Rational Mispricing with Unpredictable Demand Shocks*

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November 25, 2016

*I would like to thank René Garcia, Raman Uppal, and participants at EDHEC PhD Seminar, 2016, for helpful comments.

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Abstract

Movements in prices depend both on innovations to cashflows and changes in discount rates, which can be modelled as fluctuations in the cross-sectional distribution of wealth across an unchanging set of investment objectives. This paper explores the risk that arises when investors do not have perfect information about the wealth distribution, and, as a result, cannot forecast prices accurately. To take into account this risk, investors plan their consumption for all realisations of the wealth distribution according to their subjective beliefs. This makes markets highly incomplete, and derivative assets become non-redundant. Derivatives serve a dual purpose: they allow investors to adjust consumption for different realisations of the wealth distribution, and provide information required to implement optimal allocation decisions. Asset prices, and expected returns, depend on the sensitivity of stochastic discount factor and assets’ payoffs to the wealth distribution. Prices of derivatives deviate from the expected cost of creating synthetic derivatives through dynamic trading, creating apparent mispricings between derivatives and primary assets. The imprecise information about the wealth distribution can induce an additional demand for dynamic trading, so that passive investment strategies are no longer optimal. Our results also have implications for arbitrage activity, informational efficiency of prices, and the role of financial innovation.

JEL Classification: G10, G11, G12, G13, G14, G23

Keywords: demand shocks, mispricing, limits to arbitrage, derivatives, incomplete markets, informational efficiency of prices.
1 Introduction

Standard models of asset pricing depend on arbitrage to rule out deviations of asset prices from their fundamental values implied by the models. Any deviation from fundamental values, it is argued, will create opportunities for excess risk-adjusted returns (arbitrage opportunities), which all investors will seek to exploit. The pursuit of arbitrage opportunities will in turn cause prices to converge to their fundamental values, eliminating any mispricing. According to this argument, the presence of a few ‘deep-pocketed’ arbitrageurs, i.e. arbitrageurs with sufficient wealth, with superior information about assets’ cash flows is all that is needed to rule out any deviation of asset prices from their true fundamental values. This argument, however, has been recently contested in the growing literature on limits to arbitrage, especially after the global financial crisis of 2008, which is difficult to explain without assuming sustained and significant mispricing of assets relative to their fundamental values. This literature explores the possibilities of asset-mispricing by studying the ‘limits to arbitrage’ that arise due to costs of identifying arbitrage opportunities, financial constraints faced by arbitrageurs, coordination costs amongst arbitrageurs, and arbitrageurs’ incentives.

Shleifer and Vishny (1997) study a setting where asymmetric information between end-investors and specialized arbitrageurs limits arbitrageurs’ ability to correct mispricing, by restricting their access to cash. Gromb and Vayanos (2002, 2009) show that collateral constraints limit arbitrageurs’ ability to eliminate price discrepancies between two identical assets. Liu and Longstaff (2004) shows that in the presence of margin constraints, it is often optimal for risk-averse arbitrageurs to underinvest in arbitrage by taking smaller positions than allowed by margin constraints. Sotes-Paladino and Zapatero (2015) show that convex incentives of money-managers can result in substantial over-investment in overpriced securities. Abreu and Brunnermeier (2002, 2003), and Liu and Mello (2011) show that the coordination risk amongst rational arbitrageurs can induce them to limit their arbitrage positions.

However, Ljungqvist and Qian (2014) suggests that arbitrageurs’ capital and short-sale constraints often do not prohibit arbitrage, even in some extreme situations.
Arbitrageurs can coordinate with other arbitrageurs and shareholders of the mispriced securities by publicly revealing their information, through highly detailed and free reports, and induce them to trade against the mispricing. This allows arbitrageurs to overcome their own limited capital and short-sale constraints. Dass, Massa, and Patgiri (2008) empirically examines the relation between mutual fund compensation contracts and their tendency to invest in bubble stocks, and find that the funds whose contracts link compensation more strongly to fund-performance reduced their holdings of bubble stocks. This suggests that the aforementioned limits to arbitrage based on constraints, incentives, and coordination risk cannot fully account for arbitrageurs’ inability to correct mispricings. Moreover, these limits to arbitrage generally do not explain why misprings arise in the first place, especially the ones that are more systematic and persistent, such as index arbitrage opportunities (Neal (1996)), violations of put-call parity (Ofek, Richardson, and Whitelaw (2004)), and relative mispricing between primes and scores (Jarrow and O’HARA (1989)).

In this paper, we explore a setting in which asset prices depend on cross-sectional distribution of wealth and investment objectives (preferences, constraints, etc.), which we refer to as the wealth distribution henceforth, and mispricings arise rationally due to each investor’s imprecise information about the wealth distribution.\(^1\) Our focus on imprecision about the wealth distribution is motivated by the observation that a significant amount of variation in prices is driven not by cash-flow news, but by changes in investors’ demand for assets (Shiller (1983), Campbell (1991), Campbell and Ammer (1993), Koijen and Yogo (2015)). In the variance decomposition of Campbell (1991), about one-third of the variation in unexpected stock returns is driven by changes in expected discount rates, and another one-third is driven by covariance of expected dividends and expected discount rates. In Koijen and Yogo (2015), a decomposition of stock returns into supply- and

\(^1\)For convenience, we refer to this cross-sectional distribution of wealth, preferences, and constraints, simply as the wealth distribution. In fact, if the set of investment objectives, which includes both preferences and constraints, is assumed to include all possible objectives, then the only relevant variable left for asset prices is the wealth distribution. That is, we define wealth distribution as the distribution of wealth across a fixed set of investment objectives, instead of a distribution of wealth across a set of agents. Moreover, we use the terms ‘wealth distribution’ and ‘consumption distribution’ interchangeably, because the two have a one-to-one correspondence.
demand-side effects reveals that 91% of the variation in stock returns is driven by demand-side effects, which are independent of changes in cashflow and other observed characteristics of assets. Grossman (1995) refers to this price variation as ‘allocational changes in prices’.

In an equilibrium model, these changes in prices can be expressed as fluctuations in the wealth distribution across an unchanging set of investment objectives. If each investor could forecast the wealth distribution precisely, the changes in equilibrium discount rates would be predictable contingent on cashflow shocks, and would not constitute a distinct risk. In such a setting, hedging against cashflow shocks would eliminate all uncertainty from prices. However, with imprecise knowledge of the wealth distribution, investors’ price forecasts contingent on cashflow shocks would be imprecise, and investors would not be able to hedge all the risk by hedging only against cashflow shocks. If the residual uncertainty in prices is economically significant, then investors may optimally choose to incorporate this uncertainty into their allocation decisions, to minimise their utility losses due to inaccurate price forecasts. For instance, in the autoregressive model of expected returns used in Campbell (1991), for a persistence parameter of 0.9, a 1% change in expected returns can cause a capital gain or loss of about 10%. Our objective in this paper is to highlight the important qualitative implications of this additional risk arising from such unforecastable shocks to the wealth distribution.

Following Grossman (1988), we consider a setting where some of the investors face a funding-ratio constraint. In the presence of a funding-ratio constraint, asset prices depend on an average level of funding-ratio of constrained investors, which may not be known to all investors in the economy. Thus, uninformed investors forecast future prices based on their imprecise estimate of the average level of funding-ratio in the economy. However, each uninformed investor knows that its estimate may be imprecise, and takes into account this imprecision in its price forecasts. Thus, even if the bias in each uninformed investor’s price estimate cancels out across a large number of uninformed investors, making the average (across uninformed investors) price forecast unbiased, the effect of imprecision in each investor’s estimate will not cancel out, and will impact equilibrium prices. Our framework allows us to shed
light on the uncertainty arising from the imprecision in the wealth distribution, and its qualitative effects on equilibrium prices and allocations.

We show that price realisation contingent on a cashflow shock depends on the realisation of the wealth distribution, which uninformed investors cannot forecast accurately, and plan their consumption for various realisations of prices that are possible under their subjective probability measure. This makes markets highly incomplete under the uninformed informed investors’ subjective probability measure, as the wealth distribution is a continuous variable and can take infinitely many values, and uninformed investors do not have enough assets to adjust their consumption for every possible realisation of the wealth distribution. Thus, the uninformed investors’ subjective price for an asset is the average of their subjective prices across all states that are realisable according to their beliefs about the wealth distribution. The change in the subjective price of any asset can be expressed as a shift in the level of discount rate, a shift in the beta of the asset, and a shift in the asset’s expected payoff. While the shift in the discount rate affects all assets identically, the shifts in an asset’s beta and payoff are asset-specific. Hence, different assets become mispriced relative to each other, depending on the sensitivities of their payoffs to the wealth distribution. The mispricings between assets can be understood as a compensation for the liquidity risk arising from the imprecision about the wealth distribution, which affects different assets differently.

Due to market incompleteness, derivative assets are no longer redundant, as derivatives can allow investors to adjust their consumption for different possible realisations of the wealth distribution. Derivatives with different payoff functions will have a different sensitivity to the wealth distribution. As a result, prices of different derivatives will deviate more or less from the expected cost of replicating these derivatives using dynamic trading strategies, which provides a rationale for systematic mispricings in different type of derivatives. In addition, the prices of riskfree and risky assets no longer provide sufficient information to implement optimal allocation decisions, as optimal allocations depend on nonlinear covariances between assets’ payoffs and the stochastic discount factor. Thus, derivatives can be valuable both for smoothing consumption for different realisations of the wealth
distribution, and for obtaining information that is required to implement optimal allocations.

In addition to mispricings between derivatives and primary assets, the imprecision about the wealth distribution can also create cross-sectional mispricings in primary assets. For instance, assets with higher price to dividend ratios will be more sensitive to the wealth distribution, and will be more mispriced, creating cross-sectional differences in expected returns between high- and low-price to dividend ratio assets. This can create cross-sectional differences in expected returns between assets with different growth opportunities (value and growth stocks), and sizes (small and big cap stocks), which are likely to have different price to dividend ratios. Similarly, assets with higher price to dividend ratios will exhibit higher variance, which can provide a reason for why longer maturity assets may exhibit excess variance compared to shorter-maturity assets (Giglio and Kelly (2015)).

The mispricings evolve with the state of the economy, and intensify upon negative shocks to dividends and dividend growth rates. Upon negative shocks to dividends, future asset payoffs (prices) become more sensitive to the wealth distribution, increasing the mispricing. Upon negative shocks to dividend growth rates, the stochastic discount factor become more sensitive to the wealth distribution, again resulting in higher mispricing.

Investors with better information about the wealth distribution, which we refer to as arbitrageurs, trade against investors with more imprecise information, and absorb their demand. Thus, the presence of informed investors reduces the impact of uninformed investors’ imprecision about the wealth distribution on equilibrium prices. The equilibrium price deviation is a fraction, which is equal to the uninformed investors’ share of aggregate consumption, of the deviation in subjective prices of uninformed investors.

