**Theorem 4** Denote two different host bond prices at time \( t \) by \( p^{(1)} \) and \( p^{(2)} \) and two different firm’s asset value at time \( t \) by \( v^{(1)} \) and \( v^{(2)} \). The following properties hold for the \( f_{\text{CBCD}}(p, v, t) \), \( f_{\text{CBD}}(p, v, t) \) and \( f_{\text{CBC}}(p, v, t) \).

1. \( p^{(1)} > p^{(2)} \Rightarrow f_{\text{CBCD}}(p^{(1)}, v, t) < f_{\text{CBCD}}(p^{(2)}, v, t) \).
2. \( p^{(1)} > p^{(2)} \Rightarrow f_{\text{CBD}}(p^{(1)}, v, t) < f_{\text{CBD}}(p^{(2)}, v, t) \).
3. \( v^{(1)} < v^{(2)} \Rightarrow f_{\text{CBCD}}(p, v^{(1)}, t) < f_{\text{CBCD}}(p, v^{(2)}, t) \).
4. \( v^{(1)} < v^{(2)} \Rightarrow f_{\text{CBD}}(p, v^{(1)}, t) > f_{\text{CBD}}(p, v^{(2)}, t) \).

We here display the proof of the first item in **Theorem 4** based on the corollaries and lemma listed in Appendix A and B. The technique applied is similar to that utilized to prove Inequality (17). The other items of this theorem can also be proved by mimicking the same method, and these proofs will be displayed in Appendix C.2.

**Proof.** In the beginning of the following proof, we recall that given that the state at time \( t \) is \((p, v)\), \( \tau_{\text{CBD}} \) is the bond holder’s optimal conversion time subject to the bond issuing firm’s call or default time \( \tau_{\text{CD}} \), which is also the optimal stopping time at this state.

Let \( p^{(1)} \) and \( p^{(2)} \) be two possible host bond prices at time \( t \) and we consider the case \( p^{(1)} > p^{(2)} \). Given that the state at time \( t \) is \((p^{(1)}, v)\), let \( \tau^{(1)}_{\text{CBD}}, \tau^{(1)}_{\text{CD}} \in [t, T] \), be the optimal conversion time subject to the strategy to call or default the bond \( \tau^{(2)}_{\text{CD}} \), where \( \tau^{(2)}_{\text{CD}} \in [t, T] \), which is a feasible but not the optimal stopping time at this state. However, given that the state at time \( t \) is \((p^{(2)}, v)\), the \( \tau^{(2)}_{\text{CD}} \) is the optimal call or default time subject to the strategy to convert the bond \( \tau^{(1)}_{\text{CBD}} \), which is a feasible but not the optimal stopping time at this state. Thus, we have

\[
\begin{align*}
f_{\text{CBCD}}(p^{(1)}, v, t) &- f_{\text{CBCD}}(p^{(2)}, v, t) \\
&= \mathbb{E} \left[ \beta^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} (z V^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} - p^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} I_{\tau^{(1)}_{\text{CBD}} < \tau^{(2)}_{\text{CD}}, \tau^{(1)}_{\text{CBD}} < \tau^{(1)}_{\text{CBD}}}) - \beta^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} (P^{(1)}_{t, \text{CBD}} - k (V^{(1)}_{t, \text{CBD}}, \tau^{(1)}_{\text{CBD}})) I_{\tau^{(1)}_{\text{CBD}} < \tau^{(1)}_{\text{CBD}}}) F_t \right] \\
&\quad - \mathbb{E} \left[ \beta^{(2)}_{t, \tau^{(2)}_{\text{CBD}}} (z V^{(2)}_{t, \tau^{(2)}_{\text{CBD}}} - p^{(2)}_{t, \tau^{(2)}_{\text{CBD}}} I_{\tau^{(2)}_{\text{CBD}} < \tau^{(1)}_{\text{CBD}}, \tau^{(2)}_{\text{CBD}} < \tau^{(2)}_{\text{CBD}}}) - \beta^{(2)}_{t, \tau^{(2)}_{\text{CBD}}} (P^{(2)}_{t, \text{CBD}} - k (V^{(2)}_{t, \text{CBD}}, \tau^{(2)}_{\text{CBD}})) I_{\tau^{(2)}_{\text{CBD}} < \tau^{(2)}_{\text{CBD}}}) F_t \right] \\
&\quad + \mathbb{E} \left[ \beta^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} (z V^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} - p^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} I_{\tau^{(1)}_{\text{CBD}} < \tau^{(2)}_{\text{CD}}, \tau^{(1)}_{\text{CBD}} < \tau^{(1)}_{\text{CBD}}}) - \beta^{(1)}_{t, \tau^{(1)}_{\text{CBD}}} (P^{(1)}_{t, \text{CBD}} - k (V^{(1)}_{t, \text{CBD}}, \tau^{(1)}_{\text{CBD}})) I_{\tau^{(1)}_{\text{CBD}} < \tau^{(1)}_{\text{CBD}}}) F_t \right] \\
&\quad \not\leq f_{\text{CBD}}(p^{(1)}, v, t) - f_{\text{CBD}}(p^{(2)}, v, t).
\end{align*}
\]
Part A of Inequality (21) is greater than \( f_{CBD}(p^{(1)}, v, t) \) because, rather than the conversion strategy \( \tau_{CBD}^{(1)} \), the bond holder’s optimal strategy is \( \tau_{CBD}^{(1)} \), subject to the bond issuer’s suboptimal strategy \( \tau_{CBD}^{(2)} \). Given that the state at time \( t \) is \( (p^{(1)}, v) \). However, part B is less than \( f_{CBD}(p^{(2)}, v, t) \) because, rather than the call or default strategy \( \tau_{CBD}^{(2)} \), the bond issuer’s optimal strategy is \( \tau_{CBD}^{(2)} \), subject to the bond holder’s suboptimal strategy \( \tau_{CBD}^{(1)} \) given that the state at time \( t \) is \( (p^{(2)}, v) \). Thus, the Inequality (21) is established. Equation (22) is obtained after rearranging Inequality (21). According to Corollary 3 and 5,

\[
\beta^{(1)}_{t, \tau_{CBD}^{(1)}}, \frac{\partial }{\partial v} V^{(1)}_{t, \tau_{CBD}^{(1)}} - \beta^{(2)}_{t, \tau_{CBD}^{(1)}}, \frac{\partial }{\partial v} V^{(2)}_{t, \tau_{CBD}^{(1)}} = 0, \\
-\beta^{(1)}_{t, \tau_{CBD}^{(1)}}, \frac{\partial }{\partial p} P^{(1)}_{t, \tau_{CBD}^{(1)}} + \beta^{(2)}_{t, \tau_{CBD}^{(1)}}, \frac{\partial }{\partial p} P^{(2)}_{t, \tau_{CBD}^{(1)}} < 0
\]

almost surely, ensuring part C is negative almost surely. Besides, with Corollary 3 and 6, we also know

\[
\beta^{(2)}_{t, \tau_{CBD}^{(2)}}, \frac{\partial }{\partial v} P^{(2)}_{t, \tau_{CBD}^{(2)}} - \beta^{(1)}_{t, \tau_{CBD}^{(2)}}, \frac{\partial }{\partial v} P^{(1)}_{t, \tau_{CBD}^{(2)}} < 0,
\]

and that makes part D positive, negative or equal to zero. Thus, we cannot confirm \( f_{CBD}(p^{(1)}, v, t) - f_{CBD}(p^{(2)}, v, t) < 0 \).

The proof for the \( f_{CBD}(p, v, t) \) is basically the same with all above except that \( \kappa(V_{*}, \tau) = V_{*} \). That is, given that the state at time \( t \) is \( (p^{(1)}, v) \), let \( \tau_{CBD}^{(1)} \in [t, T] \), be the optimal conversion time subject to the strategy to default the bond \( \tau_{CBD}^{(2)} \in [t, T] \), which is feasible but not the optimal stopping time at this state. However, given that the state at time \( t \) is \( (p^{(2)}, v) \), the \( \tau_{CBD}^{(2)} \) is the optimal default time subject to the strategy to convert the bond \( \tau_{CBD}^{(1)} \), which is a feasible but not the optimal stopping time at this state. Thus, we have

\[
f_{CBD}(p^{(1)}, v, t) - f_{CBD}(p^{(2)}, v, t) \\ \leq \mathbb{E} \left[ \beta^{(1)}_{t, \tau_{CBD}^{(1)}}, \frac{\partial }{\partial v} V^{(1)}_{t, \tau_{CBD}^{(1)}} - \beta^{(2)}_{t, \tau_{CBD}^{(1)}}, \frac{\partial }{\partial v} V^{(2)}_{t, \tau_{CBD}^{(1)}} \right] \mathbb{I}_{\left\{ \tau_{CBD}^{(1)} < \tau_{CBD}^{(2)} \right\}} + \mathbb{E} \beta^{(2)}_{t, \tau_{CBD}^{(2)}}, \frac{\partial }{\partial v} P^{(2)}_{t, \tau_{CBD}^{(2)}} - \beta^{(1)}_{t, \tau_{CBD}^{(2)}}, \frac{\partial }{\partial v} P^{(1)}_{t, \tau_{CBD}^{(2)}} \mathbb{I}_{\left\{ \tau_{CBD}^{(2)} < \tau_{CBD}^{(1)} \right\}} < 0
\]

Equation (24) is established according to Corollary 5: Corollary 3 confirms \( \beta^{(2)}_{t, \tau} P^{(2)}_{t, \tau} - \beta^{(1)}_{t, \tau} P^{(1)}_{t, \tau} < 0 \) almost surely, ensuring that \( f_{CBD}(p^{(1)}, v, t) - f_{CBD}(p^{(2)}, v, t) < 0 \).

We especially highlight the Inequality (23) that shows it is the existence of the call provision that makes the embedded game option \( f_{CBD}(p, v, t) \) in callable convertible insensitive to the changes of interest rates.

Similar to \( f_{CB}(p, v, t) \), \( f_{CBD}(p, v, t) \) decreases with the underlying host bond price at a slower rate. However, the presence of a call provision may reverse this relation and extend the upper bound of the put delta inequality. This paves the way for clarifying how the bond issuing firm’s call decision influences the bond holder’s conversion decision. On the other hand, similar to \( f_{CB}(p, v, t) \), \( f_{CBD}(p, v, t) \) increases with the firm’s asset value, but the rate of increase is bounded within \([0, 1]\) rather than \([0, z]\), because the presence of the default option makes this game option more sensitive toward the
firm’s creditworthiness. With all of the items and Equation (9) and (15), this theorem summarizes that $p_{\text{CBCD}}$ increases with the firm’s asset value at a slower rate. However, unlike $p_{\text{CB}}$ and $p_{\text{CBD}}$, $p_{\text{CBCD}}$ may be insensitive toward the changes in the levels of interest rates, because $f_{\text{CBCD}}(p,v,t)$ is insensitive toward to the changes in host bond prices due to the presence of the call provision.

### 3.2.2 Optimal Call and Conversion Policy

Before arguing the optimal call and conversion policy, we first determine the region in terms of the firm’s asset value that both of the option holders may have incentives to keep the callable convertible bond alive.

**Proposition 1** Let $t \in [0,T)$. If it is optimal for both of the bond issuing firm and bond holder to keep the callable convertible bond alive at time $t$, then the corresponding firm’s asset value at time $t$ must be $v \in (0, \frac{k}{z})$.

The brief proof of this proposition is listed below.

**Proof.**

If it is optimal for both of the firm and bond holder to keep the callable convertible bond alive at time $t$, $t \in [0,T)$, the corresponding value of the embedded game option obeys

$$f_{\text{CBCD}}(p,v,t) > zv - p,$$
$$f_{\text{CDCB}}(p,v,t) > p - \kappa(v,t).$$

Then it follows

$$zv - p < f_{\text{CBCD}}(p,v,t) < \kappa(v,t) - p,$$

for $f_{\text{CBCD}}(p,v,t) \equiv -f_{\text{CDCB}}(p,v,t)$. That implies $zv < \kappa(v,t)$ when the callable convertible bond is alive at time $t$. Specifically, $v > 0$ for the case $\kappa(v,t) = k_l \wedge v = v$ (i.e., the default case), and $v < \frac{k_l}{z}$ for the case $\kappa(v,t) = k_l$ (i.e., the call case).