This mispricing due to imprecision in the wealth distribution also creates a risk for arbitrageurs. If arbitrageurs do not take into account the effect of uninformed investors’ imprecision about the wealth distribution on asset prices, their forecasts will be biased, causing a loss of consumption. To avoid this loss, arbitrageurs can try to infer the amount of imprecision in uninformed investors’ forecasts from
current mispricing. This reduces their risk of making biased price forecasts. However, if uninformed investors have imprecise information about both the wealth distribution and cashflows, then arbitrageurs may not be able to distinguish the two effects, and may fail to benefit from mispricings even when they arise due to imprecise dividend forecasts, which they could profitably eliminate. That is, to the extent that an observed mispricing may be explained by imprecision about the wealth distribution, which will persist over time, arbitrageurs may remain reluctant to trade against it. Hence, even mispricings that arise due to imprecise dividend forecast, which would not have survived otherwise, may persist, as long as these mispricings are within the range that can be explained by uninformed investors’ imprecision about the wealth distribution.

Finally, the imprecision in the wealth distribution would create an additional demand for dynamic trading, and passive investment strategies would no longer be optimal. For an exogenously given initial endowment of traded assets, the optimal consumption stream that an investor can achieve depends on the wealth distribution. With imprecise knowledge about the wealth distribution, an investor would set its optimal consumption stream as some sort of a weighted average of consumption streams that are possible for different realisations of the wealth distribution, and, hence, the optimal consumption stream would depend on the investor’s imprecision about the wealth distribution. In order to obtain this optimal consumption stream, investors would have to trade dynamically to expose their consumption to realised prices, as dividends are independent of the wealth distribution.

The rest of the paper is organised as follows: Section 2 discusses related literature. Section A provides a qualitative discussion of why the wealth distribution matters. Section 3 describes our economic setting. Section 4 formalises the model in a setting with three investors and imprecise information about the wealth distribution. Section 5 extends the basic model to a setting with many investors, and discusses why the imprecision about the wealth distribution may survive in a model with learning. Section 6 concludes.
2 Related Literature

The main focus of this paper is on the imperfect knowledge of the cross-sectional distribution of wealth and preferences, which is also the primary distinction between this work and the existing literature. The existing asset pricing models with a representative agent (e.g. Lucas (1978), Campbell and Cochrane (1999), Bansal and Yaron (2004)) assume preferences for the representative agent, and, thus, questions regarding underlying investor heterogeneity and imperfect knowledge about the nature and extent of heterogeneity cannot be analysed in these settings. In heterogeneous models (e.g. see Dumas (1989); Detemple and Murthy (1994); Bhamra and Uppal (2014) and Hommes (2005); Curcuru, Heaton, Lucas, and Moore (2010) for a review), the focus is typically on understanding the effects of heterogeneity, in contrast to investors’ imperfect knowledge about the extent of heterogeneity, and investors are implicitly assumed to have perfect information about the each others’ preferences and wealth, for analytical tractability. However, as argued in Section 1, the investor heterogeneity may have a significant impact on asset prices through its effect on equilibrium discount rates, and the knowledge about the extent of heterogeneity may play as significant a role as the knowledge about the assets’ cashflows.

The closest to this paper are Gallmeyer, Seppi, and Hollifield (2005), Kraus and Smith (1989), and Grossman (1988). Gallmeyer, Seppi, and Hollifield (2005) develops a model of demand discovery where agents learn about other investors’ future demand through trading activity. In contrast to this, our focus is not on the effects of demand discovery but on the effects of incorporating uncertainty about this demand in making optimal decisions, which yields distinct implications by rendering markets incomplete. The market incompleteness in our model is also what primarily distinguishes this paper from Kraus and Smith (1989), where the wealth distribution is assumed to take one of two possible states, and no cashflow news is assumed to arrive at the intermediate date, so that the markets remain complete.

In this regard, our paper is closer to Grossman (1988), which explores the effect of uncertainty about the demand for portfolio insurance on market volatility, when
uncertainty about the demand for insurance interacts with the cashflow news, in a setting with a representative portfolio insurer and a representative liquidity provider. The portfolio insurer tries to replicate a put option by increasing (decreasing) its allocation to the risky asset in good (bad) states of the world. The liquidity provider provides liquidity to the portfolio insurer, and the price of the risky asset depends on liquidity provider’s ability to do so. When the liquidity providers does not know the extent of portfolio insurer’s demand accurately, it may fail to provide the required liquidity, making prices more volatile. The main effect explored in Grossman (1988) is the bias in liquidity provider’s estimate of the demand for portfolio insurance in different states of the world, which biases the liquidity provider’s price forecast. Our focus, in contrast, is not on the bias in forecasted prices, but on the imprecision in this forecast. We show that in a setting with a large number of investors, the effect of bias in each investor’s price forecast can cancel out across investors, as some investors may overestimate prices and some may underestimate prices, absorbing each other’s demand and leaving equilibrium prices unchanged. However, the effect of imprecision in each investor’s price forecast does not cancel out even with a large number of investors, and has distinct implications for asset prices that do not arise due to biased price forecasts alone. Moreover, in Grossman (1988), the focus is on investors’ ability to replicate their optimal consumption stream, but the optimal consumption stream itself is taken as exogenous. We show that investors’ optimal consumption stream itself will be affected by the imprecision in their knowledge of the wealth distribution, creating additional demand for derivative assets that would not arise if their consumption stream is taken as exogenous.

The uncertainty about the wealth distribution leads to uncertainty about asset return dynamics, as the distribution of asset returns conditional on cashflow shocks is not accurately known. Thus, our approach can be interpreted as the one where investors take a Bayesian approach towards imprecisely known model variables, as is the case in the literature on ambiguity and model uncertainty (Ju and Miao (2012), Epstein and Schneider (2008)). The main difference from this literature on model uncertainty is that the uncertainty we are concerned with is related not to the cashflow dynamics or production technology, but to the distribution of
wealth and investment objectives in the economy, and we take a simpler Bayesian approach towards uncertainty about imprecisely known model variables in contrast to a more general non-Bayesian approach that allows investors to treat uncertainty about the knowledge of a variable’s distribution differently from the uncertainty arising from the known distribution of the variable.

Because investors are assumed not to have perfect knowledge of the distribution of conditional return moments, an obvious question to ask is if investors can acquire this information from observed prices. Hence, this paper shares some elements with the literature on noisy rational expectations equilibrium (Hellwig (1980), Admati (1985)). Similar to the noisy rational expectations equilibrium (NREE) literature, we are concerned with how prices may or may not be able to convey information about the wealth distribution, but in contrast to this literature, we are not concerned with finding the equilibrium price function that conveys information sought by investors, while simultaneously clearing the market. Instead, our focus is on how the equilibrium price function is affected when investors incorporate the imprecision in the distributional parameters of forecasted payoffs. While the need to incorporate imprecision in the distributional parameters of forecasted payoffs can arise even in a static two-date setting, which is standard in NREE models, this need is magnified in a dynamic multi-date setting. In a static two-date setting, prices can accurately aggregate information across investors, and provide a more informative signal about assets’ payoff than the private signal of any individual investor, thus minimising imprecision in investors’ estimates of distributional parameters. In a multi-date setting, prices aggregate information about future payoffs across shocks to exogenous fundamentals, and, thus, prices reveal information only about the average payoff at the next date. As a result, investors’ forecasts of assets’ payoffs upon a given shock to fundamentals may remain imprecise. In other words, prices would not necessarily reveal more informed investors’ information regarding asset payoffs conditional on shocks to fundamentals, as long as markets are not complete in a static sense (i.e. Arrow-Debreu securities are traded for all possible states of the world).

Hence, as long as markets are not complete in a static sense, imprecision about the wealth distribution can survive, which serves as a risk that is distinct from the
cashflow risk, and can be interpreted as a form of liquidity risk, as it only affects investors’ consumption (and, hence, their utility) through dynamic trading. This form of liquidity risk is most closely related to notion of liquidity that measures financial assets’ ability to transport wealth across time, similar to Holmström and Tirole (2001), and is distinct from the liquidity risk that may arise from uncertainty about transaction costs (Acharya and Pedersen (2005), Amihud, Mendelson, and Pedersen (2006)), or from exogenous liquidity shocks (Huang (2003)), and has different implications for asset prices and asset allocations. In Holmström and Tirole (2001), illiquidity refers to the limits on the aggregate value of financial assets in the future, which determines the amount of wealth that can be transferred between different periods, and arises from agency problems between investors and entrepreneurs. In our setting, illiquidity refers to the uncertainty about the future value of financial assets, and arises due to imprecise knowledge of the wealth distribution. These two notions of illiquidity can have a similar effect. A higher illiquidity, in the sense of Holmström and Tirole (2001), limits financial assets’ ability to finance future consumption or investment needs, by decreasing the amount of wealth that can be transported to the future. A higher illiquidity, in the sense of this paper, also affects financial assets’ ability to finance future consumption, by increasing the uncertainty in their payoffs.

This additional risk arising from the imprecise knowledge of the wealth distribution is priced in equilibrium and affects asset prices, which may look anomalous relative to standard asset pricing models that do not incorporate this risk. This relates our paper to the literature on mispricings, e.g. Basak and Croitoru (2000, 2006), and De Long, Shleifer, Summers, Robert, and Waldmann (1990). In Basak and Croitoru (2000, 2006), mispricings arise due to position limits faced by investors, and in De Long, Shleifer, Summers, Robert, and Waldmann (1990), mispricings arise due to noise traders. The main difference with these papers is the source of mispricing. In contrast to these papers, misprings in our paper are neither due to constraints nor due to any irrational behaviour, but arise because investors take into account the imprecision in their price forecasts in valuing assets. Because forecasted prices of different assets can have a different degree of imprecision
depending on their dividend stream, this can cause some assets to be over- or under-valued relative to others.

The presence of this additional risk makes markets incomplete, and creates a role for derivatives, and, hence, our paper also contributes to the literature on the role of derivatives in financial markets. Detemple and Selden (1991) study the interaction between primary and derivative asset markets when primary market is dynamically incomplete. Cao (1999) study the effect of introducing a derivative on the acquisition of information in an incomplete market. Carr and Madan (2001) study optimal allocations with derivative assets in a setting where the demand for derivatives is driven by investors’ inability to trade continuously in the underlying assets. In this strand of literature, the demand for derivative assets generally arises due to transaction costs or because there are not enough assets to hedge the cashflow shocks perfectly. Our contribution to this literature is to provide a new source of market incompleteness that arises from investors’ imprecise knowledge about the wealth distribution. This can cause markets to be incomplete even in the absence of any transaction costs, and even when there are enough primary assets to hedge cashflow shocks, and creates a demand for derivatives. Derivatives are used to hedge as well as speculate on this uncertainty about the wealth distribution. Derivatives with different payoff functions of the same underlying assets may have different exposures to the uncertainty about the wealth distribution, and may earn different liquidity risk premia, creating systematic mispricings between derivatives and primary assets.

3 Economic Setting

We consider a discrete-time, pure-exchange economy with a single perishable consumption good that serves as the numeraire, one riskfree and one risky asset, and one derivative on the risky assets. The economy is populated by three groups of investors, one of which faces a funding-ratio constraint. Constraints are indexed by $l \in 0, 1$, where $l = 0$ denotes the budget constraint, and $l = 1$ denotes a funding-ratio constraint. The traded assets are indexed by $n \in \{0, 1, 1\}$, where $n = 0, 1, 2$
denote the riskfree, risky, and derivative assets, respectively. The payoff of each asset is denoted by $X_n$, and the supply of each asset is denoted by $Z_n$. Investors are indexed by $m \in [0, M]$. The risky asset is a claim to a stochastic dividend, $d_{n,t}$, with a positive net supply that is normalized to one, while the riskfree asset is a one-period riskfree claim to one unit of the consumption good and is in zero net supply.

All investors optimise their utility from intertemporal consumption, and know the true dynamics of the dividend processes of all assets. Investors are assumed to have identical preferences, and differ from each other only due to their constraints and their information regarding the consumption (wealth) distribution in the economy. The consumption-share of each investor is denoted by $\omega_m$, and the vector of consumption-shares, $\omega$, serves as the state variable. Vectors of variables across investors are denoted by an underline, $\underline{\mathbf{x}}$, and the vectors of variables across assets are denoted by a boldface, $\mathbf{x}$. For example, the vector of consumption shares across investors is denoted by $\omega$; the vector of prices for traded assets is denoted by $\mathbf{P}$; and the matrix of allocations to all traded assets by all investors is denoted by $\underline{\mathbf{\theta}}$.

3.1 Preferences

We assume that all investors have identical constant relative risk aversion (CRRA) preferences with power utility. That is, the utility function for each investor can be written as

$$u_m(c_{m,t}) = \frac{c_{m,t}^{1-\gamma}}{1-\gamma},$$

(1)

where $c_{m,t}$ is the amount of consumption at time $t$, and $\gamma$ is the risk aversion parameter, which is assumed to be equal for all investors.
3.2 Objective Function

In its general form, we write the objective function of the $m^{th}$ investor as

$$V_{m,t} = \max_{c_{m,t}} \left[ \sum_{\tau=t}^{T} \beta^{T-\tau} u_m(c_{m,\tau}) \right]$$

$$= \max_{c_{m,t}, \theta_{m,t}} \left( u_m(c_{m,t}) + \beta E_{m,t} [V_{m,t+1}(W_{m,t+1})] \right),$$

where $\beta$ denotes subjective time-discount rate, which is assumed to be equal for all investors, and $W_{m,t}$ denotes the total wealth of the $m^{th}$ investor at time $t$.

Forming the Lagrangian for this constrained optimisation problem, we get

$$L_{m,t} = u_m(c_{m,t}) + \beta E_{m,t} [V_{m,t+1}(W_{m,t+1})]$$

$$+ \lambda_{m,t}^{bc} \left( \sum_n \theta_{m,t-1} X_{n,t} - c_{m,t} - \sum_n \theta_{m,t} P_{n,t} \right)$$

$$+ \lambda_{m,t}^{fr} \left( \sum_n \theta_{m,t} P_{n,t} - F_{m,t}^{\min} \right)$$

where $P_{n,t}$ denotes the price of the $n^{th}$ asset, $X_{n,t}$ is the payoff of the $n^{th}$ asset, $\lambda_{m,t}^{bc}$ denotes the Lagrange coefficient for the budget constraint, $\lambda_{m,t}^{fr}$ denotes the Lagrange coefficient for the funding ratio constraint, $F_{m,t}^{\min}$ denotes the minimum wealth that the funding-ratio constraint requires, and is given by the present value of an exogenously given consumption stream

$$F_{m,t}^{\min} = \phi_{m}^{\min} E_{m,t} \left[ \sum_{\tau=1}^{T-t} m_{m,t+\tau} c_{m,t+\tau}^{\min} \right],$$

where $c_{t:T}^{\min}$ is the exogenous consumption stream, $\phi_{m}^{\min}$ is the fraction of the present value of exogenous consumption stream that the constrained investor has to hold, and $m_{m,t+\tau}$ is the stochastic discount factor of the $m^{th}$ investor for cashflows at time $t + \tau$, which is determined endogenously.
The equilibrium conditions for optimal consumptions and allocations of each investor can be obtained as the first order Karuch-Kuhn-Tucker (KKT) conditions using each investor’s Lagrangian. The first order condition w.r.t. consumption is

\[ \lambda_{m,t}^{bc} = u'_m(c_{m,t}). \]  

(6)

The first order condition w.r.t. \( \lambda_{m,t}^{bc} \) is

\[ c_{m,t} + \sum_n \theta_{m,t}^n P_{n,t} = \sum_n \theta_{m,t-1}^n X_{n,t}. \]  

(7)

The first order condition w.r.t. the allocation to \( n^{th} \) asset is given by

\[ P_{m,n,t} = E_{m,t} \left[ V_{m,t+1} X_{n,t+1} \right] \]  

(8)

where \( P_{m,n,t} \) denotes the price of \( n^{th} \) asset computed under \( m^{th} \) investor’s expectation operator. In equilibrium all investors have to agree on the prices of traded assets, and therefore

\[ P_{m,n,t} = P_{m',n,t}, \text{ where } m \neq m'. \]  

(9)

The complementary slackness conditions for the funding-ratio constraint is

\[ \lambda_{m,t}^{fr} \left[ \theta_{m,t}^0 P_{0,t} + \theta_{m,t}^1 P_{1,t} - \phi_{m,t} F_{m,t}^{emin} \right] = 0. \]  

(10)

The envelope theorem yields

\[ \frac{dV_{m,t}}{dW_{m,t}} = \frac{\partial I_{m,t}}{\partial W_{m,t}} = \lambda_{m,t}^{bc}. \]  

(11)

The optimality conditions for unconstrained investors can be obtained by setting \( \lambda_{m,t}^{fr} \) to zero in the set of equilibrium conditions written above (Equations (6) to (10)).
The market clearing conditions for all assets can be written as

$$\sum_{m=0}^{M} \theta_{m,t}^n = Z_n. \quad (12)$$

We define the consumption share of \(m\)th investor as

$$\omega_{m,t} \equiv \frac{c_{m,t}}{\sum_m c_{m,t}},$$

and use the vector of consumption shares, \(\omega_t\), across investors as a state variable.

Now we discuss how the wealth distribution affects equilibrium (see Appendix A for a more detailed discussion). The simplest setting in which the wealth distribution can have an effect is a three-date setting. Therefore, we consider three dates \(T\), \(T-1\), and \(T-2\). At the terminal date \(T\), prices of all assets are zero, and, hence, assets’ payoffs are equal to their dividends. Investors choose their optimal consumption given a realisation of assets’ dividends. Denoting optimal quantities with a caret, \(\hat{}\), the optimal consumptions at time \(T\) can be written as

$$\hat{c}_{m,T} = \sum_{n} \theta_{m,T-1}^n d_{n,T}. \quad (13)$$

Because each investor’s allocations at time \(T-1\) and dividend realisations at time \(T\) are known to the investor, the optimal consumption can be determined without any knowledge of the wealth distribution.

At time \(T-1\), investors make optimal decisions about both consumptions and allocations.

**Proposition 3.1.** At time \(T-1\), the wealth distribution affects the optimal allocation and consumption decisions only through its effect on time \(T-1\) asset prices. That is

$$\hat{c}_{m,T-1} \equiv \hat{c}_{m,T-1} (P_{T-1}, \lambda_{m,T-1})$$

$$\hat{\theta}_{m,T-1}^n \equiv \hat{\theta}_{m,T-1}^n (P_{T-1}, \omega_{m,T-1}, \lambda_{m,T-1}).$$
Therefore, asset prices at time $T - 1$ convey sufficient information for investors to implement their optimal decisions at time $T - 1$.

In turn, time $T - 1$ prices can be written as a function of the optimal consumptions, $\xi_{T-1}$, and shadow prices of constraints, $\Lambda_{T-1}$, at time $T - 1$. If we consider all possible values for the shadow prices of constraints, and define the wealth distribution, $\Omega$, as the distribution of wealth across this fixed vector of shadow prices of constraints

$$\Omega \equiv \Omega(w, \Lambda),$$  \hspace{1cm} (14)

then the equilibrium prices can be written as a function of the wealth distribution alone

$$P_{T-1} \equiv P_{T-1}(\Omega_{T-1}).$$

At time $T - 2$, the optimal consumption and allocation decisions depend not only on prices at time $T - 2$, which are observable, but also on prices at time $T - 1$, which are not observable.

**Proposition 3.2.** At time $T - 2$, the wealth distribution affects optimal allocation decisions both through its effect on current, time $T - 2$, prices, but also its effect on future, time $T - 1$, prices. That is

$$\hat{c}_{m,T-2} \equiv \hat{c}_{m,T-2}(P_{T-2}, \lambda_{m,T-2}, \Omega_{T-1})$$
$$\hat{\theta}^n_{m,T-2} \equiv \hat{\theta}^n_{m,T-2}(P_{T-2}, \omega_{m,T-2}, \lambda_{m,T-2}, \Omega_{T-1}).$$

Therefore, asset prices at time $T - 2$ do not convey sufficient information for investors to implement their optimal decisions at time $T - 1$.

Thus, at time $T - 2$, investors need to forecast the wealth distribution, $\Omega_{T-1}$, at time $T - 1$ to obtain a forecast of time-$T - 1$ prices, and implement their optimal decisions. Therefore, the imprecision about the wealth distribution at time $T - 2$ will affect agents’ optimal decisions through its effect on their price forecasts.
4 Three-Investor Model

The simplest setting in which the imprecision about the wealth distribution can play a role is the one with three investors, three dates, and one constraint. With only two investors, there can be no imprecision about the wealth distribution across investors if the aggregate dividend (consumption) and the price of the aggregate dividend (aggregate wealth) is known, as each investor can infer other investor’s wealth by knowing its own wealth. And with only two dates, the imprecision about the wealth distribution has no effect on asset prices. The presence of a constraint makes asset prices depend on the wealth distribution, so that the imprecision about the wealth distribution can affect agents’ allocation decisions.

We consider one funding-ratio constrained investor, and two unconstrained investors. All investors have access to one riskfree asset in zero net supply, one risky asset in positive net supply, and one derivative asset in zero net supply. The derivative asset allows us to explore the demand for derivative due to uncertainty about the wealth distribution. When the markets are complete, such a derivative would be redundant, and the equilibrium would be unaffected by whether or not the derivative trades. However, the markets become incomplete when there is uncertainty about the wealth distribution, and the derivative can affect equilibrium. To avoid notational ambiguity, we denote the three dates as $T$, $T-1$, and $T-2$, and the three investors as 0, 1, and 2.