This proposition suggests that it must be optimal to default at time $t$ once $v = 0$ and optimal to call or convert once $v = \frac{k_l}{z}$. Specifically, the latter case indicates that it is always optimal for the bond issuing firm to exercise the call option once $v = \frac{k_l}{z}$, because the bond value once call is at most equal to $k_l = zv$. This captures Ingersoll (1977a) and Brennan and Schwartz (1977)’s at-the-money call policy. On the other hand, subject to this call policy, it is always optimal for the bond holder to convert the bond once $v = \frac{k_l}{z}$, because the conversion value is at least equal to the call price. That allows us to characterize part of the bond holder’s conversion policy in terms of the firm’s asset value. We denote this conversion policy subject to the at-the-money call by $v_{\text{CB}}^*$, $v_{\text{CB}}^* \equiv \frac{k_l}{z}$ for any host bond price $p > 0$. Thus with this policy, observable conversion of a callable convertible may be irrelevant to the level of interest rate due to the presence of the call provision that grants the firm to right the minimize the bond value. Besides, out-of-the-money conversion observed by Jensen and Pedersen (2016) is possible, because the moneyness of the conversion option, $zv_{\text{CB}}^* - p$, may be negative once the option is compulsively exercised.

Following the conversion policy subject to the at-the-money call implied by **Proposition 1**, we then shift our focus on the optimal exercise policies obeyed within the region $(0, v_{\text{CB}}^*)$. These policies
will be summarized in **Theorem 5** and 6. The corresponding proofs are based on **Theorem 4** and will be listed in **Appendix C.2**.

**Theorem 5** Let \( t \in [0, T] \) and consider \( 0 < v < v_{CB^*} \).

1. Take the firm’s asset value at time \( t \) as given. If there is any host bond price at time \( t \) such that it is optimal to exercise the game option given that the state at time \( t \) is \((p,v)\), then there exists a critical host bond price, \( b_{CB^*}(v,t) \), such that it is optimal to convert the bond at time \( t \) if and only if \( p \leq b_{CB^*}(v,t) \), and there exists another critical host bond price, \( b_{CD^*}(v,t) \), such that it is optimal to call or default the bond at time \( t \) if and only if \( p \geq b_{CD^*}(v,t) \).

2. Take the host bond price at time \( t \), \( p > 0 \), as given. If there is any firm’s asset value at time \( t \) such that it is optimal to exercise the call option given that the state at time \( t \) is \((p,v)\), then there exists a critical firm’s asset value, \( k_t \leq \tilde{v}_{CD^*}(p,t) < v_{CB^*} \), such that it is optimal to call the bond at time \( t \) if and only if \( v \geq \tilde{v}_{CD^*}(p,t) \). Besides, if there is any firm’s asset value such that it is optimal to exercise the default option, then there exists another critical firm’s asset value, \( 0 \leq v_{CD^*}(p,t) \leq k_t \), such that it is optimal to default the bond at time \( t \) if and only if \( v \leq v_{CD^*}(p,t) \).

The existence of \( b_{CB^*}(v,t) \) confirms the presence of premature voluntary conversions of callable convertibles as found in Finnerty (2015). Besides, the existence of \( b_{CD^*}(v,t) \) and \( \tilde{v}_{CD^*}(p,t) \) on the other hand confirm the presence of out-of-the-money calls as in Cowan et al. (1993); Grundy and Verwijmeren (2012); King and Mauer (2014); Bechmann et al. (2014), and it points out the orientation that out-of-the-money calls are usually triggered when the level of interest rate is relatively low as observed in Bechmann et al. (2014). We then argue the shape of these boundaries as follows.

**Theorem 6** For each \( t \in [0, T] \)

1. \( v^{(1)} < v^{(2)} \leq k_t < v_{CB^*} \Rightarrow b_{CD^*} \left(v^{(1)}, t\right) \leq b_{CD^*} \left(v^{(2)}, t\right) \). \hspace{1cm} (Default case)

2. \( k_t < v^{(1)} < v^{(2)} < v_{CB^*} \Rightarrow b_{CD^*} \left(v^{(1)}, t\right) \geq b_{CD^*} \left(v^{(2)}, t\right) \). \hspace{1cm} (Call case)

3. \( v < v_{CB^*} \Rightarrow b_{CB^*} \left(v, t\right) \geq b_{CB} \left(v, t\right) \).

4. \( v < v_{CB^*} \Rightarrow b_{CD^*} \left(v, t\right) \leq b_{CD} \left(v, t\right) \).

Part 1 and 2 of **Theorem 6** describe the default and call policies followed by a firm with a convertible bond. Similar to the policies obeyed by the otherwise identical firm with a nonconvertible bond as displayed in **Theorem 8** in **Appendix A**, the critical host bond price increases with the firm’s asset value as \( v \leq k_t \) but decreases with the asset value as \( v > k_t \). This implies that it on the one hand requires lower interest rate to make a healthier firm default but on the other hand requires lower interest rate to make an unhealthier firm announce a call. If the firm simultaneously owns the call and default options, exercising one of them will destroy the other and this destruction makes the firm delay its decision if it behaves to maximize its equity holder’s value; the delay would get salient as the value destroyed is greater. This captures the interaction revealing the firm’s tradeoff. However, part 4 of this theorem clarifies that the presence of the conversion option precipitates the call decision. This is because, instead of pre-empting default, exercising the call option destroys the conversion option that may trigger value dilution injurious to the existing equity holder as found in Bechmann et al. (2014). To maximize the bond value subject to this call policy, the bond holder would convert its bond earlier by obeying the policy \( v_{CB^*} \) and \( b_{CB^*} (v,t) \) in part 3 as observed in Jensen and Pedersen (2016) to prevent loss of conversion value; this precipitation would get salient as the
Critical Firm Asset Value (v)
Critical Host Bond Price (p)

Call: CD
Convert: CB
Default: CD, CDCB
Call: CDCB
Convert: CDCB

\[ p = k \]
\[ v = k/z \]

Figure 1: Optimal default, call, and conversion boundaries. Optimal default, call, and conversion boundaries are displayed for three otherwise identical 3-year callable defaultable (CD), pure convertible (CB), and callable defaultable convertible (CDCB) bonds with coupon rates 3% and face value 100. The call prices for both of the two callable bonds are their par value. The horizontal dot line is the level of the host bond price \( p = k \), and the vertical dot line is the level of the bond issuing firm’s asset value \( v = k/z \) and \( z = 0.7 \). The black solid line corresponds to \( b_{CD}(v,t) \) and \( b_{CD}^*(v,t) \), the default case. The light gray dash line corresponds to \( b_{CD}(v,t) \), the call case. The dark gray solid line corresponds to \( b_{CD}^*(v,t) \). For host bond prices below \( b_{CD}(v,t) \) and \( b_{CD}^*(v,t) \), it is optimal to keep the bonds. Otherwise, it is optimal to call or default the bonds. The dark gray dash line corresponds to \( b_{CB}(v,t) \). The light gray solid line corresponds to \( b_{CB}^*(v,t) \) and \( v_{CB}^* = k/z \). For host bond prices below \( b_{CB}(v,t) \) and \( b_{CB}^*(v,t) \) and for the firm’s asset value on the right of \( v_{CB}^* \), it is optimal to convert the bonds. Otherwise, it is optimal to keep the bonds. The call and default policies minimize the bond values, whereas the conversion policy maximizes the bond values. The instantaneous short-term rate follows \( dr_t = (\theta_t - ar_t) dt + \sigma d\tilde{Z}_t \). The mean reversion rate for \( r_t \) is \( a = 0.5 \) and the volatility is \( \sigma = 0.1 \). The bond issuing firm’s payout ratio \( \gamma = 0.04 \) and the firm value volatility \( \phi = 0.2 \). The correlation between the dynamics of the asset value and instantaneous short-term rate \( \rho = 0 \).

value destroyed is greater. This captures the interaction revealing the conflict of interest between the firm and bond holder. Finally, we especially note that \( b_{CD}(v,t) > \kappa(v,t) \) as proved in AC, whereas \( b_{CD}^*(v,t) \) might be less than \( k_t \) according to part 4 of this theorem. Figure 1 illustrates the results created by a two-dimensional tree according to the technique in Dai et al. (2013).

3.3 Empirical Implications

Compared with the results in AC and those in Section 3.1, we theoretically document the interaction effect revealing the conflict of interest between the bond issuing firm and bond holder behaving to maximize their own value: the presence of the conversion option precipitates call and, in response to this call policy, the presence of the call provision precipitates conversion; the more valuable the the call and conversion options, the more salient the precipitation. If this interaction effect holds, out-of-the-money calls of convertible bonds is possible, and we can further infer that this type of call is for pre-empting conversion rather than for pre-empting default though either call or conversion can eliminate the possibility of default. In addition, this effect provides another tunnel to explain Jensen
and Pedersen (2016)’s observation about early conversion decision. Based on the results illustrated in Figure 1, we summarize the empirical implications as follows.

**Hypothesis 1.** A callable bond would be called earlier in its call period due to the presence of the conversion option given the level of interest rate, its coupon rate, its issuing firm’s credit quality, and its average call price during the call period. Note that the higher the coupon rate and the better the firm’s credit quality, the earlier the call timing; the higher the level of interest rate and the average call price, the later the call timing.

Subject to the call policy for callable convertible bonds addressed in Hypothesis 1, the conversion policy would be developed as the following hypothesis.

**Hypothesis 2.** A convertible bond would be converted earlier in its conversion period due to the presence of the call option given the level of interest rate, its coupon rate, its issuing firm’s credit quality, and the dividend per bond if converted. Note that the better the firm’s credit quality, the earlier conversion timing. However, the level of interest rate, the coupon rate, and the dividend per bond if converted may be insensitive to the conversion timing.

We especially notice that higher coupon rates lead to higher possibility of call but result in less possibility to trigger conversion. That thus implies higher possibility of bond mature if firms want to force conversion. Similarly, higher dividend per bond if converted lead to less possibility of call but results in higher possibility to trigger conversion that dilutes existing equity holder’s value. That thus implies higher possibility of bond mature if firms behave to maximize the equity holder’s value.

## 4 Empirical Tests

### 4.1 Data

To examine the empirical implications derived from our theoretical framework, we collect the information about the features of dollar-denominated callable nonconvertible, noncallable convertible and callable convertible corporate bonds issued during the period January 1990 – December 2010 from Mergent Fixed Income Securities Database. That includes bond issue dates, maturity dates, principal amounts, coupon rates, ratings with corresponding rating dates, whether the bonds are called, converted and mature, call information (e.g., the first call dates, call effective dates, call price schedules and call frequency), conversion information (e.g., the first conversion dates, conversion effective dates, conversion prices, conversion commodities and quantity of conversion commodities) and other covenant details. We then search for the call announcement date through ABI/INFORM Complete in ProQuest system. Besides, we obtain the information on constant maturity Treasury rates based on Federal Reserve Board’s H.15 Report from Federal Reserve Bank Reports and gather the details on dividends paid by convertible bond issuers from Compustat.

We focus on the fixed and non-resettable coupon-bearing bonds with $1000 par value and 30/360 day count convention. We eliminate puttable, exchangeable or pay-in-kind bonds (see Sarkar, 2003; King and Mauer, 2014) or bonds with credit enhancement. Besides, to avoid the call decision that may be influenced by bond covenants, we exclude the callable bonds with sinking fund, make whole, maintenance and replacement fund, and sudden death par provisions and those with indexed principal redemptions. In addition, to concentrate on the bonds convertible to the common stocks of issuing
firms, we also exclude the bonds convertible to cash, preferred stocks of issuing firms, stocks of other firms, and others. Finally, we choose non-perpetual bonds that can be called or converted at any time within the stated call and conversion periods, and we select only those that are already mature, called and converted to construct our sample.

The remaining bond samples are separated into two groups: the callable and convertible samples. The former sample includes 2687 callable nonconvertibles and 362 callable convertibles that are called or mature, whereas the latter one includes 34 noncallable convertibles and 223 callable convertibles that are converted and mature. We remove the bonds with incomplete information. For example, we eliminate the callable bonds missing valid call price schedules (i.e., the complete set of call prices and dates), and call announcement and effective dates from the callable sample. Specifically, the time spans between announcement and effective dates are limited to 90 days. In addition, we eliminate the convertible bonds missing the information about the issuing firms’ dividend payments, and conversion effective dates from the convertible sample. The final callable sample comprises 2432 callable nonconvertibles (399 mature and 2033 called bonds) and 301 callable convertibles (115 mature and 186 called bonds). The final convertible sample comprises 24 noncallable convertibles (23 mature and 1 converted bonds) and 141 callable convertibles (105 mature and 36 converted bonds; 29 of the 36 converted bonds are identified as forced conversion). Table 1 reports the distribution of the final sample by year issued.

<table>
<thead>
<tr>
<th>Year</th>
<th>Callable Sample</th>
<th>Convertible Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Callable</td>
<td>Callable Convertible</td>
</tr>
<tr>
<td>1990–1995</td>
<td>862</td>
<td>31</td>
</tr>
<tr>
<td>1996–2000</td>
<td>934</td>
<td>152</td>
</tr>
<tr>
<td>2001–2005</td>
<td>421</td>
<td>101</td>
</tr>
<tr>
<td>2006–2010</td>
<td>215</td>
<td>17</td>
</tr>
<tr>
<td>Total</td>
<td>2432</td>
<td>301</td>
</tr>
</tbody>
</table>

Table 1: Distribution of the bond sample by year issued during the period 1990–2010. The bond sample is separated into two subsamples: the callable sample and convertible sample. The callable sample consists of callable nonconvertible and callable convertible bonds issued during the period 1990–2010, and we include only the bond issues that are already called and mature. The convertible sample contains noncallable convertible and callable convertible bonds issued during the same period, and we include only the bond issues that are already converted and mature.

### 4.2 Variables and Descriptive Statistics

Notice that life spans and the lengths of call periods of callable nonconvertible bonds are usually longer than those of callable convertibles. To make the call or conversion time comparable, we standardize the option exercise time by defining the dependent variable, the ratio of time span (\( \text{RatioTS} \)), as follows:

\[
\text{RatioTS} = \frac{\text{Maturity Date} - \text{Effective Date}}{\text{Maturity Date} - \text{First Date}},
\]

where the \text{Effective Date} stands for call/conversion effective date or bond maturity date, and the \text{First Date} represents the first call or conversion date. Note that this ratio is bounded within \([0, 1]\): it equals 1 as the \text{Effective Date} equals the \text{First Date} and equals 0 as the \text{Effective Date} equals the \text{Maturity Date}. Thus, the greater the ratio, the earlier the bond is called or converted in the stated call or converted period.

The following explanatory variables are used in the empirical tests to proxy the parameters of our
We choose 2-year Treasury rate as the proxy for the reference interest rate level, because most of our convertible bonds are short-term or middle-term. We then select the rates in month \(-5\) given that the month 0 is the call, conversion or maturity effective month and treat them as the determinants of the outcomes of our bond samples (i.e., called, converted or mature).\(^8\) The coupon rate is the constant coupon rate on the bond. With our theoretical models, it can thus be predicted that greater \(\text{RatioTS}\) follows the lower interest rate level or the higher coupon rate for the callable sample (i.e., bonds called or mature). However, following the discussion in Section 3.2.2 and in Hypothesis 2, \(\text{RatioTS}\) is insensitive to the interest rate level and the coupon rate for the convertible sample.

Our bond samples include both rated and unrated bonds, and we identify them using the unrated indicator as in Kroszner and Rajan (1994): 0 if the bond is rated and 1 if otherwise. For the rated bonds, we use \(\text{S&P}'s\), Moody’s or Fitch’s bond ratings as proxies for bond issuing firms’ credit quality,\(^9\) and we imitate King and Mauer (2000) to cardinalize the bond ratings as AAA = 1, AA+ = 2... and D = 25. For the unrated bonds that are not rated by \(\text{S&P}\), Moody and Fitch, we follows Lemmon and Roberts (2010)’s observation and regard them as the bonds with the worst rating in the junk bond rating category though this treatment may overly degrade their true credit quality.\(^10\) With the cardinalized ratings and according to our model, it can thus be predicted that greater \(\text{RatioTS}\) follows the lower bond credit rating for both of the callable and convertible samples. Following this prediction, if treating unrated bonds as the worst rating junk bonds overly degrades their credit quality on average, the unrated indicator would be positively related to \(\text{RatioTS}\).

Note that we select the callable bond samples that can be called at any time within the stated call periods. The average levels of call prices during the call periods are then determined by calculating the summation of each call price multiplied by the ratio of the corresponding call period to whole stated call period under 30/360 day count convention. Regarding the dividend per bond, we follow Sarkar (2003) to approximate it as the dividend income per convertible bond if converted for the most recent year prior to the conversion or maturity effective year. Notice that we focus only on the bond samples with $1000 par value. Thus, our model predict that greater \(\text{RatioTS}\) follows the lower average levels of call price during the call period for the callable sample. However, similar to the coupon rate, \(\text{RatioTS}\) is insensitive to the dividend per bond if converted for the convertible sample as discussed in Hypothesis 2.

Finally, for the callable bond samples, we use the convertible indicator to identify whether the callable bond is convertible: 0 if the callable bond is nonconvertible and 1 if otherwise. Similarly, for the convertible bond samples, we use the convertible indicator to identify whether the convertible bond is convertible: 0 if the convertible bond is noncallable and 1 if otherwise. With the indicators, the interaction effect implied by our model then predicts that the coefficient estimates on them is positively related to \(\text{RatioTS}\) for both of the callable and convertible samples given the aforementioned exploratory variables. That is, compared with callable nonconvertible bonds, callable convertible bonds are prone to be called by bond issuing firm in the stated call period. Likewise, compared with noncallable convertible bonds, callable convertible bonds are prone to be converted by bond holders in the stated

---

\(^8\) King and Mauer (2000) select the rates in month \(-2\) or \(-3\) given that the month 0 is the call announcement month, and we limit the time spans between announcement and effective dates to 90 days and choose the rates in month \(-5\) given that the month 0 is the call, conversion or maturity effective month.

\(^9\) That is on the premise that the three rating agencies use similar rating criteria.

\(^10\) According to Molyneux and Shamroukh (1996), the junk bond market consists of bonds rated Ba1 or lower by Moody’s, BB+ or lower by \(\text{S&P}'s\), or unrated. Also, Lemmon and Roberts (2010) empirically identify that speculative grade firms are more profitable than unrated firms “on average” and that they are significantly financially healthier than unrated firms as indicated by higher Altman Z-scores.
conversion period. Table 2 summarizes the sample characteristics.

2.A. Callable Sample

<table>
<thead>
<tr>
<th></th>
<th>Callable</th>
<th>Callable Convertible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity (year)</td>
<td>13.27</td>
<td>6.74</td>
</tr>
<tr>
<td>Call period (year)</td>
<td>8.40</td>
<td>3.73</td>
</tr>
<tr>
<td>Ratio of time span</td>
<td>0.64</td>
<td>0.40</td>
</tr>
<tr>
<td>Treasury rate (%)</td>
<td>2.76</td>
<td>3.13</td>
</tr>
<tr>
<td>Coupon rate (%)</td>
<td>8.32</td>
<td>5.21</td>
</tr>
<tr>
<td>Bond rating</td>
<td>12.11</td>
<td>18.57</td>
</tr>
<tr>
<td>Average call price (%)</td>
<td>101.01</td>
<td>101.28</td>
</tr>
</tbody>
</table>

2.B. Convertible Sample

<table>
<thead>
<tr>
<th></th>
<th>Convertible</th>
<th>Callable Convertible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity (year)</td>
<td>5.64</td>
<td>6.50</td>
</tr>
<tr>
<td>Conversion period (year)</td>
<td>5.60</td>
<td>6.38</td>
</tr>
<tr>
<td>Ratio of time span</td>
<td>0.00</td>
<td>0.15</td>
</tr>
<tr>
<td>Treasury rate (%)</td>
<td>0.97</td>
<td>3.23</td>
</tr>
<tr>
<td>Coupon rate (%)</td>
<td>4.26</td>
<td>5.40</td>
</tr>
<tr>
<td>Bond rating</td>
<td>20.33</td>
<td>21.51</td>
</tr>
<tr>
<td>Dividend per bond</td>
<td>4.53</td>
<td>6.85</td>
</tr>
</tbody>
</table>

Table 2: Descriptive statistics of variables used in the regressions. This table displays the descriptive statistics of variables that would be used in the regression, including mean, median and standard deviation (SD). The panel A is for the callable sample and B is for the convertible sample. The maturity is the bond initial maturity in years. The call period is the length of the period in years during which a callable bond may be called. The conversion period is the length of period in years during which a convertible bond may be converted. The ratio of time span defined for the callable sample is the ratio of time span between the call effective (maturity) date and maturity date to the length of the call period, and that for the convertible sample is the ratio of time span between the bond conversion effective (maturity) date and maturity date to the length of the conversion period. The bond rating is the issuing firm’s cardinalized S&P’s (or Moody’s or Fitch’s) bond rating, and AAA = 1,..., D = 25. The Treasury rate in percent per annum is the 2-year Treasury rate in month −5 given that the month 0 is the call/conversion effective month or the bond maturity month. The coupon rate is the constant coupon rate on a bond issue. The average call price is the weighting average of the stated call prices in percentage of bond par value. The dividend per bond is the dividend income per bond (if converted) for the most recent year before the conversion effective month or the bond maturity month.

4.3 Regression Results

Table 3 displays the predicted signs following the aforementioned section and the regressions of RatioTS on Convertible/Callable indicator, Treasury rate, Coupon rate, Bond rating, Average call price/Dividend per bond, and Unrated indicator for the callable sample of 2733 bonds and the convertible sample of 165 bonds. In regression (2) and (4), we further take the financial indicator as the control variable to identify whether the bond is issued by a highly regulated financial company.

For the callable sample, as predicted, regression (1) indicates that RatioTS is significantly negatively related to Treasury rate, Bond rating, and Average call price but is significantly positively related to Coupon rate. Though the sample statistics in Table 2 shows the ratio of time span for the callable convertible bonds is less than that of the callable nonconvertible bonds on average, it can observed through this regression that the Convertible indicator is significantly positively related to RatioTS by taking the aforementioned exploratory variables as given. That implies callable convertible bonds are prone to be called by bond issuing firm in the stated call period. This pattern is the same even if we take the financial indicator as the control variable in regression (2). On the other hand, for the convertible sample, the regression (3) indicates that the RatioTS is significantly
Table 3: Regression of ratio of time span on explanatory variables. The dependent variable is the ratio of time span. For the callable sample, it is the ratio of time span between the call effective (maturity) date and maturity date to the length of the call period. For the convertible sample, it is the ratio of time span between the bond conversion effective (maturity) date and maturity date to the length of the conversion period. The exploratory variables are defined as follows. The convertible indicator equals to one once the callable bond is convertible, and zero otherwise. The callable indicator equals to one once the convertible bond is callable, and zero otherwise. The Treasury rate is the 2-year rate in month. The coupon rate is the constant coupon rate on the bond. The bond rating is the issuing firm’s cardinalized S&P’s (or Moody’s or Fitch’s) bond rating, and AAA = 1, ..., D = 25. The average call price is the weighting average of the stated call prices in percentage of bond par value. The dividend per bond is the dividend income per bond if converted for the most recent year before the conversion or maturity effective month. The unrated indicator equals to one if the bond is unrated, and zero otherwise. The financial indicator equals to one if the bond is issued by a financial firm, and zero otherwise. T-statistics are listed in parentheses. Asterisks indicate significance levels: *, **, *** signify the 10%, 5% and 1% levels using a two-tailed test.