While a three-investor model is less realistic than a multi-investor model both because it has only three investors, and because it lends a special status to the constrained investor, as asset prices depend on the information related to constrained investor, who would then naturally have superior information to forecast prices compared to the other investors. However, the three-investor model is notationally simple, and the main results that are the focus of our paper, do not depend on these special features of the three-investor model. Therefore, we first present the model and main results in a three-investor model, and then in Section 5.1, we show that these special features of the three-investor model go away in a setting with large number of investors, but the results obtained in the three-investor model can be extended to this setting in a straightforward manner.
In the next two sections, we characterise the equilibrium in this three-investor setting with perfect and imperfect information about the wealth distribution, respectively. We denote the optimal quantities (allocations, consumptions, and prices) in the unconstrained case, which corresponds to a model without demand shocks, with a ring, \( \widehat{\cdot} \), the optimal quantities in the case of perfect information, which corresponds to a model with forecastable demand shocks, with a caret, \( \overset{\sim}{\cdot} \), and the optimal quantities in the case of imperfect information, which corresponds to a model with unforecastable demand shocks, with a tilde, \( \tilde{\cdot} \). The difference in the optimal quantities between imperfect and perfect information is denoted by \( \Delta \widehat{\xi} \equiv \widehat{\xi} - \overset{\sim}{\xi} \).

### 4.1 The Benchmark Model with Perfect Information

In this section, we solve for the equilibrium when all investors have perfect information about the wealth distribution. We use it as our benchmark model, and characterise the impact of imprecisely known wealth distribution in terms of deviations from this perfect-information benchmark in Section 4.2.

#### 4.1.1 Allocations and Prices at Time \( T - 1 \)

Given that there are no allocation decisions to be made at time \( T \), we start by solving the optimal allocation decisions at time \( T - 1 \), which finance consumptions at time \( T \).

**Proposition 4.1.** The equilibrium allocations to the riskfree asset are given by

\[
\overset{0}{\theta}_{0,T-1} = \overset{0}{\theta}_{1,T-1} = \overset{0}{\theta}_{2,T-1} = 0. \tag{15}
\]

The equilibrium allocations to the risky asset for \( m = \{0, 2\} \) (unconstrained) investors are given by

\[
\overset{m}{\theta}_{m,T-1} = \begin{cases} 
\omega_{m,T-1}, & \text{if } \omega_{1,T-1} > \omega_{T-1}^{\text{min}} \\
\omega_{m,T-1} \frac{1 - \omega_{T-1}^{\text{min}}}{1 - \omega_{1,T-1}}, & \text{if } \omega_{1,T-1} \leq \omega_{T-1}^{\text{min}}
\end{cases} \tag{16}
\]

18
and the equilibrium allocation to the risky asset for \( m = 1 \) (constrained) investor is given by

\[
\hat{\theta}_{1,T-1} = \begin{cases} 
\omega_{1,T-1}, & \text{if } \omega_{1,T-1} > \omega_{T-1}^{\min} \\
\omega_{T-1}^{\min}, & \text{if } \omega_{1,T-1} \leq \omega_{T-1}^{\min}.
\end{cases}
\] (17)

where \( \omega_{T-1}^{\min} \) denotes the threshold consumption share of the funding-ratio constrained investor below which the constraint becomes binding, and is given by

\[
\omega_{T-1}^{\min} = \phi_{T-1}^{\min} E_{T-1} \left[ \frac{c_{T}}{d_{1,T}} \right].
\] (18)

We assume \( \phi_{T-1}^{\min} c_{T}^{\min} < E_{T-1}[d_{1,T}] \), so that \( \omega_{T-1}^{\min} < 1 \). The corresponding consumptions at time \( T \) are given by

\[
\tilde{c}_{m,T} = \hat{\theta}_{m,T-1} d_{1,T},
\] (19)

Once, the optimal consumptions at time \( T \) are known, the prices the at time \( T - 1 \) can be obtained as a function of consumption shares at time \( T - 1 \), by discounting their time-\( T \) payoffs

\[
P_{n,T-1} = \beta E_{T-2} \left[ \frac{c_{0,T-1}^{\gamma}}{c_{0,T-1}^{\gamma}} X_{n,T} \right],
\] (20)

where \( X_{n,T} \) is the payoff of the \( n^{th} \) asset at time \( T \), and \( \beta \frac{c_{0,T-1}^{\gamma}}{c_{0,T-1}^{\gamma}} \) is the stochastic discount factor of \( m = 0 \) investor, which is equal to the stochastic discount factors of \( m = 1 \) and \( m = 2 \) investors.
Corollary 1. Prices of the riskfree and risky assets are given by

\[
\begin{align*}
\hat{P}_{0,T-1} &= \begin{cases} 
\beta E_{T-1}[d_1^T, \gamma], & \text{if } \omega_{1,T-1} > \omega_{T-1}^{\text{min}} \\
\beta \left( \frac{1-\omega_{T-1}^{\text{min}}}{1-\omega_{1,T-1}} \right)^{-\gamma} E_{T-1}[d_1^T, \gamma], & \text{if } \omega_{1,T-1} \leq \omega_{T-1}^{\text{min}}
\end{cases} \\
\hat{P}_{1,T-1} &= \begin{cases} 
\beta E_{T-1}[d_1^T, \gamma], & \text{if } \omega_{1,T-1} > \omega_{T-1}^{\text{min}} \\
\beta \left( \frac{1-\omega_{T-1}^{\text{min}}}{1-\omega_{1,T-1}} \right)^{-\gamma} E_{T-1}[d_1^T, \gamma], & \text{if } \omega_{1,T-1} \leq \omega_{T-1}^{\text{min}}
\end{cases}
\end{align*}
\]

(21)

(22)

Corollary 2. The effect of the constraint can be summarised in one variable \( \Omega_{1,T-1} = \frac{1-\omega_{T-1}^{\text{min}}}{1-\omega_{1,T-1}} \) as

\[
f_\omega(\Omega_{1,T-1}) = \min (1, \Omega_{1,T-1}) = \begin{cases} 
1, & \text{if } \omega_{1,T-1} > \omega_{T-1}^{\text{min}} \\
\Omega_{1,T-1}, & \text{if } \omega_{1,T-1} \leq \omega_{T-1}^{\text{min}}
\end{cases}
\]

(23)

(24)

The equilibrium quantities can then be written more succinctly in terms of their unconstrained counterparts, which we denote by \( \hat{\theta}, \hat{c}, \hat{P} \), as

\[
\begin{align*}
\hat{\theta}_{m,T-1}^0 &= \hat{\theta}_{m,T-1}^0 = 0 \\
\hat{\theta}_{m,T-1}^1 &= \hat{\theta}_{m,T-1}^1 f_\omega(\hat{\Omega}_{1,T-1}), \text{ for } m = \{0, 2\} \\
\hat{\theta}_{1,T-1}^1 &= \hat{\theta}_{1,T-1}^1 f_\omega(\hat{\Omega}_{1,T-1}) + 1 - f_\omega(\hat{\Omega}_{1,T-1}) \\
\hat{c}_{m,T} &= \hat{c}_{m,T} f_\omega(\hat{\Omega}_{1,T-1}) \\
\hat{P}_{n,T-1} &= \hat{P}_{n,T-1} f_\omega(\hat{\Omega}_{1,T-1})
\end{align*}
\]

(25)

(26)

(27)

(28)

(29)

When \( \hat{\omega}_{1,T-1} > \omega_{T-1}^{\text{min}} \), the funding-ratio constraint is not binding, and, as a result, \( f_\omega(\hat{\Omega}_{1,T-1}) = 1 \), which makes all the equilibrium quantities—allocations, consumptions, and prices—indeedependent of the wealth distribution. But when \( \hat{\omega}_{1,T-1} < \omega_{T-1}^{\text{min}} \), the funding-ratio constraint is binding, and the equilibrium quantities deviate from their unconstrained counterparts, and the degree of this deviation is determined by \( \hat{\Omega}_{1,T-1} \), which depends on the distribution of consumption across investors, \( \omega_{T-1} \), with different investment objectives, characterised by \( \omega_{T-1}^{\text{min}} \).
Hence, $\hat{\Omega}_{1,T-1}$ is a function of the wealth distribution across investment objectives, and is what we formally refer to when we talk about the wealth distribution.

Given that the markets are complete in the case of perfect information, the price of any derivative at time $T - 1$ can be written as

$$P_{V',T-1} = \beta E_{T-1} \left[ \frac{c^\gamma_{0,T}}{c^\gamma_{0,T-1}} f(d_{1,T}) \right].$$

**Corollary 3.** If the payoff of the derivative is a linear function of the payoff of the riskfree and risky assets,

$$f(d_{0,T}, d_{1,T}) = a_0 d_{0,T} + a_1 d_{1,T}, \quad (30)$$

as is the case for a futures contract, then the price of the derivative at time $T - 1$ is a linear combination of the prices of the riskfree and risky assets

$$\hat{P}_{V',T-1} = a_0 \hat{P}_{0,T-1} + a_1 \hat{P}_{1,T-1}. \quad (31)$$

Thus, the price of a linear derivative at time $T - 1$ can be obtained by simply observing the prices of riskfree and risky assets.

### 4.1.2 Allocations and Prices at Time $T - 2$

Given that our focus in this paper is to understand the effect of imprecision about the wealth distribution, and not the funding-ratio constraint, we make a simplifying assumption of $c_{T-1}^{\min} = 0$, so that once the constrained investor holds sufficient wealth at time $T - 1$, the constraint is satisfied, and the full effect of the constraint is incorporated in $f_\omega(\hat{\Omega}_{1,T-1})$.

**Proposition 4.2.** If $c_{T-1}^{\min} = 0$, the optimal consumptions at time $T - 1$ can be written as a function of time $T - 2$ consumption share as

$$\hat{c}_{m,T-1} = \omega_{m,T-2} d_{1,T-1}. \quad (32)$$
Hence \( \hat{\Omega}_{1,T-1} \) can be written as

\[
\hat{\Omega}_{1,T-1} = \frac{1 - \hat{\omega}_{1,T-1}}{1 - \omega_{T-1}^{min}} = \frac{1 - \omega_{1,T-2}}{1 - \omega_{T-1}^{min}}.\tag{33}
\]

Given optimal consumptions at time \( T - 1 \), equilibrium prices at time \( T - 2 \) can be obtained as a function of consumptions at time \( T - 2 \)

\[
\hat{P}_{n,T-2} = \beta E_{T-2} \left[ \frac{c^{\gamma}_{0,T-1}}{c^{\gamma}_{0,T-2}} \hat{X}_{n,T-1} \right] \tag{34}
\]

**Corollary 4.** Equilibrium prices at time \( T - 2 \) are given by

\[
\hat{P}_{0,T-2} = \beta E_{T-2} \left[ \frac{d_{1,T-1}^{-1}}{d_{1,T-2}^{-1}} \right] \tag{35}
\]

\[
\hat{P}_{1,T-2} = \frac{\beta}{d_{1,T-2}^{\gamma}} E_{T-2} \left[ d_{1,T-1}^{(1-\gamma)} + \beta d_{1,T-1}^{(1-\gamma)} f_{\omega}^{-\gamma}(\hat{\Omega}_{1,T-1}) \right] \tag{36}
\]

\[
\hat{P}_{1',T-2} = \frac{\beta^2}{d_{1,T-2}^{\gamma}} E_{T-2} \left[ f_{\omega}^{-\gamma}(\hat{\Omega}_{1,T-1}) \left( a_0 d_{1,T-1}^{-\gamma} + a_1 d_{1,T-1}^{(1-\gamma)} \right) \right] \tag{37}
\]

\[
= E_{T-2} \left[ \frac{d_{1,T-1}^{-1}}{d_{1,T-2}^{-1}} \left( a_0 \hat{P}_{0,T-1} + a_1 \hat{P}_{1,T-1} \right) \right]. \tag{38}
\]

These prices provide the fundamental values of assets that will prevail in the case of prefect information. The price of the linear derivative at time \( T - 2 \) is no longer a linear combination of observed prices of riskfree and risky assets, as was the case at time \( T - 1 \) (Equation (31)). Therefore, to determine the equilibrium price of the derivative, investors either need to know the Arrow-Debreu prices for time-\( T \) dividend realisations (Equation (37)), or they need to forecast the price of riskfree and risky assets at time \( T - 1 \) (Equation (38)). In the case of perfect information, \( f_{\omega}^{-\gamma}(\hat{\Omega}_{1,T-1}) \) is known, and, hence, the prices of riskfree and risky assets time \( T - 1 \) (conditional on the dividend realisation at time \( T - 1 \)) can be forecasted by knowing the dividend dynamics. Thus, the cost of creating a ‘synthetic derivative’ using the riskfree and risky assets that replicates the cash flows of the derivative, is known.
conditional on the shocks to dividends, and the equilibrium price of the traded derivative is equal to the expected cost of creating a synthetic derivative.

**Corollary 5.** The optimal allocations at time \( T - 2 \), for \( m = \{0, 2\} \) investors, can be written as

\[
\hat{\theta}_{m,T-2}^1 = \frac{\Delta_{u,d} \left( d_{1,T-1} + f_{1,T-1}^{1-\gamma}(\hat{\Omega}_{1,T-1})\hat{P}_{1,T-1} \right)}{\Delta_{u,d} \left( d_{1,T-1} + f_{1,T-1}^{1-\gamma}(\hat{\Omega}_{1,T-1})\hat{P}_{1,T-1} \right)}
\]

\[
\hat{\theta}_{m,T-2}^0 = \omega_{m,T-2} \left( d_{1,T-1,u} + f_{1,T-1,u}^{1-\gamma}\hat{P}_{1,T-1,u} \right) - \hat{\theta}_{m,T-2}^1 \left( d_{1,T-1,u} + f_{1,T-1,u}^{1-\gamma}\hat{P}_{1,T-1,u} \right),
\]

where \( \Delta_{u,d}(X) = X_u - X_d \), and \( u, d \) denote the two nodes at time \( T - 1 \) corresponding to positive and negative innovations in the aggregate dividend, respectively.

For \( m = 1 \) investor, the optimal allocations at time \( T - 2 \) are given by

\[
\hat{\theta}_{1,T-2}^1 = \frac{\Delta_{u,d} \left( d_{1,T-1} + f_{1,T-1}^{1-\gamma}(\hat{\Omega}_{1,T-1})\hat{P}_{1,T-1} \right)}{\Delta_{u,d} \left( d_{1,T-1} + f_{1,T-1}^{1-\gamma}(\hat{\Omega}_{1,T-1})\hat{P}_{1,T-1} \right)} + \Delta_{u,d} \left( f_{1,T-1}^{1-\gamma}(1 - f_{1,T-1})\hat{P}_{1,T-1} \right)
\]

\[
\hat{\theta}_{1,T-2}^0 = \omega_{1,T-2} \left( d_{1,T-1,u} + f_{1,T-1,u}^{1-\gamma}\hat{P}_{1,T-1,u} \right) - \hat{\theta}_{1,T-2}^1 \left( d_{1,T-1,u} + f_{1,T-1,u}^{1-\gamma}\hat{P}_{1,T-1,u} \right)
\]

\[
+ (1 - f_{1,T-1,u}) f_{1,T-1,u}^{1-\gamma}\hat{P}_{1,T-1,u},
\]

When \( \hat{\omega}_{1,T-2} > \{\omega_{T-1,d}^{\min}, \omega_{T-1,u}^{\min}\} \), optimal allocations at time \( T - 2 \) reduce to

\[
\hat{\theta}_{m,T-2}^0 = \hat{\theta}_{m,T-2}^1 = 0
\]

\[
\hat{\theta}_{m,T-2}^0 = \hat{\theta}_{m,T-2}^1 = \omega_{m,T-2},
\]

which is independent of the constrained investor’s share of consumption. However, when \( \hat{\omega}_{1,T-2} \not\approx \{\omega_{T-1,d}^{\min}, \omega_{T-1,u}^{\min}\} \), the optimal allocations of all investors depend on the wealth distribution through \( \Omega_{1,T-2} \). Moreover, in contrast to optimal allocations at time \( T - 1 \), which can be written solely in terms of prices at time \( T - 1 \), the allocations at time \( T - 2 \) cannot be written solely in terms of the time \( T - 2 \) prices. The equilibrium price of the risky asset at time \( T - 2 \) is a function of the \( f_{\omega} \)-weighted
expected value of time-$T - 1$ prices, $E_{T - 2} \left[ \hat{P}_{1,T-1} f_{\omega}^{-\gamma} \right]$, while optimal allocations require knowing dispersion in $f_{\omega}$-weighted time-$T - 1$ prices, $\Delta_{\omega,d} \left( f_{\omega}^{1-\gamma} \hat{P}_{1,T-1} \right)$. Therefore, the prices of riskfree and risky assets at time $T - 2$ do not convey sufficient information for investors to implement their optimal allocations, and, as a result, the optimal allocations are affected by investors’ imprecision about the wealth distribution.

It should be noted that if the derivative was traded and priced according to Equation (37), then investors would have sufficient information to estimate $f_{\omega}^{-\gamma}$ for both dividend realisations at time $T - 1$. However, in such a case, the dividend would be redundant, and likely would not be traded. Therefore, Equation (37) would only be valid if all investors priced assets as if they had perfect information about the wealth distribution, and the derivative was indeed redundant, and, therefore, cannot be used to estimate the wealth distribution accurately, if investors do not believe that they have perfect information about the wealth distribution, in which case the price of the derivative would be affected by investors’ imprecision about $f_{\omega}^{-\gamma}$, as we discuss in the next section.

## 4.2 The Model with Imprecise Information

The imprecise information about the wealth distribution only affects equilibrium at time $T - 2$, which we characterise in this section. The Equation (29) shows that asset prices at time $T - 1$ are affected by $\hat{\Omega}_{1,T-1}$, which we referred to as the wealth distribution. In general, investors may be unaware of the true functional relation between prices and the wealth distribution. In which case, investors would have to form beliefs both about the functional relation between prices and the wealth distribution, $\hat{P}_{T-1} \left( \hat{\Omega}_{1,T-1} \right)$, and the wealth distribution, $\hat{\Omega}_{1,T-1}$. For simplicity, we assume in this section that investors have accurate beliefs about the functional relation, and are only uncertain about $\hat{\Omega}_{1,T-1}$.

When $\Omega_{1,T-1}$ is imprecisely known to some investors, they can try to infer $\Omega_{1,T-1}$ from observed prices at time $T - 2$, but these prices depend both on the average level, $E_{T-2}[\Omega_{1,T-1}]$, and the covariance, $\text{Cov}_{T-2} (d_{1,T}, \Omega_{1,T-1})$, of $\Omega_{1,T-1}$ with the
dividend at time $T - 1$. Therefore, investors would remain uncertain about $\Omega_{1,T-1}$ at each node of time $T - 1$, which is required to implement their optimal allocations given in Corollary 8.

Additional uncertainty about $\Omega_{1,T-1}$ may arise if $c^{\text{min}}$ is stochastic. In this case, the constrained investor’s consumption at time $T - 1$, and hence $\Omega_{1,T-1}$ will be affected by shocks to $c_T^{\text{min}}$ at time $T - 1$. That is, shocks to $\Omega_{1,T-1}$ may be driven both by shocks to cashflows and by shocks to parameters related to investment objectives (in this case $c^{\text{min}}$). Therefore, we assume that investors set their allocations at time $T - 2$ according to their subjective forecasts of time-$T - 1$ prices, based on their subjective forecasts of $\Omega_{1,T-1}$, and take into account the imprecision in their forecast.

Thus, the subjective price of each asset can be written explicitly as

$$P_{n,T-2}^m = E \left[ \beta \frac{u'(\sum_n \theta_{m,T-2}^n X_{n,T-1} - F_{m,T-1}) X_{n,T-1}}{u'(c_{m,T-2})} \mid \mathcal{F}_{m,T-2} \right]$$

$$= E_{T-2} \left[ \int g_m(\Omega_{1,T-1}) u'(\sum_n \theta_{m,T-2}^n X_{n,T-1} - F_{m,T-1}) X_{n,T-1} \frac{d\Omega_{1,T-1}}{u'(c_{m,T-2})} \right]$$

where $\mathcal{F}_{m,T-2}$ denotes the $m^{th}$ investor’s information set, which consists of information about realised dividends, prices, and information about the distribution of consumption across investors, $\Omega_{1,T-2}$, and exogenous consumption stream of the constrained investor, $c_{m,T-2}^{\text{min}}$. Based on its information set, the uninformed investor forms beliefs about $\Omega_{1,T-1}$, which are represented by $g_m(\Omega_{1,T-1})$, which denotes the subjective probability density of $\Omega_{1,T-1}$ for $m^{th}$ investor.

Given that $\Omega_{1,T-1}$ depend on variables related to $m = 1$ investor, we assume that $m = 1$ investor can perfectly forecast $\Omega_{1,T-1}$. To allow the possibility of some investors having perfect information about other investors, we assume $m = 0$ also has perfect information about $\Omega_{1,T-1}$. Hence, only $m = 2$ investor has imprecise information about $\Omega_{1,T-1}$, and we assume its subjective forecast of $\Omega_{1,T-1}$
is distributed according to

$$\Pr(X \leq \Omega_{1,T-1} \leq X + dX \mid \mathcal{F}_{2,T-2}) \equiv g(X)dX$$ (45)$$

where $X \in [\Omega_{1,T-1}^{\min}, \Omega_{1,T-1}^{\max}]$, and $\Omega_{1,T-1} = E_{2,T-2}[\Omega_{1,T-1}]$ denotes the $m = 2$ investor’s average estimate of $\Omega_{1,T-1}$.