5 Conclusion

This paper constructs a valuation framework based on a structural model of credit risk for a callable convertible bond associated with the option execution decisions revealing the conflict of interest between the bond issuing firm and the bond holder. We then document that this conflict stimulates the interaction precipitating call and conversion decision as the two option holders behave to maximize negatively related to Bond rating. Taking the relation as given, we find that the Callable indicator is significantly positively related to RatioTS, suggesting callable convertible bonds are prone to be converted by bond holders in the stated conversion period. This pattern is again the same even if we take the financial indicator as the control variable in regression (4). Notice especially that the conversion timing is thus insensitive to the level of interest rate once the interaction effect holds (i.e., both of the Convertible and Callable indicators are significantly positively related to RatioTS).
their own value at expense of the other. The precipitation would be more salient as the values of the two options appreciate. Out-of-the-money call or conversion is even triggered following this interaction effect, and it is the effect that makes the conversion policy insensitive to the level of interest rate. We thus address that a bond issuing firm calls early to pre-empt conversion and the corresponding bond holder converts early to pre-empt redemption rather than pre-empting default though either call or conversion can eliminates the possibility of default. These empirical implications can better capture Bhattacharya (2012)’s insight into the observable early call decision and can provide another tunnel to explain Jensen and Pedersen (2016)’s observation about the early conversion decision. We finally consolidate these implications by carrying out empirical tests with twenty years of callable nonconvertible, noncallable convertible, and callable convertible bond data.

References


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AC theoretically analyze the optimal call and default policy for the issuer of a callable defaultable bond when the interest rates and issuing firm’s asset value are characterized as continuous-time Markov processes. Within this framework, they first decompose an unit-face-value callable defaultable bond into an otherwise identical host bond minus a combined option to call or default. To highlight the interaction between the call and default decision, they compare the policy to call or default with the call and default policy followed by the otherwise identical firm with a pure callable and pure defaultable bond outstanding, respectively. In this section, we provide a quick review of AC’s results that will facilitate our analyses on callable convertible bonds.

Let \( \{ r^{(1)}_t \} \quad \tau \geq t \) and \( \{ r^{(2)}_t \} \quad \tau \geq t \) denote two otherwise identical processes of short-term interest rates with initial values \( r^{(1)}_t \leq r^{(2)}_t \). The no-crossing property in Karatzas and Shreve (1987) demonstrates that \( \tilde{P} \left[ r^{(1)}_t \leq r^{(2)}_t, 0 \leq t < \infty \right] = 1 \).

AC apply this no-crossing property to prove the no-crossing properties for \( \{ \beta^{(1)}_{t, \tau} \} \quad \tau \geq t \), \( \{ P^{(1)}_\tau \} \quad \tau \geq t \), \( \{ \beta^{(2)}_{t, \tau} \} \quad \tau \geq t \), and \( \{ V^{(1)}_\tau \} \quad \tau \geq t \) listed as the following corollaries. Given \( t \geq 0 \),

**Corollary 1** Let \( \{ \beta^{(1)}_{t, \tau} \} \quad \tau \geq t \) and \( \{ \beta^{(2)}_{t, \tau} \} \quad \tau \geq t \) be the processes of two discount factors corresponding to two different initial short-term interest rates \( r^{(1)}_t \) and \( r^{(2)}_t \), respectively. Then

\[
\beta^{(1)}_{t, \tau} < \beta^{(2)}_{t, \tau} \Rightarrow \tilde{P} - a.s. \quad \forall \tau \in [t, \infty).
\]

Note that \( \tilde{P} - a.s. \) is the abbreviation of \( \tilde{P} - almost surely \). The definition of a host bond price, Equation (4), and Corollary 1 imply

**Corollary 2** \( r^{(1)}_t \leq r^{(2)}_t \Rightarrow P^{(1)}_\tau \geq P^{(2)}_\tau, \tilde{P} - a.s. \forall \tau \in [t, T]. \)

Associating Corollary 1 with Corollary 2, we have

**Corollary 3** \( r^{(1)}_t < r^{(2)}_t \Rightarrow \beta^{(1)}_{t, \tau} P^{(1)}_\tau > \beta^{(2)}_{t, \tau} P^{(2)}_\tau, \tilde{P} - a.s. \forall \tau \in [t, T]. \)

With the dynamics of an issuing firm’s asset value, Equation (2), the asset value at time \( \tau \) can be derived as

\[
V_\tau = V_0 e^{\int_t^\tau r_s ds - \frac{1}{2} \int_t^\tau \sigma^2_s ds + \int_t^\tau \phi_s d\tilde{W}_s},
\]

and the no-crossing property implies that

**Corollary 4** \( r^{(1)}_t < r^{(2)}_t \Rightarrow V^{(1)}_\tau \leq V^{(2)}_\tau, \tilde{P} - a.s. \forall \tau \in [t, T]. \)

Following Corollary 3 given \( P_t \equiv p \), AC further propose and verify

**Lemma 1** \( r^{(1)}_t \leq r^{(2)}_t \Rightarrow \tilde{E} \left[ \beta^{(2)}_{t, \tau} P^{(2)}_\tau - \beta^{(1)}_{t, \tau} P^{(1)}_\tau \right] \geq p^{(2)} - p^{(1)}, \tilde{P} - a.s. \forall \tau \in [t, T]. \)
To analyze the issuing firm’s policy to call, to default and to call or default the bond, AC first derive the properties of the three embedded options based on the aforementioned corollaries and lemma (see Appendix A in AC) as follows.

**Theorem 7** Denote two different host bond prices at time $t$ by $p^{(1)}$ and $p^{(2)}$ and two different firm’s asset value at time $t$ by $v^{(1)}$ and $v^{(2)}$. The following properties hold for the $f_X(p,v,t)$ as $X = C, D,$ and $CD$.

1. $p^{(1)} > p^{(2)} \Rightarrow f_X(p^{(1)}, v, t) > f_X(p^{(2)}, v, t)$.
2. $v^{(1)} < v^{(2)} \Rightarrow f_X(p, v^{(1)}, t) \geq f_X(p, v^{(2)}, t)$.
3. $p^{(1)} \neq p^{(2)} \Rightarrow 0 < \frac{f_X(p^{(1)}, v, t) - f_X(p^{(2)}, v, t)}{p^{(1)} - p^{(2)}} \leq 1$. (Call delta inequality)
4. $v^{(1)} \neq v^{(2)} \Rightarrow -1 \leq \frac{f_X(p, v^{(1)}, t) - f_X(p, v^{(2)}, t)}{v^{(1)} - v^{(2)}} < 0$. (Put delta inequality)

According to Equation (6), the three embedded options can be treated as the call options on the host bond price, and the call delta inequality entails that the values of these options increase with the underlying host bond price at a slower rate. Similarly, the three embedded options can be regarded as the put options on the issuing firm’s asset value, and the put delta inequality entails that the values of these options decrease with the underlying firm’s asset value also at a slower rate. AC then argue the range of the value $f_{CD}(p,v,t)$ as follows:

**Proposition 2**

$$f_C(p,v,t) \lor f_D(p,v,t) \leq f_{CD}(p,v,t) \leq f_C(p,v,t) + f_D(p,v,t).$$

The first inequality describes that the value of the combined option is greater than that of each simple call or default option, because the combined option has a lower strike price (see Equation (6)). However, the value of the combined option is less than the sum of the two simple options due to the interaction effect (see Kim et al., 1993). That is, exercising the call option destroys the coexisting default option, and vice versa. This proposition then paves the way for clarifying how the call provision influences default risk.

To theoretically characterize the issuing firm’s call and default policy, AC first confirm the existence of the critical host bond price, $b_X(v,t)$, to optimally call or default the bond at time $t$ for $X = C, D,$ or $CD$. That is, given the firm’s asset value at time $t$, $v$, there exists an boundary to optimally exercise the embedded option if and only if the host bond price $p \geq b_X(v,t)$ (i.e., when the interest rate is low enough). Meanwhile, they establish the critical boundaries in terms of the firm’s asset value, $v_X(p,t)$, to optimally call or default the bond at time $t$ for $X = D$ or $CD$. That is, given the host bond price at time $t$, $p$, it is optimal to default if and only if the firm’s asset value $v \leq v_X(p,t)$ (i.e., when the firm is unhealthy enough), and it is optimal to call if and only if $v \geq v_X(p,t)$ (i.e., when the firm is healthy enough). After confirming the existence of these critical boundaries, AC then sketch the shapes for them as follows:

**Theorem 8** For each $t \in [0, T)$

1. $v^{(1)} < v^{(2)} \Rightarrow b_D(v^{(1)}, t) \leq b_D(v^{(2)}, t)$.
2. $p^{(1)} > p^{(2)} \Rightarrow v_D(p^{(1)}, t) \geq v_D(p^{(2)}, t)$. 

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3. \( v^{(1)} < v^{(2)} \leq k_t \Rightarrow b_{CD} (v^{(1)}, t) \leq b_{CD} (v^{(2)}, t) \). (Default case)

4. \( k_t < v^{(1)} < v^{(2)} \Rightarrow b_{CD} (v^{(1)}, t) \geq b_{CD} (v^{(2)}, t) \). (Call case)

5. \( v \leq k_t \Rightarrow b_{CD} (v, t) \geq b_D (v, t) \).

6. \( v > k_t \Rightarrow b_{CD} (v, t) \geq b_C (v, t) \).

Part 1 and 2 of \textbf{Theorem 8} describe the default policy followed by the issuing firm with a pure defaultable bond outstanding. The critical host bond price increases with the firm’s asset value. This implies that it requires lower interest rates (or a higher prevailing debt obligation) to make a healthier firm default (i.e., a healthier firm is less likely to default). Similarly, the critical firm’s asset value increases with the host bond price. This implies that a firm is less likely to default in the high interest rate environment. This captures the empirical observations in Duffee (1998) that the bond credit spreads are negatively related to the level of interest rate. Though part 3 states that the default policy followed by the otherwise identical firm with a callable defaultable bond obeys the same pattern with part 1, part 5 clarifies that the existence of the call option postpones the firm’s default decision (because it requires lower interest rate to make this firm default). This is because exercising the firm’s default option destroys the value of its call option and that delays its default timing. Similarly, part 6 clarifies that the existence of the default option postpones the firm’s call decision, because exercising the firm’s call option destroys the value of its default option and that delays its call timing. This interaction effect between call and default decisions is empirically confirmed by Jacoby and Shiller (2010).

\section*{Appendix B \ Extended Corollaries and Proofs}

Following the corollaries and lemma in \textbf{AC}, we provide another three useful corollaries facilitating the analyses in this paper. First, we extend \textbf{Corollary 4} to the following \textbf{Corollary 5} and \textbf{6}.

\textbf{Corollary 5} Let \( \left( \beta_{t, \tau}^{(1)} \right)_{\tau \geq t} \) and \( \left( \beta_{t, \tau}^{(2)} \right)_{\tau \geq t} \) be the processes of the discount factors corresponding to two different initial short-term interest rates \( r_t^{(1)} \) and \( r_t^{(2)} \), respectively. Then, for an arbitrary time \( t \in [0, T] \),

\[ r_t^{(1)} \leq r_t^{(2)} \Rightarrow \beta_{t, \tau}^{(1)} V_{t}^{(1)} = \beta_{t, \tau}^{(2)} V_{t}^{(2)} , \quad \tilde{\mathbb{P}} - \text{a.s.} \forall \tau \in [t, T] . \]

\textbf{Proof.} The \( V_{t}^{(1)} \) can be written as

\[ V_{t}^{(1)} = V_t e^{\int_t^\tau r_s^{(1)} ds - \int_t^\tau \gamma_s ds - \frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s d\tilde{W}_s} . \]

Multiplying \( \beta_{t, \tau}^{(1)} \) defined in \textbf{Equation (3)} on both sides of the above equation, we have

\[ \beta_{t, \tau}^{(1)} V_{t}^{(1)} = e^{-\int_t^\tau r_s^{(1)} ds} \left( V_t e^{\int_t^\tau r_s^{(1)} ds - \int_t^\tau \gamma_s ds - \frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s d\tilde{W}_s} \right) \]

\[ = V_t e^{-\int_t^\tau \gamma_s ds - \frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s d\tilde{W}_s} \]

\[ = \beta_{t, \tau}^{(2)} V_{t}^{(2)} . \]
Corollary 6 Let $\kappa(V_\tau, \tau)$ be $k_\tau$, $V_\tau$, or $k_\tau \wedge V_\tau$. Then, for an arbitrary time $t \in [0, T]$,

$$r_i^{(1)} \leq r_i^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} k_\tau \left(V^{(1)}_\tau, \tau\right) \geq \beta_{t,\tau}^{(2)} k_\tau \left(V^{(2)}_\tau, \tau\right), \forall \tau \in [t, T].$$

**Proof.** Consider the case $\kappa(V_\tau, \tau) = k_\tau \wedge V_\tau$. If $k_\tau \leq V^{(1)}_\tau < V^{(2)}_\tau$, then according to Corollary 1,

$$\beta_{t,\tau}^{(1)} k_\tau \left(V^{(1)}_\tau, \tau\right) = \beta_{t,\tau}^{(1)} k_\tau \geq \beta_{t,\tau}^{(2)} k_\tau = \beta_{t,\tau}^{(2)} \left(V^{(2)}_\tau, \tau\right).$$

This also confirms the case when $\kappa(V_\tau, \tau) = k_\tau$. If $V^{(1)}_\tau < V^{(2)}_\tau \leq k_\tau$, then according to Corollary 5,

$$\beta_{t,\tau}^{(1)} \left(V^{(1)}_\tau, \tau\right) = \beta_{t,\tau}^{(2)} \left(V^{(1)}_\tau, \tau\right) = \beta_{t,\tau}^{(2)} \left(V^{(2)}_\tau, \tau\right).$$

The case $\kappa(V_\tau, \tau) = V_\tau$ is also confirmed. If $V^{(1)}_\tau < k_\tau < V^{(2)}_\tau$, then

$$\beta_{t,\tau}^{(1)} \left(V^{(1)}_\tau, \tau\right) = \beta_{t,\tau}^{(2)} \left(V^{(1)}_\tau, \tau\right) = \beta_{t,\tau}^{(2)} \left(V^{(2)}_\tau, \tau\right).$$

Next, we calculate the conditional expectation of the random variable $\beta_{t,\tau} V_\tau$.

**Corollary 7** $\mathbb{E}[\beta_{t,\tau} V_\tau | \mathcal{F}_t] = e^{-\int_t^\tau \gamma_s ds} V_t, \forall t \in [t, T], t \geq 0.$

**Proof.** Equation (26) in the proof of Corollary 5 can be rearranged as

$$\beta_{t,\tau} V_\tau = e^{-\int_t^\tau \gamma_s ds} \left(V_t e^{-\frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s dW_s} \right).$$

Take conditional expectation given $\mathcal{F}_t$ on both sides of above equation, we have

$$\mathbb{E}[\beta_{t,\tau} V_\tau | \mathcal{F}_t] = \mathbb{E}\left[e^{-\int_t^\tau \gamma_s ds} \left(V_t e^{-\frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s dW_s} \right) \bigg| \mathcal{F}_t\right]$$

$$= e^{-\int_t^\tau \gamma_s ds} \left(\mathbb{E}\left[e^{-\frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s dW_s} \bigg| \mathcal{F}_t\right]\right)$$

$$= e^{-\int_t^\tau \gamma_s ds} \left(e^{-\frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s dW_s} \right)$$

(27)

$$= e^{-\int_t^\tau \gamma_s ds} V_t.$$

(28)

Because $e^{-\int_t^\tau \gamma_s ds}$ is deterministic and $V_t$ is $(\mathcal{F}_t)$-measurable, they can be taken out from expectation as in Equation (27). The property of an exponential martingale process (see Shreve, 2004) leads to Equation (28), and $e^{-\frac{1}{2} \int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s dW_s} = 1.$

**Appendix C** Proofs

C.1 Proofs of Theorems in Section 3.1

The proof of Theorem 1 is based not only on AC's technique and the no-crossing properties listed as corollaries and lemma in Appendix A but on our extended corollaries listed in Appendix B.

**Proof of Theorem 1.**

1. Let $p^{(1)}$ and $p^{(2)}$ be two possible host bond prices at time $t$ and we consider the case $p^{(1)} > p^{(2)}$. Given that the state at time $t$ is the host bond price $p^{(1)}$ and firm's asset value $v$, $(p^{(1)}, v)$, let the stopping
time $\tau$, $\tau \equiv \tau_{CB}^{(1)}$ and $\tau \in [t, T]$, be the optimal conversion time. For another possible state $\left(p^{(2)}, v\right)$, $\tau$ is a feasible but not the optimal conversion time. Thus, we have

$$f_{CB} \left(p^{(1)}, v, t\right) - f_{CB} \left(p^{(2)}, v, t\right) \leq \tilde{E} \left[\beta_{t,\tau}^{(1)} \left(zV_{t}^{(1)} - P_{t}^{(1)}\right)^{+} - \beta_{t,\tau}^{(2)} \left(zV_{t}^{(2)} - P_{t}^{(2)}\right)^{+} \right]_{\mathcal{F}_t}. $$

According to Corollary 3 and 5, the premise $r_t^{(1)} < r_t^{(2)}$ implies $\beta_{t,\tau}^{(1)} P_{t}(1) > \beta_{t,\tau}^{(2)} P_{t}(2)$ and $\beta_{t,\tau}^{(1)} V_{t}^{(1)} = \beta_{t,\tau}^{(2)} V_{t}^{(2)}$ for $\tau \in [t, T]$. Then,

$$\beta_{t,\tau}^{(1)} \left(zV_{t}^{(1)} - P_{t}^{(1)}\right)^{+} - \beta_{t,\tau}^{(2)} \left(zV_{t}^{(2)} - P_{t}^{(2)}\right)^{+} $$

implying $\left(z\beta_{t,\tau}^{(1)} V_{t}^{(1)} - \beta_{t,\tau}^{(1)} P_{t}^{(1)}\right)^{+} - \left(z\beta_{t,\tau}^{(2)} V_{t}^{(2)} - \beta_{t,\tau}^{(2)} P_{t}^{(2)}\right)^{+} < 0$ with positive probability. This ensures

$$\tilde{E} \left[\beta_{t,\tau}^{(1)} \left(zV_{t}^{(1)} - P_{t}^{(1)}\right)^{+} - \beta_{t,\tau}^{(2)} \left(zV_{t}^{(2)} - P_{t}^{(2)}\right)^{+} \right]_{\mathcal{F}_t} < 0$$

and confirms $f_{CB} \left(p^{(1)}, v, t\right) - f_{CB} \left(p^{(2)}, v, t\right) < 0$.

2. Let $v^{(1)}$ and $v^{(2)}$ be two possible firm’s asset values at time $t$ and we consider the case $v^{(1)} < v^{(2)}$. Given $r_t$, $\forall t \in [0, T]$, Equation (25) suggests $V_{t}^{(1)} < V_{t}^{(2)}$, $\forall t \in [t, T]$. Given that the state at time $t$ is $\left(p, v^{(1)}\right)$, let the stopping time $\tau$, $\tau \equiv \tau_{CB}^{(1)}$ and $\tau \in [t, T]$, be the optimal conversion time. For another possible state $\left(p, v^{(2)}\right)$, $\tau$ is a feasible but not the optimal conversion time. Thus, we have

$$f_{CB} \left(p, v^{(1)}, t\right) - f_{CB} \left(p, v^{(2)}, t\right) < 0.$$ (29)

$V_{t}^{(1)} < V_{t}^{(2)}$ implies $\left(zV_{t}^{(1)} - P_{t}\right)^{+} - \left(zV_{t}^{(2)} - P_{t}\right)^{+} < 0$ almost surely and $\left(zV_{t}^{(1)} - P_{t}\right)^{+} - \left(zV_{t}^{(2)} - P_{t}\right)^{+} < 0$ with positive probability. That ensures the Inequality (29) and confirms $f_{CB} \left(p, v^{(1)}, t\right) - f_{CB} \left(p, v^{(2)}, t\right) < 0$.

3. We consider the case $p^{(1)} > p^{(2)}$ at time $t$, which implies $r_t^{(1)} < r_t^{(2)}$. Part 1 of Theorem 1 confirms the inequality of the right hand side. To establish the inequality of the left hand side, we must confirm $\tilde{f} \left(p^{(1)}, v, t\right) < \tilde{f} \left(p^{(2)}, v, t\right)$. Let the stopping time $\tau$, $\tau \equiv \tau_{CB}^{(2)}$ and $\tau \in [t, T]$, be the optimal conversion time given that the state at time $t$ is $\left(p^{(2)}, v\right)$. For another possible state $\left(p^{(1)}, v\right)$, $\tau$ is a feasible but not the optimal conversion time. Then, it follows

$$\tilde{E} \left[\beta_{t,\tau}^{(1)} \left(zV_{t}^{(1)} - P_{t}^{(1)}\right)^{+} - \beta_{t,\tau}^{(2)} \left(zV_{t}^{(2)} - P_{t}^{(2)}\right)^{+} \right]_{\mathcal{F}_t} < 0.$$ (30)

To establish Equation (30), we first apply Corollary 2 and 4 that $P_{t}^{(1)} > P_{t}^{(2)}$ and $V_{t}^{(1)} \leq V_{t}^{(2)}$ given $r_t^{(1)} < r_t^{(2)}$ and $\tau \in [t, T]$. That confirms $zV_{t}^{(2)} - P_{t}^{(2)} > zV_{t}^{(1)} - P_{t}^{(1)}$ almost surely. Given $zV_{t}^{(2)} - P_{t}^{(2)} > 0$,
To establish Corollary 7 follows from the truth that

\[ \text{Equation (36)} \]

possible state \( \tau \) \( f \) side, we must confirm Theorem 1 confirms the inequality of the left hand side. To establish the inequality of the right hand side, we must confirm \( f_{CB} \left( p, v^{(1)}, t \right) - f_{CB} \left( p, v^{(2)}, t \right) > \left( v^{(1)} - v^{(2)} \right) \). Let the stopping time \( \tau, \tau \equiv \tau^{(2)} \) and \( \tau \in \left[ t, T \right] \), be the optimal conversion time given that the state at time \( t \) is \( \left( p, v^{(2)} \right) \). For another possible state \( \left( p, v^{(1)} \right) \), \( \tau \) is a feasible but not the optimal conversion time. Then, it follows

\[
\begin{align*}
& f_{CB} \left( p, v^{(1)}, t \right) - f_{CB} \left( p, v^{(2)}, t \right) \\
& \quad \geq \mathbb{E} \left[ \beta_{t, \tau} \left( zV_{t}^{(1)} - P_{t} \right)^{+} - \beta_{t, \tau} \left( zV_{t}^{(2)} - P_{t} \right)^{+} \bigg| \mathcal{F}_{t} \right] \\
& \quad = \mathbb{E} \left[ \beta_{t, \tau} \left( zV_{t}^{(1)} - P_{t} \right)^{+} - \beta_{t, \tau} \left( zV_{t}^{(2)} - P_{t} \right)^{+} \right] \cdot I_{\left\{ zV_{t}^{(2)} > P_{t} \right\}} \bigg| \mathcal{F}_{t} \right] \\
& \quad \geq \mathbb{E} \left[ \beta_{t, \tau} \left( V_{t}^{(1)} - V_{t}^{(2)} \right) \right] \cdot I_{\left\{ zV_{t}^{(2)} > P_{t} \right\}} \bigg| \mathcal{F}_{t} \right] \\
& \quad > z \left( v^{(1)} - v^{(2)} \right) .
\end{align*}
\]

To establish Equation (36), we first ensure \( zV_{t}^{(2)} - P_{t} > zV_{t}^{(1)} - P_{t} \) almost surely given \( zV_{t}^{(1)} < zV_{t}^{(2)} \). Given \( zV_{t}^{(2)} - P_{t} > 0, zV_{t}^{(1)} - P_{t} \), can be positive, negative or equal to zero, but \( \left( zV_{t}^{(1)} - P_{t} \right)^{+} \) take away the negative part. Thus, we obtain Equation (36) and establish Inequality (37). Inequality (38) follows from the truth that \( V_{t}^{(1)} - V_{t}^{(2)} < 0 \) almost surely; Equation (39) can be obtained according to Corollary 7; \( 0 < z e^{- \int_{t}^{\tau} \gamma_{v} \, ds} < 1 \), and \( v^{(1)} - v^{(2)} < 0 \) implies Inequality (40). 