This difference in the precision of investors’ forecast of $\Omega_{1,T-1}$ creates arbitrage opportunities for more informed investors. These investors can profit by effectively selling derivative claims to uninformed investors that promise consumption for those values of $\Omega_{1,T-1}$ which informed investors know will not realise, or are less probable, but uninformed investors do not. However, as long as uninformed investors recognise that there are more informed investors, they will recognise that they may over- or under-estimate the value of certain claims sold by informed investors due to their imprecise information, and may not wish to trade such claims.

For instance, consider a derivative with payoff of the form $f(d_{1,t}, P_{1,t})$, which pays only upon certain price realisations contingent on a dividend realisation at time $t$. Such a claim can even be worthless, or close to worthless, if the price realisations upon which it pays are not possible, or very unlikely, given the actual wealth distribution. Therefore, uninformed investors would know that they may overestimate the value of such a claim, and would not be willing to take long positions in such claims. And if informed investors take long positions in such claims, they would simply reveal their information to uninformed investors. Thus, such claims may remain untraded, as no one would have the incentive to trade them.

The only derivative assets that may be traded are the ones for which the payoff can be written as a function of the aggregate dividend shocks, $f(d_t)$. For these claims, uninformed investors will not know whether they are overestimating or underestimating their value, and will be willing to take long and short positions in such assets. Moreover, because these claims pay for all price realisations contingent on a dividend realisation, they would not necessarily reveal more informed investors’ information about the wealth distribution. Hence, such claims would be more attractive to all investors, and more likely to be traded. Therefore, we limit our
attention to claims with payoff functions of the form \( f(d_t) \), which is general enough to consider standard derivatives, such as futures, and call and put options.

In general, it is not necessary to assume that informed investors know \( \Omega_{1,T-1} \) with full precision. More generally, one can assume that informed investors knowledge of \( \Omega_{1,T-1} \) is also imprecise, but less imprecise than the uninformed investor’s knowledge of \( \Omega_{1,T-1} \). In this case, all investors’ forecasts of \( \Omega_{1,T-1} \) will follow a probability distribution for all possible values of \( \Omega_{1,T-1} \). For more informed investors, the probability distribution would be more concentrated around the true value, and for less informed investors the probability distribution will be more spread out. More informed investors would still be able to profit by selling claims to uninformed investors for consumption in those states to which informed investors assign less probability than uninformed investors. In this section, however, we abstract away from introducing multiple probability distributions, and assume that the informed investors’ probability distribution is sufficiently concentrated, so that it can be approximated by a delta function. In Section 5, we extend this model to more general probability distributions, and show that the main insights of the three-investor model are unaffected by this change.

Now, we look at how the imprecision about the wealth distribution affects the uninformed investor’s optimal behaviour.

\(^2\)Our assumption that some investors assign non-zero probabilities to states that will not realise implies that their pricing measure is not equivalent to the true pricing measure. Grossman (1988) also propose that when investors have incomplete information about the demand for portfolio insurance, stock price volatility should be modelled as a stochastic variable even when the true volatility is deterministic (conditional on perfect information), implying that investors’ subjective probability measures will not be equivalent to the true probability measure. A more general justification for such an assumption can be found in Kurz (1994b,a), which provides a general and detailed discussion of how investors may use ‘wrong’ probability measures for forecasting prices, if they do not know the structural relations of the economy, and try to infer their price forecasting model from observed data. Kurz (1994b) shows that in such a setting, investors’ forecasting measure can be written as a combination of two measures, one of which is equivalent to the true probability measure, and the other is orthogonal to the true measure. Therefore, in general, different agents’ probability measures may neither be equivalent to each other’s nor to the objective measure, and markets become inherently incomplete (Kurz (2008)).
Proposition 4.3. Subjective prices for \( m = 2 \) investor can be written explicitly in terms of \( f_\omega \) as

\[
P_{n,T-2}^2 = \frac{\beta}{c_{2,T-2}} E_{T-2} \left[ \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} g_2 (\Omega_{1,T-1}) (A_{2,0} + A_{2,1} f_\omega^{-\gamma} + A_{2,2} f_\omega^{1-\gamma})^{-\gamma} X_{n,T-1}(\Omega_{1,T-1}) d\Omega_{1,T-1} \right],
\]

(46)

and for \( m = \{0, 1\} \) investors as

\[
P_{n,T-2}^m = \frac{\beta}{c_{2,T-2}} E_{T-2} \left[ (A_{m,0} + A_{m,1} f_\omega^{1-\gamma})^{-\gamma} X_{n,T-1}(\Omega_{1,T-1}) \right]
\]

where \( X_{n,T-1}(\Omega_{1,T-1}) \) denotes the \( n \text{th} \) asset’s payoff at time \( T - 1 \)

\[
X_{0,T-1}(\Omega_{1,T-1}) = 1
\]

\[
X_{1,T-1}(\Omega_{1,T-1}) = (d_{1,T-1} + \hat{P}_{1,T-1} f_\omega^{-\gamma}(\Omega_{1,T-1}))
\]

\[
X_{1',T-1}(\Omega_{1,T-1}) = (a_0 \hat{P}_{0,T-1} + a_1 \hat{P}_{1,T-1}) f_\omega^{\gamma}(\Omega_{1,T-1}),
\]

and coefficients \( A_{m,n} \) are independent of \( \omega'_{1,T-1} \)

\[
A_{(0,2),0} = \theta_{(0,2),T-2}^0 + \theta_{(0,2),T-2}^1 d_{1,T-1}
\]

\[
A_{(0,2),1} = a_0 \theta_{(0,2),T-2}^0 \hat{P}_{0,T-1} + (\theta_{(0,2),T-2}^1 + a_1 \theta_{(0,2),T-2}^2) \hat{P}_{1,T-1}
\]

\[
A_{(0,2),2} = -\theta_{(0,2),T-1}^1 \hat{P}_{1,T-1}
\]

\[
A_{1,0} = \theta_{1,T-2}^0 + \theta_{1,T-2}^1 d_{1,T-1}
\]

\[
A_{1,1} = a_0 \theta_{1,T-2}^0 \hat{P}_{0,T-1} + (\theta_{1,T-2}^1 + a_1 \theta_{1,T-2}^2 - 1) \hat{P}_{1,T-1}
\]

\[
A_{1,2} = -\left( \theta_{1,T-1}^1 - 1 \right) \hat{P}_{1,T-1}.
\]

Proposition 4.3 shows that, in general, the wealth distribution affects investors’ subjective prices both through their stochastic discount factors and assets’ payoffs. In the special case where allocations to the derivative asset are zero, as they are in the case of perfect information, and there is no rebalancing of allocations to the risky assets, i.e. \( \theta_{m,T-2}^{(1,1')} = \theta_{m,T-1}^{(1,1')} \), the stochastic discount factor becomes independent of the wealth distribution, \( f_\omega \). In this special case, the effect of imprecision in
the asset’s payoff, $X_{n,T-1}(\Omega_{1,T-1})$, also cancels out, and the uninformed investors’ subjective prices converge to the perfect information case, if their average forecast of $\Omega_{1,T-1}$ is unbiased. Therefore, the imprecision in the wealth distribution only affects agents’ subjective prices when there is a need for rebalancing, consistent with our interpretation of the risk arising from imprecision in the wealth distribution as a form of liquidity risk. Conversely, the desire to expose consumption to shocks to the wealth distribution itself creates a demand for rebalancing, even when there would have been no demand for rebalancing in the case of perfect information, as we will later see when we characterise optimal allocation strategies.

**Corollary 6.** We define the variance of uninformed investor’s forecast of $\Omega_{1,T-1}$ as

$$
\Delta^2_{\Omega_{1,T-1}} \equiv \int_{\Omega_{1,T-1}^{\text{max}}}^{\Omega_{1,T-1}^{\text{min}}} g_2(\Omega_{1,T-1}) (\Omega_{1,T-1} - \overline{\Omega}_{1,T-1})^2 \, d\Omega_{1,T-1}, \tag{47}
$$

where $\overline{\Omega}_{1,T-1}$ is the uninformed investor’s average forecast of $\Omega_{1,T-1}$. For small values of $\Delta_{\Omega_{1,T-1}}$, the subjective prices of the uninformed investor is given by

$$
P^2_{n,T-2} = \mathbb{E}_{T-2} \left[ \frac{\Delta^2_{\Omega_{1,T-1}}}{\overline{m}_2} X_{n,T-1} \left( 1 + \frac{\Delta^2_{\Omega_{1,T-1}}}{\overline{m}_2} \left( \frac{\overline{m}_2'}{\overline{m}_2} + 2 \frac{\overline{m}_2'}{\overline{m}_2} \frac{X_n'_{T-1}}{X_{n,T-1}} + \frac{X_n''_{T-1}}{X_{n,T-1}} \right) \right) + \cdots \right] 
$$

$$
= \overline{P}^2_{n,T-2} + \frac{\Delta^2_{\Omega_{1,T-1}}}{\overline{m}_2} \frac{\partial^2 P^2_{n,T-2}}{\partial \overline{\Omega}^2_{1,T-1}} + \cdots,
$$

where $\overline{m}, \overline{X}_{n,T-1}, \overline{m}', \overline{X}'_{n,T-1}, \cdots$ are all evaluated at $\overline{\Omega}_{1,T-1}$.

Corollary 6 allows us to see how imprecision about the wealth distribution affects the subjective asset prices of $m = 2$ (uninformed) investor. With imprecision in the wealth distribution, asset prices are a function not only of the covariation of the pricing kernel with the asset’s payoff, but also of higher order derivatives, i.e. the full distribution of the pricing kernel and the asset’s payoff, and assets can be either over- or under-priced depending on the contributions of different derivatives of these variables. To the first order, the imprecision about the wealth distribution affects asset prices through
\[ \Delta^2_{\Omega_1, T-1} \frac{m_2^2}{m_2} \], which measures the curvature of the investor’s stochastic discount factor. This term is the same for all assets, and can be interpreted as the shift in the level of stochastic discount factor due to imprecise information. The sign of this term depends on the convexity of the pricing kernel, and the more convex the pricing kernel, the higher the shift in the level of stochastic discount factor, and the more overpriced assets will be;

\[ \Delta^2_{\Omega_1, T-1} \frac{m_2^2}{m_2} \frac{X_n,T-1}{X_n,T-1} \], which measures the covariation of the pricing kernel with the asset’s payoff. This term can be interpreted as the change in the beta of the asset due to imprecise information. The sign of this term, as we will show later, depends on the sign of rebalancing in the risky asset allocations. For a decrease in allocations, the covariation of the pricing kernel with the asset’s payoff will be negative, and the term will compensate for the excess risk introduced by imprecise wealth distribution;

\[ \Delta^2_{\Omega_1, T-1} \frac{X_n,T-1}{X_n,T-1} \], which measures the curvature of the asset’s payoff. This term depends on the functional form of the payoff of the asset, and can be interpreted as the change in the level of the asset’s expected payoff due to imprecise information. The contribution of this term can also be either positive or negative depending on the convexity of the asset’s payoff, and assets with more convex payoffs as a function of the wealth distribution will demand higher prices.

The overall effect on the uninformed investor’s subjective price will depend on the relative contribution of these three terms, and it may be negative or positive. This may effectively make some investors to appear to be over- or under-confident, even in the absence of any behavioural biases. Moreover, the uninformed investor’s stochastic discount factor for the asset’s expected payoff upon each dividend realisation effectively becomes asset specific

\[
m_{2}^{\text{eff}} = m_2 \left(1 + \frac{\Delta^2_{\Omega_1, T-1}}{2} \left( \frac{m_2^2}{m_2} \frac{X_n,T-1}{X_n,T-1} + \frac{X_n,T-1}{X_n,T-1} \right) \right).
\]
As a result, different assets may be discounted differently in equilibrium, and may appear to be mispriced relative to each other.

Equating subjective prices, given in Proposition 4.3, across investors along with market clearing conditions for each asset provides us $N \times M$ equations to solve for $N \times M$ unknown allocations at time $T - 2$. In the case of imprecise information, an agreement on the prices of risk-free and risky assets does not allow investors to equate their subjective stochastic discount factors for every realisation of the wealth distribution at time $T - 1$, and, hence, the markets are no longer complete, and derivatives are no longer redundant.

Due to the complex nonlinear dependence of subjective prices on asset allocations, we linearise these equations around $\widehat{\theta}_{T - 2}$ using the first-order Taylor expansion:

$$P_{m,T-2}^n(\theta_{m,T-2}) = P_{m,T-2}^n(\widehat{\theta}_{m,T-2}) + \sum_{j=0,1,1'} (\theta^j_{m,T-2} - \widehat{\theta}^j_{m,T-2}) \frac{\partial P_{m,T-2}^n(\widehat{\theta}_{m,T-2})}{\partial \theta^j_{m,T-2}}. \tag{48}$$

Thus, the set of linearised equilibrium conditions can be written as

$$P_{0,T-2}^0(\widehat{\theta}_{0,T-2}) + \sum_j (\theta^j_{0,T-2} - \widehat{\theta}^j_{0,T-2}) \frac{\partial P_{0,T-2}^0(\widehat{\theta}_{0,T-2})}{\partial \theta^j_{0,T-2}}$$

$$= P_{n,T-2}^n(\widehat{\theta}_{m,T-2}) + \sum_j (\theta^j_{m,T-2} - \widehat{\theta}^j_{m,T-2}) \frac{\partial P_{n,T-2}^m(\widehat{\theta}_{m,T-2})}{\partial \theta^j_{m,T-2}}, \quad m = \{1, 2\} \tag{49}$$

$$\sum_m (\theta^m_{m,T-2} - \widehat{\theta}^m_{m,T-2}) = 0,$$

for all $n$.

**Proposition 4.4.** With imprecise information, investors’ subjective prices evaluated at their optimal allocations in the case of perfect information are related

---

3These non-linear equations can be solved numerically, if an exact solution is required. Our purpose in this paper is primarily to gain insights about the main qualitative effects of imprecise wealth distribution, and a first-order Taylor expansion is more suitable than a numerical analysis for that purpose.
\[
\hat{P}_{n,T-2}(\hat{\theta}_{0,T-2}) = P_{n,T-2}^0(\hat{\theta}_{0,T-2}) = P_{n,T-2}^1(\hat{\theta}_{0,T-2}) = P_{n,T-2}^2(\hat{\theta}_{0,T-2}) - \Delta \hat{P}_{n,T-2}^2,
\]

(50)

where

\[
\Delta \hat{P}_{n,T-2}^2 = \bar{\delta}_{n,T-2} + \frac{\Delta^2 \Omega_{1,T-1}^1}{2} \frac{\partial^2 (\hat{P}_{n,T-2}^2)}{\partial \Omega_{1,T-1}^2} + \cdots,
\]

(51)

where \( \bar{\delta}_{n,T-2} = \hat{P}_{n,T-2} - \hat{P}_{n,T-2}^2 \) is the bias in the price introduced due to bias in the estimate of \( \Omega_{1,T-1} \).

That is, for \( m = \{0, 1\} \) investors, who have perfect information about the wealth distribution, subjective prices at \( \hat{\theta}_{m,T-2} \) are the same as equilibrium prices in the case of perfect information, while the subjective prices of \( m = 2 \) investor are modified by \( \Delta \hat{P}_{n,T-2}^2 \) due to the investor’s imprecision about the wealth distribution.

**Corollary 7.** The derivatives of investors’ subjective prices with respect to their allocations are related as

\[
\frac{\partial P_{n,T-2}^1(\hat{\theta}_{1,T-2})}{\partial \hat{\theta}_{1,T-2}^1} \approx \frac{\hat{c}_{0,T-2}}{\hat{c}_{1,T-2}} \frac{\partial P_{n,T-2}^0(\hat{\theta}_{0,T-2})}{\partial \hat{\theta}_{0,T-2}^0},
\]

(51)

\[
\frac{\partial P_{n,T-2}^2(\hat{\theta}_{2,T-2})}{\partial \hat{\theta}_{2,T-2}^2} = \frac{\partial \hat{P}_{n,T-2}^2}{\partial \hat{\theta}_{2,T-2}^2} + \frac{\partial \Delta \hat{P}_{n,T-2}^2}{\partial \hat{\theta}_{2,T-2}^2} = \frac{\hat{c}_{0,T-2}}{\hat{c}_{2,T-2}} \frac{\partial P_{n,T-2}^0}{\partial \hat{\theta}_{0,T-2}^0} + \frac{\partial \Delta \hat{P}_{n,T-2}^2}{\partial \hat{\theta}_{2,T-2}^2}.
\]

(52)
Using Proposition 4.4 and Corollary 7, the linearised equilibrium conditions given in Equation (49) can be written more succinctly in matrix form as

\[
\begin{align*}
\partial_{\hat{\theta}} \hat{P}_{T-2}^0 \cdot \Delta \hat{\theta}_{0,T-2} &= \frac{\hat{\omega}_{0,T-2}}{\hat{\omega}_{1,T-2}} \partial_{\hat{\theta}} \hat{P}_{T-2}^0 \cdot \Delta \hat{\theta}_{1,T-2} \\
\partial_{\hat{\theta}} \hat{P}_{T-2}^0 \cdot \Delta \hat{\theta}_{0,T-2} &= \left( \frac{\hat{\omega}_{0,T-2}}{\hat{\omega}_{2,T-2}} \partial_{\hat{\theta}} \hat{P}_{T-2}^0 + \partial_{\hat{\theta}} \Delta \hat{P}_{T-2}^2 \right) \cdot \Delta \hat{\theta}_{2,T-2} + \Delta \hat{P}_{T-2}^2 \quad (53)
\end{align*}
\]

\[
\Delta \hat{\theta}_{0,T-2} + \Delta \hat{\theta}_{1,T-2} + \Delta \hat{\theta}_{2,T-2} = 0,
\]

where

\[
\Delta \hat{\theta}_{m,T-2} = \begin{bmatrix}
\Delta \hat{\theta}_{0,m,T-2} \\
\vdots \\
\Delta \hat{\theta}_{N,m,T-2}
\end{bmatrix}, \quad \Delta \hat{P}_{T-2}^2 = \begin{bmatrix}
\Delta \hat{P}_{0,T-2}^2 \\
\vdots \\
\Delta \hat{P}_{N,T-2}^2
\end{bmatrix}
\]

\[
\partial_{\hat{\theta}} \hat{P}_{T-2}^m = \begin{bmatrix}
\frac{\partial \hat{P}_{2,0,T-2}}{\partial \theta_{m,T-2}} & \ldots & \frac{\partial \hat{P}_{2,N,T-2}}{\partial \theta_{m,T-2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \hat{P}_{N,0,T-2}}{\partial \theta_{m,T-2}} & \ldots & \frac{\partial \hat{P}_{N,N,T-2}}{\partial \theta_{m,T-2}}
\end{bmatrix}, \quad \partial_{\hat{\theta}} \Delta \hat{P}_{T-2}^m = \begin{bmatrix}
\frac{\partial \Delta \hat{P}_{2,0,T-2}}{\partial \theta_{m,T-2}} & \ldots & \frac{\partial \Delta \hat{P}_{2,N,T-2}}{\partial \theta_{m,T-2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \Delta \hat{P}_{N,0,T-2}}{\partial \theta_{m,T-2}} & \ldots & \frac{\partial \Delta \hat{P}_{N,N,T-2}}{\partial \theta_{m,T-2}}
\end{bmatrix}.
\]

The first two matrix equations in System 53 equate the changes in subjective prices of \( m = 0 \) investor with \( m = 1 \) and \( m = 2 \) investors, so that all investors agree on equilibrium prices. The left hand side of the first two equations in System 53 is the change in \( m = 0 \) investor’s subjective price, which arises due to change in its allocations, while the right hand side of these two equations reflect changes in \( m = 1 \) and \( m = 2 \) investors, respectively. The last equation in System 53 is the market clearing condition for all the traded assets, and indicates that the changes in allocations arising due to imprecision about the wealth distribution have to aggregate to zero, meaning that investors have to absorb each other’s demands.
Proposition 4.5. The optimal allocations at time $T - 2$ in the case of imprecise information can then be written as

\[
\begin{align*}
\Delta \hat{\theta}_{0,T-2} &= \hat{\omega}_{0,T-2} \left[ \partial_{\hat{\theta}} \hat{P}^2_{T-2} + (\hat{\omega}_0, T-2 + \hat{\omega}_{1,T-2}) \partial_{\hat{\theta}} \Delta \hat{P}^2_{T-2} \right]^{-1} \cdot \Delta \hat{P}^2_{T-2} \\
\Delta \hat{\theta}_{1,T-2} &= \hat{\omega}_{1,T-2} \left[ \partial_{\hat{\theta}} \hat{P}^2_{T-2} + (\hat{\omega}_0, T-2 + \hat{\omega}_{1,T-2}) \partial_{\hat{\theta}} \Delta \hat{P}^2_{T-2} \right]^{-1} \cdot \Delta \hat{P}^2_{T-2} \\
\Delta \hat{\theta}_{2,T-2} &= - (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \left[ \partial_{\hat{\theta}} \hat{P}^2_{T-2} + (\hat{\omega}_0, T-2 + \hat{\omega}_{1,T-2}) \partial_{\hat{\theta}} \Delta \hat{P}^2_{T-2} \right]^{-1} \cdot \Delta \hat{P}^2_{T-2}.
\end{align*}
\]