The proof of Theorem 2 is based on AC’s technique and the items listed in Theorem 1.

**Proof of Theorem 2.** In the following proof, we note that the continuation region at time \( t \) for the conversion option is the open set

\( U \equiv \left\{ \left( p, v, t \right) \in R^{+} \times R^{+} \times [0, T] : f_{CB} \left( p, v, t \right) > (zv - p)^{+} \right\} \),

and that for all \( t \in [0, T) \), \( f_{CB} \left( p, v, t \right) > 0 \).

1. Let \( p^{(1)} \) and \( p^{(2)} \) be two possible host bond prices at time \( t \) and we consider the case \( p^{(1)} > p^{(2)} \). Suppose it is optimal to continue at \( p^{(2)} \), we show that it is then optimal to continue at \( p^{(1)} \). According to the put delta inequality in Theorem 1, we have

\[
\frac{f_{CB} \left( p^{(2)}, v, t \right) - f_{CB} \left( p^{(1)}, v, t \right)}{p^{(2)} - p^{(1)}} \geq -1
\]

\[ \Rightarrow f_{CB} \left( p^{(1)}, v, t \right) \geq f_{CB} \left( p^{(2)}, v, t \right) + p^{(2)} - p^{(1)} .
\]
Because it is optimal to continue at \( p^{(2)} \), we further have
\[
f_{CB} \left( p^{(1)}, v, t \right) \geq f_{CB} \left( p^{(2)}, v, t \right) + p^{(2)} - p^{(1)}
\]
\[
> (zv - p^{(2)})^+ + p^{(2)} - p^{(1)}
\]
\[
\geq (zv - p^{(2)}) + p^{(2)} - p^{(1)}
\]
\[
= zv - p^{(1)}.
\]

Then, given \( f_{CB} (p^{(1)}, v, t) > 0 \) for all \( t \in [0, T) \), we confirm \( f_{CB} (p^{(1)}, v, t) > (zv - p^{(1)})^+ \), ensuring that it is also optimal to continue at \( p^{(1)} \). Let \( b_{CB}(v, t) \) be the infimum of the host bond price \( p \) such that \( (p, v, t) \in U \). The point \( (b_{CB}(v, t), v, t) \) is not in continuation region \( U \) because the region is open. Thus, \( f_{CB} (b_{CB}(v, t), v, t) = zv - b_{CB}(v, t) > 0 \), implying \( b_{CB}(v, t) < zv \).

2. Let \( v^{(1)} \) and \( v^{(2)} \) be two possible firm’s asset values at time \( t \) and we consider the case \( v^{(1)} < v^{(2)} \). Suppose it is optimal to continue at \( v^{(2)} \), we show that it is then optimal to continue at \( v^{(1)} \). Using the call delta inequality in Theorem 1, we have
\[
\frac{f_{CB} (p, v^{(1)}, t) - f_{CB} (p, v^{(2)}, t)}{v^{(1)} - v^{(2)}} < z
\]
\[
\Rightarrow f_{CB} (p, v^{(1)}, t) > f_{CB} (p, v^{(2)}, t) + z \left( v^{(1)} - v^{(2)} \right).
\]

Because it is optimal to continue at \( v^{(2)} \), we then have
\[
f_{CB} \left( p, v^{(1)}, t \right) \geq f_{CB} \left( p, v^{(2)}, t \right) + zv^{(1)} - zv^{(2)}
\]
\[
> (zv^{(2)} - p)^+ + zv^{(1)} - zv^{(2)}
\]
\[
\geq (zv^{(2)} - p) + zv^{(1)} - zv^{(2)}
\]
\[
= zv^{(1)} - p.
\]

Then, given \( f_{CB} (p, v^{(1)}, t) > 0 \) for all \( t \in [0, T) \), we confirm \( f_{CB} (p, v^{(1)}, t) > (zv^{(1)} - p)^+ \), ensuring that it is also optimal to continue at \( v^{(1)} \). Let \( v_{CB}(p, t) \) be the supremum of the firm’s asset value \( v \) such that \( (p, v, t) \in U \). The point \( (p, v_{CB}(p, t), t) \) is not in continuation region \( U \) because \( U \) is open. Thus, \( f_{CB} (p, v_{CB}(p, t), t) = zv_{CB}(p, t) - p > 0 \), implying \( v_{CB}(p, t) > p \) given \( 0 < z < 1 \).

The proof of Theorem 3 is also based on the items listed in Theorem 1.

Proof of Theorem 3.
1. Consider the host bond price at time \( t \) is \( p \). We want to confirm that if \( p > b_{CB} (v^{(2)}, t) \) (i.e., the conversion option is in the continuation region \( U \)), then \( p > b_{CB} (v^{(1)}, t) \) as well. According to the call delta inequality in Theorem 1 given \( p > b_{CB} (v^{(2)}, t) \), we have
\[
f_{CB} \left( p, v^{(1)}, t \right) \geq f_{CB} \left( p, v^{(2)}, t \right) + zv^{(1)} - zv^{(2)}
\]
\[
> (zv^{(2)} - p)^+ + zv^{(1)} - zv^{(2)}
\]
\[
\geq (zv^{(2)} - p) + zv^{(1)} - zv^{(2)}
\]
\[
= zv^{(1)} - p.
\]

Thus, \( f_{CB} (p, v^{(1)}, t) > (zv^{(1)} - p)^+ \) because \( f_{CB} (p, v^{(1)}, t) > 0 \) for all \( t \in [0, T) \).
2. Consider the firm’s asset value at time \( t \) is \( v \). We want to confirm that \( zv < zv_{CB} \left( p^{(2)}, t \right) \), then \( zv < zv_{CB} \left( p^{(1)}, t \right) \) as well. According to the put delta inequality in Theorem 1 given \( zv < zv_{CB} \left( p^{(2)}, t \right) \)

\[
 f_{CB} \left( p^{(1)}, v, t \right) \geq f_{CB} \left( p^{(2)}, v, t \right) + p^{(2)} - p^{(1)} \\
 > (zv - p^{(2)})^+ + p^{(2)} - p^{(1)} \\
 \geq (zv - p^{(2)}) + p^{(2)} - p^{(1)} \\
 = zv - p^{(1)}.
\]

Thus, \( f_{CB} \left( p^{(1)}, v, t \right) > (zv - p^{(1)})^+ \) because \( f_{CB} \left( p^{(1)}, v, t \right) > 0 \) for all \( t \in [0, T) \).

C.2 Proofs of Theorems in Section 3.2

The technique to confirm \( f_{CB}(p, v, t) - f_{CBCD}(p, v, t) \geq 0 \) is similar to that to confirm \( f_{CBC}(p, v, t) - f_{CBCD}(p, v, t) \geq 0 \) in Section 3.2.1.

**Proof of Inequality (17).** First we note that \( f_{CB}(p, v, t) \) is the value of the game option at time \( t \) associated with the bond holder’s optimal conversion time \( \hat{\tau}_{CB} \) subject to the bond issuing firm’s optimal default time \( \hat{\tau}_{D^*} \), and \( \hat{\tau}_{CB}, \hat{\tau}_{D^*} \in [t, T] \). We on the other hand note that \( f_{CBCD}(p, v, t) \) is the value of another game option associated with the bond holder’s optimal conversion time \( \hat{\tau}_{CB} \) subject to the firm’s optimal call or default time \( \hat{\tau}_{CD^*} \), and \( \hat{\tau}_{CD^*} \in [t, T] \). Then, given that the state at time \( t \) is the pair of the host bond price and issuing firm’s asset value \( (p, v) \), let \( \tau_{D^*}, \tau_{D^*} \in [t, T] \), be the optimal time to default the noncallable convertible bond subject to the strategy to convert the bond \( \tau_{CD^*}, \tau_{CD^*} \in [t, T] \), which is a feasible but not optimal stopping time at this state. However, given the same state, the \( \tau_{CB} \) is the optimal time to convert the otherwise identical defaultable convertible bond subject to the strategy to default the bond \( \tau_{CD^*}, \tau_{CD^*} = \tau_{D^*} \), which is on the other hand a feasible but not optimal stopping time at this state. Thus, we have

\[
 f_{CB}(p, v, t) - f_{CBCD}(p, v, t) \\
 \geq \mathbb{E} \left[ \beta_{t, \tau_{CB}} \left( zV_{r_{CB}}, - P_{r_{CB}} \right) I_{(\tau_{CB}, < \tau_{D^*})} - \beta_{t, \tau_{D^*}} \left( P_{\tau_{D^*}}, - V_{\tau_{D^*}} \right) I_{(\tau_{D^*}, < \tau_{CD^*})} \right]_{\mathcal{F}_t} \\
 - \mathbb{E} \left[ \beta_{t, \tau_{CB}} \left( zV_{r_{CB}}, - P_{r_{CB}} \right) I_{(\tau_{CB}, < \tau_{D^*})} - \beta_{t, \tau_{D^*}} \left( P_{\tau_{D^*}}, - \kappa (V_{\tau_{D^*}}, \tau_{D^*}) \right) I_{(\tau_{D^*}, < \tau_{CD^*})} \right]_{\mathcal{F}_t} \\
 = \mathbb{E} \left[ (\beta_{t, \tau_{D^*}} \left( P_{\tau_{D^*}}, - \kappa (V_{\tau_{D^*}}, \tau_{D^*}) \right)) - \beta_{t, \tau_{D^*}} \left( P_{\tau_{D^*}}, - V_{\tau_{D^*}} \right) \right]_{\mathcal{F}_t} \\
 \geq 0. \tag{41}
\]

Part A of Inequality (41) is less than \( f_{CB}(p, v, t) \) because, rather than the optimal conversion strategy \( \hat{\tau}_{CB} \) subject to firm’s optimal call strategy \( \hat{\tau}_{C^*} \), \( \tau_{CB} \) is the suboptimal strategy subject to the firm’s optimal strategy \( \tau_{D^*} \) to default the noncallable convertible given that the state at time \( t \) is \( (p, v) \). However, part B is greater than \( f_{CBCD}(p, v, t) \) because, rather than the optimal conversion strategy \( \hat{\tau}_{CB} \) subject to firm’s optimal call or default strategy \( \hat{\tau}_{CD^*} \), \( \tau_{CB} \) is the optimal strategy subject to the firm’s suboptimal strategy to default the callable convertible given the same state at time \( t \). Thus, Inequality (41) is established. Equation (42) is obtained after rearranging Inequality (41). The value \( \beta_{t, \tau_{D^*}} \left( V_{\tau_{D^*}}, - \kappa (V_{\tau_{D^*}}, \tau_{D^*}) \right) \) in Equation (42) is nonnegative almost
surely because $\kappa(V_{T^D}, \tau_{D^*})$ is the minimum of $V_{T^D}$ and $k_{T^D}$, and that makes the expectation value in Equation (43) nonnegative. This thus ensures $f_{CBD}(p, v, t) - f_{CBD}(p, v, t) \geq 0$. ■

The proof of the first item in Theorem 4 is provided in Section 3.2.1. The remaining parts of the proof are listed as follows.

Proof of Theorem 4.

2. Let $v^{(1)}$ and $v^{(2)}$ be two possible firm’s asset value at time $t$ and we consider the case $v^{(1)} < v^{(2)}$. Given $r_t, \forall t \in [0, T]$, Equation (25) suggests $V_{u}^{(1)} < V_{u}^{(2)}, \forall u \in [t, T]$. Besides, given that the state at time $t$ is $(p, v^{(1)})$, let $\tau_{CB}^{(1)}(1), \tau_{CD}^{(1)}(1) \in [t, T]$, be the optimal conversion time subject to the strategy to call or default the bond $\tau_{CD}^{(2)}$, which is a feasible but not the optimal stopping time at this state. Given that the state at time $t$ is $(p, v^{(2)})$, let $\tau_{CB}^{(2)}(2), \tau_{CD}^{(2)}(2) \in [t, T]$, is the optimal call or default time subject to the strategy to convert the bond $\tau_{CD}^{(1)}$, which is a feasible but not the optimal stopping time at this state. For the subcase $k_t \leq v^{(1)} < v^{(2)}$ and $t = \tau_{CB}^{(2)} < \tau_{CB}^{(1)}$, we have $f_{CBD}(p, v^{(1)}, t) - f_{CBD}(p, v^{(2)}, t) = 0$. For other subcases, we have

\[
\begin{align*}
    f_{CBD}(p, v^{(1)}, t) & - f_{CBD}(p, v^{(2)}, t) \\
    \leq & \mathbb{E} \left[ \beta_{t, \tau_{CB}^{(1)}}(1) \left( zV_{t}^{(1)} - P_{t}^{(1)} \right) I_{\{v^{(1)} \leq \tau_{CD}^{(2)}(1) \}} - \beta_{t, \tau_{CB}^{(2)}}(2) \left( P_{t}^{(2)} - k \left( V_{t}^{(2)} - \tau_{CD}^{(2)}(2) \right) \right) I_{\{v^{(2)} < \tau_{CD}^{(2)}(2) \}} \right]_{\mathcal{F}_t} \\
    - & \mathbb{E} \left[ \beta_{t, \tau_{CB}^{(1)}}(1) \left( zV_{t}^{(2)} - P_{t}^{(1)} \right) I_{\{v^{(2)} \leq \tau_{CD}^{(2)}(1) \}} - \beta_{t, \tau_{CB}^{(2)}}(2) \left( P_{t}^{(2)} - k \left( V_{t}^{(2)} - \tau_{CD}^{(2)}(2) \right) \right) I_{\{v^{(2)} < \tau_{CD}^{(2)}(2) \}} \right]_{\mathcal{F}_t} \\
    = & \mathbb{E} \left[ \beta_{t, \tau_{CB}^{(1)}}(1) \left( zV_{t}^{(2)} - P_{t}^{(1)} \right) I_{\{v^{(1)} \leq \tau_{CD}^{(2)}(1) \}} - \beta_{t, \tau_{CB}^{(2)}}(2) \left( zV_{t}^{(2)} - P_{t}^{(2)} \right) \right]_{\mathcal{F}_t} \\
    + & \mathbb{E} \left[ \beta_{t, \tau_{CB}^{(1)}}(1) \left( P_{t}^{(2)} - k \left( V_{t}^{(2)} - \tau_{CD}^{(2)}(2) \right) \right) I_{\{v^{(2)} \leq \tau_{CD}^{(2)}(1) \}} - \beta_{t, \tau_{CB}^{(2)}}(2) \left( P_{t}^{(2)} - k \left( V_{t}^{(2)} - \tau_{CD}^{(2)}(2) \right) \right) \right]_{\mathcal{F}_t} \\
    < & 0.
\end{align*}
\]

Part E of Inequality (44) is greater than $f_{CBD}(p, v^{(1)}, t)$ because, rather than the conversion strategy $\tau_{CB}^{(1)}$, the bond holder’s optimal strategy is $\tau_{CB}^{(1)}$, subject to the bond issuer’s suboptimal strategy $\tau_{CD}^{(2)}$, given that the state at time $t$ is $(p, v^{(1)})$. However, part F is less than $f_{CBD}(p, v^{(2)}, t)$ because, rather than the call or default strategy $\tau_{CD}^{(2)}$, the bond issuer’s optimal strategy is $\tau_{CD}^{(1)}$, subject to the bond holder’s suboptimal strategy $\tau_{CB}^{(1)}$, given that the state at time $t$ is $(p, v^{(2)})$. Thus, the Inequality (44) is established. Equation (45) is obtained after rearranging Inequality (44). First, we confirm $\beta_{t, \tau_{CB}^{(1)}}(1) (V_{t}^{(1)} - \tau_{CD}^{(2)}(1)) - \beta_{t, \tau_{CB}^{(2)}}(2) (V_{t}^{(2)} - \tau_{CD}^{(2)}(2)) < 0$ almost surely for $V_{t}^{(1)} < V_{t}^{(2)}$, ensuring part G is negative almost surely. Besides, $\beta_{t, \tau_{CB}^{(1)}}(1) V_{t}^{(1)} - \beta_{t, \tau_{CB}^{(2)}}(2) V_{t}^{(2)} \leq 0$ almost surely for $\kappa (V_{t}^{(1)} - \tau_{CD}^{(2)}(2)) \leq \kappa (V_{t}^{(2)} - \tau_{CD}^{(2)}(2))$, ensuring part H is equal to or less than 0 almost surely. Thus, we confirm $f_{CBD}(p, v^{(1)}, t) - f_{CBD}(p, v^{(2)}, t) < 0$, concluding that $f_{CBD}(p, v^{(1)}, t) - f_{CBD}(p, v^{(2)}, t) \leq 0$.

3. We consider the case $p^{(1)} > p^{(2)}$ at time $t$, which implies $r_t^{(1)} < r_t^{(2)}$. To establish this inequality, we need to confirm $f_{CBD}(p^{(1)}, v, t) - f_{CBD}(p^{(2)}, v, t) \geq p^{(2)} - p^{(1)}$. Given that the state at time $t$ is $(p^{(1)}, v)$, let $\tau_{CB}^{(2)}(2)$, be a feasible but not the optimal conversion time subject to the strategy to call or
default the bond \( x_{CD}^{(1)} \), which is the optimal stopping time at this state. Given that the state at time \( t \) is \((p^{(2)}, v^{(2)})\), \( \tau_{CD}^{(1)} \) is a feasible but not the optimal call or default time subject to the strategy to convert the bond \( x_{CD}^{(2)} \), which is the optimal stopping time at this state. Then, we have

\[
\begin{align*}
&\text{Corollary 5} \\
&\text{Inequality (47)} \\
&\text{is established. Combining the indicator functions in } \\
&\text{Part L of } \tau \\
&\text{we must confirm} \\
&\text{Theorem 4} \\
&\text{4} \\
&\text{M and the inequality is changed into equality according to } \\
&\text{Corollary 5} \\
&\text{Inequality (50)} \\
&\text{Inequality (49)} \\
&\text{Part L of } \text{Inequality (47)} \\
&\text{is equal to Part J of Equation (46) because } \beta_{t}^{(1)} V_{t}^{(1)} - \beta_{t}^{(2)} V_{t}^{(2)} = 0 \\
&\text{according to Corollary 5. Part M of Inequality (47) is less than Part K of Equation (46) because } \\
&\beta_{t}^{(1)} \kappa \left( V_{t}^{(1)}, \tau_{CD}^{(1)} \right) - \beta_{t}^{(2)} \kappa \left( V_{t}^{(1)}, \tau_{CD}^{(1)} \right) \geq 0 \\
&\text{according to Corollary 6. Thus, the Inequality (47) is} \\
&\text{established. Combining the indicator functions in Inequality (47), we obtain Equation (48).} \\
&\text{Inequality (49) is established because } \beta_{t}^{(2)} P_{t}^{(2)} - \beta_{t}^{(1)} P_{t}^{(1)} \leq 0 \\
&\text{almost surely according to Corollary 3, and Inequality (50) follows from Lemma 1.} \\
&\text{The proof of } f_{CBD} \left( p^{(1)}, v, t \right) - f_{CBD} \left( p^{(2)}, v, t \right) \geq p^{(2)} - p^{(1)} \\
&\text{is basically the same with that above except the Inequality (47). If } \kappa \left( V_{t}^{(1)}, \tau_{CD}^{(1)} \right) = V_{t}^{(1)} \text{ and } \kappa \left( V_{t}^{(2)}, \tau_{CD}^{(1)} \right) = V_{t}^{(2)} \text{, Part K is equal to Part M} \\
&\text{and the inequality is changed into equality according to Corollary 5. The upper bound of the put delta inequality is confirmed by part 1 of this theorem.} \\
&\text{4. We consider the case } v^{(1)} < v^{(2)}. \text{ Equation (25) suggests } V_{u}^{(1)} < V_{u}^{(2)}, \forall u \in [t, T]. \text{ Part 2 of} \\
&\text{Theorem 4 confirms the inequality of the left hand side. To establish the inequality of the right hand side, we must confirm } f_{CBD} \left( p, v^{(1)}, t \right) - f_{CBD} \left( p, v^{(2)}, t \right) > v^{(1)} - v^{(2)} . \text{ Given that the state at time } t \\
&\text{is } (p, v^{(1)}), \text{let } \tau_{CD}^{(2)} \text{ be a feasible but not the optimal conversion time subject to the strategy to call} \\
&\text{or default the bond } \tau_{CD}^{(1)} \text{, which is the optimal stopping time at this state. Given that the state at time } t \\
&\text{is } (p, v^{(2)}), \text{ } \tau_{CD}^{(2)} \text{ is a feasible but not the optimal call or default time subject to the strategy to} \]
convert the bond \( r^{(2)}_{CB^*} \), which is the optimal stopping time at this state. Then, we have

\[
f_{CBCD} \left( p, v^{(1)}, t \right) - f_{CBCD} \left( p, v^{(2)}, t \right)
\geq \mathbb{E} \left[ \beta_{t, \tau^{(2)}_{CB^*}} \left( z^{(1)}_{V^{(1)}_{CB^*}} - P_{\tau^{(2)}_{CB^*}} \right) I_{\tau^{(1)}_{CB^*} < \tau^{(2)}_{CB^*}} \right] - \mathbb{E} \left[ \beta_{t, \tau^{(2)}_{CB^*}} \left( z^{(2)}_{V^{(2)}_{CB^*}} - P_{\tau^{(2)}_{CB^*}} \right) I_{\tau^{(1)}_{CB^*} < \tau^{(2)}_{CB^*}} \right]
+ \mathbb{E} \left[ \left( \beta_{t, \tau^{(1)}_{CB^*}} - \beta_{t, \tau^{(2)}_{CB^*}} \right) \left( V^{(1)}_{\tau^{(1)}_{CB^*}} - V^{(2)}_{\tau^{(2)}_{CB^*}} \right) I_{\tau^{(1)}_{CB^*} < \tau^{(2)}_{CB^*}} \right] \tag{51}
\]

\[
= \mathbb{E} \left[ \beta_{t, r^{(2)}_{CB^*}} \left( z^{(1)}_{V^{(1)}_{CB^*}} - z^{(2)}_{V^{(2)}_{CB^*}} \right) I_{\tau^{(1)}_{CB^*} < \tau^{(2)}_{CB^*}} \right] + \mathbb{E} \left[ \left( \beta_{t, \tau^{(1)}_{CB^*}} \kappa \left( V^{(1)}_{\tau^{(1)}_{CB^*}}, \tau^{(1)}_{CB^*} \right) - \beta_{t, \tau^{(2)}_{CB^*}} \kappa \left( V^{(2)}_{\tau^{(2)}_{CB^*}}, \tau^{(2)}_{CB^*} \right) \right) I_{\tau^{(1)}_{CB^*} < \tau^{(2)}_{CB^*}} \right] \tag{52}
\]

\[
\geq \mathbb{E} \left[ \beta_{t, \tau^{(2)}_{CB^*}} - \beta_{t, \tau^{(1)}_{CB^*}} \right] I_{\tau^{(2)}_{CB^*} > \tau^{(1)}_{CB^*}} \tag{53}
\]

\[
= \mathbb{E} \left[ \beta_{t, \tau^{(2)}_{CB^*}} - \beta_{t, \tau^{(1)}_{CB^*}} \right] I_{\tau^{(2)}_{CB^*} > \tau^{(1)}_{CB^*}} \tag{54}
\]

\[
e^{-\int_t^T \gamma ds} \left( v^{(1)} - v^{(2)} \right) \tag{55}
\]

\[
\geq v^{(1)} - v^{(2)}. \tag{56}
\]

Part P of Inequality (52) is less than part N of Equation (51) because \( 0 < z < 1 \) and \( \beta_{t, \tau^{(2)}_{CB^*}} - \beta_{t, \tau^{(1)}_{CB^*}} V^{(2)}_{\tau^{(2)}_{CB^*}} < 0 \) almost surely. Besides, part Q of Inequality (52) is not greater than part O of Equation (51) because \( \kappa \left( V^{(1)}_{\tau^{(1)}_{CB^*}}, \tau^{(1)}_{CB^*} \right) - \kappa \left( V^{(2)}_{\tau^{(2)}_{CB^*}}, \tau^{(2)}_{CB^*} \right) \leq 0 \) and \( V^{(1)}_{\tau^{(1)}_{CB^*}} - V^{(2)}_{\tau^{(2)}_{CB^*}} < 0 \). Thus, we establish Inequality (52). Combining the indicator functions in Inequality (52), we obtain Equation (53).

Inequality (54) is established because \( \beta_{t, \tau^{(1)}_{CB^*}} V^{(1)}_{\tau^{(1)}_{CB^*}} - \beta_{t, \tau^{(2)}_{CB^*}} V^{(2)}_{\tau^{(2)}_{CB^*}} < 0 \) almost surely. Equation (55) follows from Corollary 7: \( 0 < e^{-\int_t^T \gamma ds} < 1 \), and \( v^{(1)} - v^{(2)} < 0 \) implies Inequality (56).

Regarding the call delta inequality, we first prove the inequality of the left hand side. The relation \( v^{(1)} < v^{(2)} \Rightarrow f_{CBC} \left( p, v^{(1)}, t \right) \leq f_{CBC} \left( p, v^{(2)}, t \right) \) are needed to be confirmed. The proof of this relation is basically the same with that of part 2 of this theorem. The proof of the inequality of the right hand side is also basically the same with that of this part except Inequality (52). That is, part O equals to zero when \( \kappa \left( V^{(1)}_{\tau^{(1)}_{CB^*}}, \tau^{(1)}_{CB^*} \right) = \kappa \left( V^{(2)}_{\tau^{(2)}_{CB^*}}, \tau^{(2)}_{CB^*} \right) = k_{\tau^{(1)}_{CB^*}} ; \) part P is less than part N and the inequality is again established.

In the following proof of Theorem 5, we note that the continuation region at time \( t \) for the game option is the open set

\[
U^* \equiv \left\{ (p, v, t) \in R^+ \times R^+ \times [0, T) : zv - p < f_{CBCD} (p, v, t) < \kappa (v, t) - p \right\}
\]

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for $zv < \kappa(v,t)$.

**Proof of Theorem 5.**

1. Let $p^{(1)}$ and $p^{(2)}$ be two possible host bond prices at time $t$ and we consider the case $p^{(1)} > p^{(2)}$ with $f_{CBCD}(p^{(1)},v,t) < \kappa(v,t) - p^{(1)}$ and $f_{CBCD}(p^{(2)},v,t) < \kappa(v,t) - p^{(2)}$ (i.e., the case the bond issuer does not exercise the call or default option). Suppose it is optimal to continue at $p^{(1)}$ with $f^{(1)}$.

Because it is optimal to continue at $p^{(2)}$, we further have

$$f_{CBCD}(p^{(1)},v,t) \geq f_{CBCD}(p^{(2)},v,t) + p^{(2)} - p^{(1)}$$

This confirms $f_{CBCD}(p^{(1)},v,t) > zv - p^{(1)}$ and ensures that it is also optimal to continue at $p^{(1)}$. Let $b_{CB^*}(v,t)$ be the infimum of the host bond price $p$ such that $(p,v,t) \in U^*$. The point $(b_{CB^*}(v,t),v,t)$ is not in continuation region $U^*$ because the region is open. Thus, $f_{CBCD}(b_{CB^*}(v,t),v,t) = zv - b_{CB^*}(v,t)$.

On the other hand, we consider the case $p^{(1)} > p^{(2)}$ with $f_{CDCB}(p^{(1)},v,t) < p^{(1)} - zv$ and $f_{CDCB}(p^{(2)},v,t) < p^{(2)} - zv$ (i.e., the case the bond holder does not exercise the conversion option). Suppose it is optimal to continue at $p^{(1)}$ given that the firm’s asset value at time $t$ is $0 < v < v_{CB^*}$, we show that it is then optimal to continue at $p^{(2)}$. According to Equation (11) and part 3 of Theorem 4, we have

$$f_{CDCB}(p^{(2)},v,t) \geq f_{CDCB}(p^{(1)},v,t) + p^{(2)} - p^{(1)}.$$

Because it is optimal to continue at $p^{(1)}$, we further have

$$f_{CDCB}(p^{(2)},v,t) \geq f_{CDCB}(p^{(1)},v,t) + p^{(2)} - p^{(1)}$$

This confirms $f_{CDCB}(p^{(2)},v,t) > p^{(2)} - \kappa(v,t)$ and ensures that it is also optimal to continue at $p^{(2)}$. Let $b_{CD^*}(v,t)$ be the supremum of the host bond price $p$ such that $(p,v,t) \in U^*$. The point $(b_{CD^*}(v,t),v,t)$ is not in continuation region $U^*$ because the region is open. Thus, $f_{CDCB}(b_{CD^*}(v,t),v,t) = b_{CD^*}(v,t) - \kappa(v,t)$.

2. Let $v^{(1)}$ and $v^{(2)}$ be two possible firm’s asset values at time $t$ and we consider the case $k_1 < v^{(1)} < v^{(2)} < v_{CB^*}$ with $f_{CDCB}(p,v^{(1)},t) < p - zv^{(1)}$ and $f_{CDCB}(p,v^{(2)},t) < p - zv^{(2)}$ (i.e., the case the bond holder does not exercise the conversion option). Suppose it is optimal to exercise the call option at $v^{(1)}$, we show that it is then optimal to exercise the call option at $v^{(2)}$. According to
Equation (11) and part 2 of Theorem 4, we have
\[ f_{CDCB}(p, v^{(2)}, t) \leq f_{CDCB}(p, v^{(1)}, t) = p - \kappa(v^{(1)}, t) = p - k_t. \]

Besides, \( f_{CDCB}(p, v^{(2)}, t) \geq p - \kappa(v^{(2)}, t) = p - k_t \). Then, we obtain \( f_{CDCB}(p, v^{(2)}, t) = p - k_t \) and ensures that it is also optimal to exercise the call option at \( v^{(2)} \). Let \( \bar{v}_{CD^*}(p, t) \) be the minimum of the firm’s asset value at time \( t, k_t \leq \bar{v}_{CD^*}(p, t) < v_{CB^*} \), such that it is optimal to call at \((p, v, t)\).

On the other hand, we consider the case \( 0 \leq v^{(1)} < v^{(2)} \leq k_t \) with \( f_{CDCB}(p, v^{(1)}, t) < p - zv^{(1)} \) and \( f_{CDCB}(p, v^{(2)}, t) < p - zv^{(2)} \). Suppose it is optimal to continue at \( v^{(1)} \), we show that it is then optimal to continue at \( v^{(2)} \). According to Equation (11) and part 4 of Theorem 4, we have
\[ \frac{f_{CDCB}(p, v^{(1)}, t) - f_{CDCB}(p, v^{(2)}, t)}{v^{(1)} - v^{(2)}} > -1 \]
\[ \Rightarrow f_{CDCB}(p, v^{(2)}, t) > f_{CDCB}(p, v^{(1)}, t) + v^{(1)} - v^{(2)}. \]

Because it is optimal to continue at \( v^{(1)} \), we then have
\[ f_{CDCB}(p, v^{(2)}, t) \geq f_{CDCB}(p, v^{(1)}, t) + v^{(1)} - v^{(2)} \]
\[ \Rightarrow (p - v^{(1)}) + v^{(1)} - v^{(2)} = p - v^{(2)} \]

This confirms \( f_{CDCB}(p, v^{(2)}, t) > p - v^{(2)} \) and ensures that it is also optimal to continue at \( v^{(2)} \). Let \( v_{CD^*}(p, t) \) be the minimum of the firm’s asset value \( v \) such that \((p, v, t) \in U^* \). The point \((p, v_{CD^*}(p, t), t)\) is not in continuation region \( U^* \) because \( U^* \) is open. Thus, \( f_{CDCB}(p, v_{CD^*}(p, t), t) = p - v_{CD^*}(p, t) \) and \( 0 \leq v_{CD^*}(p, t) \leq k_t \).

Proof of Theorem 6.
1. Consider the host bond price at time \( t \) is \( p \) and the case \( v^{(1)} < v^{(2)} < v_{CB^*} \) with \( f_{CDCB}(p, v^{(1)}, t) < p - zv^{(1)} \) and \( f_{CDCB}(p, v^{(2)}, t) < p - zv^{(2)} \). For the scenario \( v^{(1)} < v^{(2)} \leq k_t \), we want to confirm that if \( 0 < p < b_{CD^*}(v^{(1)}, t) \) (i.e., the firm does not exercise its default option), then \( p < b_{CD^*}(v^{(2)}, t) \) as well. According to Equation (11) and part 4 of Theorem 4 given \( 0 < p < b_{CD^*}(v^{(1)}, t) \), we have
\[ f_{CDCB}(p, v^{(2)}, t) \geq f_{CDCB}(p, v^{(1)}, t) + v^{(1)} - v^{(2)} \]
\[ \Rightarrow (p - v^{(1)}) + v^{(1)} - v^{(2)} = p - v^{(2)} \]
Thus, \( f_{CDCB}(p, v^{(2)}, t) > p - v^{(2)} \), ensuring \( p < b_{CD^*}(v^{(2)}, t) \).

2. Consider the case \( v^{(1)} < v^{(2)} < v_{CB^*} \) with \( f_{CDCB}(p, v^{(1)}, t) < p - zv^{(1)} \) and \( f_{CDCB}(p, v^{(2)}, t) < p - zv^{(2)} \). We want to confirm that if \( 0 < p < b_{CD^*}(v^{(2)}, t) \) (i.e., the firm does not exercise its call option), then \( p < b_{CD^*}(v^{(1)}, t) \) as well. According to Equation (11) and part 2 of Theorem 4 given \( 0 < p < b_{CD^*}(v^{(2)}, t) \), we have
\[ f_{CDCB}(p, v^{(1)}, t) \geq f_{CDCB}(p, v^{(2)}, t) > p - k_t. \]
Thus, \( f_{CDCB}(p, v^{(1)}, t) > p - k_t \), ensuring \( p < b_{CD^*}(v^{(1)}, t) \).
3. Consider the case $f_{CB}(p,v,t) < \kappa(v,t) - p$. We want to confirm that if $p > b_{CB}(v,t)$ (i.e., the callable convertible bond holder does not exercise the conversion option), then $p > b_{CB}(v,t)$ as well. According to Inequality (12) given $p > b_{CB}(v,t)$, we have

$$f_{CB}(p,v,t) \geq f_{CB}(p,v,t) > zv - p.$$ 

Thus, $f_{CB}(p,v,t) > zv - p$, ensuring $p > b_{CB}(v,t)$.

4. Consider the case $f_{CD}(p,v,t) < p - zv$. We want to confirm that if $p < b_{CD}(v,t)$ (i.e., the firm issuing callable convertible bond does not exercise its option), then $p < b_{CD}(v,t)$ as well. According to Inequality (13) given $p < b_{CD}(v,t)$, we have

$$f_{CD}(p,v,t) \geq f_{CD}(p,v,t) > p - \kappa(v,t).$$

Thus, $f_{CD}(p,v,t) > p - \kappa(v,t)$, ensuring $p < b_{CD}(v,t)$.  

*