According to Proposition 4.5, the change in uninformed investor’s subjective price affects allocations through two terms: $\partial_{\hat{\theta}} \hat{P}^2_{T-2}$ and $\partial_{\hat{\theta}} \Delta \hat{P}^2_{T-2}$. The first term, $\partial_{\hat{\theta}} \hat{P}^2_{T-2}$, measures the slope of the uninformed investor’s demand curve under perfect information. The second term, $\partial_{\hat{\theta}} \Delta \hat{P}^2_{T-2}$, measures the change in the uninformed investor’s demand curve due to imprecise information. Because the slopes of the investors’ demand curves, in the case of perfect information, are related by Corollary 7, the changes in informed investor’s allocations can be interpreted as a sum of two contributions: 1) a contribution arising from its own demand curve, and 2) a contribution arising from the change in the uninformed investor’s demand curve. The first contribution, arising due to $\partial_{\hat{\theta}} \hat{P}^2_{T-2}$, can be interpreted as the change in investors’ allocations due to change in current equilibrium prices, as this term would be non-zero as long there is a non-zero change in equilibrium prices (as can be seen explicitly in Equation 102). The second contribution, arising due to $\partial_{\hat{\theta}} \Delta \hat{P}^2_{T-2}$, can be interpreted as the change in allocation due to a change in forecasted consumption and payoffs, as this term vanishes if these forecasted variables are independent of the wealth distribution. The change in informed investors’ allocations is increasing in their share of consumption, $\omega_{m,T-2}$, and is opposite in sign to the change in the uninformed investor’s allocations. The wealthier an informed investor is, the more aggressively it bets against the mispricing.
Using Corollary 6, the change in allocations can be rewritten as

\[
\Delta \hat{\theta}_{m,T-2} = B_m \partial_p \hat{\theta}_{m,T-2} \cdot E_{T-2} \left[ \bar{\delta}_{n,T-2} + \frac{\Delta^2}{2} \left( \bar{m}_2 \bar{X}_{T-1} + 2\bar{m}_2 \bar{X}'_{T-1} + \bar{m}_2 \bar{X}''_{T-1} \right) \right],
\]

where \( B_m \) is a constant determined by relative consumption shares of different investors, and

\[
\partial_p \hat{\theta}_{m,T-2} = \left[ \partial_\theta \hat{P}^2_{m,T-2} + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \partial_\theta \Delta \hat{P}^2_{m,T-2} \right]^{-1}.
\]

It can be seen from Equation (54) that in the case of many uninformed investors, each having its own estimate of \( \Omega_{1,T-1} \), the contribution of the average bias in subjective prices, \( \bar{\delta}_{n,T-2} \), across investors may cancel out, as the bias may be positive for some investors and negative for others. Nevertheless, the contribution of the second term, \( \left( \bar{m}_2 \bar{X}_{T-1} + 2\bar{m}_2 \bar{X}'_{T-1} + \bar{m}_2 \bar{X}''_{T-1} \right) \), which arises due to the investors’ uncertainty around their own forecasts of \( \Omega_{1,T-1} \) does not vanish. Therefore, in the rest of this analysis, we assume that there is no bias in the uninformed investor’s price forecast, i.e. \( \bar{\delta}_{n,T-2} = 0 \).

### 4.2.1 Equilibrium with One Riskfree and One Risky Asset

Now we explore the changes in investors’ allocations, and analyse the impact of imprecise information about the wealth distribution, in a standard setting where there is only one riskfree and one risky asset.

**Corollary 8.** *In the case of one riskfree and one risky asset, the optimal allocations are given by*

\[
\Delta \hat{\theta}_{m,T-2} = B_m \frac{-\gamma}{\hat{p}^2_{2,T-2} \left| \partial_\theta \hat{P}^\Delta_{m,T-2} \right|} \left[ \begin{array}{c}
\hat{P}_{1,\gamma+1} + \frac{\Delta^2_1(\hat{\omega}_0,T-2+\hat{\omega}_1,T-2)}{2} \partial^2 \hat{P}^\Delta_{1,\gamma+1} \\
\hat{P}_{0,\gamma+1} + \frac{\Delta^2_1(\hat{\omega}_0,T-2+\hat{\omega}_1,T-2)}{2} \partial^2 \hat{P}^\Delta_{0,\gamma+1}
\end{array} \right]
\]

\[
\left[ \begin{array}{c}
\Delta \hat{P}^2_{0,T-2} \\
\Delta \hat{P}^2_{1,T-2}
\end{array} \right],
\]

(56)
where $|\theta_6 \hat{P}_{m,T-2}| = \frac{\gamma^2}{\epsilon^2_{T-2}} |\hat{P}_{\gamma+1}| = \frac{\gamma^2}{\epsilon^2_{T-2}} \left[ \hat{P}_{0,T-2} \hat{P}_{1,T-2} - \left( \frac{\hat{P}_{1,T-2}}{\hat{P}_{1,T-2}} \right)^2 \right]$, ignoring contribution of terms of order $\Delta^4_{\Omega_{1,T-1}}$ and higher.

Ignoring the contribution of the $\Delta^4_{\Omega_{1,T-1}}$ term, the changes in the allocations to the riskfree asset can be written as

$$
\Delta \hat{\theta}^0_{m,T-2} = -\frac{\hat{c}_{2,T-2}}{\gamma} \frac{B_m}{|\hat{P}_{\gamma+1}|} \left[ \left( \hat{P}_{11,\gamma+1} \Delta \hat{P}^2_{0,T-2} - \hat{P}_{1,\gamma+1} \Delta \hat{P}^2_{1,T-2} \right) \right] 
= -\frac{\hat{c}_{2,T-2}}{\gamma} B_m \left[ \frac{\Delta \hat{P}^2_{0,T-2}}{|\hat{P}_{0,T-2}|} \left( \frac{\hat{P}_{1,T-2}}{|\hat{P}_{\gamma+1}|} \right)^2 \left( \frac{\Delta \hat{P}^2_{1,T-2}}{\hat{P}_{1,T-2}} - \frac{\Delta \hat{P}^2_{0,T-2}}{\hat{P}_{0,T-2}} \right) \right]. \quad (57)
$$

Similarly, the change in the risky asset allocation can be written as

$$
\Delta \hat{\theta}^1_{m,T-2} = -\frac{\hat{c}_{2,T-2}}{\gamma} \frac{B_m}{\hat{P}_{\gamma+1}} \left[ \left( -\hat{P}_{11,\gamma+1} \Delta \hat{P}^2_{0,T-2} - \hat{P}_{0,\gamma+1} \Delta \hat{P}^2_{1,T-2} \right) \right] 
= -\frac{\hat{c}_{2,T-2}}{\gamma} B_m \left[ \frac{\hat{P}_{0,T-2} \hat{P}_{1,T-2}}{|\hat{P}_{\gamma+1}|} \left( \frac{\Delta \hat{P}^2_{0,T-2}}{\hat{P}_{1,T-2}} - \frac{\Delta \hat{P}^2_{0,T-2}}{\hat{P}_{0,T-2}} \right) \right]. \quad (58)
$$

The signs of these changes in allocations of a given investor are determined by the signs of $\Delta \hat{P}^2_{0,T-2}$ and $\Delta \hat{P}^2_{1,T-2}$, as prices of different payoffs are non-negative. For the uninformed ($m = 2$) investor, $-\gamma B_m > 0$, and its allocation to the riskfree (risky) asset increases (decreases) with an increase in the subjective price of the riskfree asset, and decreases (increases) with an increase in the subjective price of the risky asset. That is, an increase in the subjective price of an asset causes the uninformed investor to increase its allocation to that asset. In contrast, informed investors decrease their allocations to this asset to absorb the excess demand of the uninformed investor. The net change in the allocations to the riskfree and risky assets for an investor can be either positive or negative, depending on the relative contribution of changes in the subjective prices of the riskfree and risky assets.

The equilibrium prices in the case of imprecise information can be obtained as

$$
P_{T-2} = \hat{P}_{T-2} + \theta_6 \hat{P}^0_{T-2} \cdot \Delta \hat{\theta}_{0,T-2}. \quad (59)
$$
which yields

\[ \Delta \hat{P}_{n,T-2} = \hat{\omega}_{2,T-2} \frac{\Delta^2_{\Omega_{1},T-1}}{2} \left( 1 + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \frac{\Delta^2_{\Omega_{1},T-1}}{2} \frac{\partial^2 \left( \hat{P}_{\gamma+1}^2 \right)^{-1}}{\partial \Omega_{1}^2} \right) \frac{\partial^2 \hat{P}_{n,T-2}^2}{\partial \Omega_{1}^2} \].

Thus the changes in equilibrium prices in the riskfree and risky asset are determined by the amount of imprecision about the wealth distribution, \( \Delta^2_{\Omega_{1},T-1} \), consumption share of the uninformed investor, \( \hat{\omega}_{2,T-2} \), sensitivity of prices to the wealth distribution, \( \frac{\partial^2 \hat{P}_{n,T-2}^2}{\partial \Omega_{1}^2} \), and the menu of securities available to investors, \( \hat{P}_{\gamma+1} \). When either one of \( \Delta^2_{\Omega_{1},T-1} \) or \( \frac{\partial^2 \hat{P}_{n,T-2}^2}{\partial \Omega_{1}^2} \) goes to zero, there is no change in the uninformed investor’s subjective price, and there is no change in the equilibrium prices. Otherwise, the equilibrium prices and subjective prices move in the same direction.

When the uninformed investor’s subjective price for an asset increases (decreases), it increases (decreases) its allocation to that asset, and, consequently, informed investors have to decrease (increase) their allocations to that asset. Because informed investors’ subjective prices for that asset have not changed, the price of the asset will have to increase (decrease) in order to induce them to change their allocations, generating a positive relation between the uninformed investor’s subjective price and the equilibrium price.

The change in equilibrium prices is a fraction, \( \hat{\omega}_2 \), of the change in uninformed investors’ subjective price, \( \Delta \hat{P}_n^2 \). That is, informed investors reduce the price impact of the change in uninformed investors’ subjective price, but cannot completely eliminate it, and the equilibrium price deviation is determined by the relative share of consumption between informed and uninformed investors. When the consumption share of uninformed investors, \( \hat{\omega}_{2,T-2} \), goes to zero, the changes in uninformed investor’s subjective prices has no effect on equilibrium prices, and when \( \hat{\omega}_{2,T-2} \) goes to one, the equilibrium prices are equal to the uninformed investors’ subjective prices.
To the order of $\Delta^2_{\Omega_1,T-1}$, the percentage mispricing, $\frac{\Delta \hat{P}_{n,T-2}}{P_{n,T-2}}$, for each asset is given by

$$\frac{\Delta \hat{P}_{0,T-2}^2}{\hat{P}_{0,T-2}^2} = \omega_{2,T-2} - \frac{\Delta^2_{\Omega_1,T-1}}{\hat{P}_{0,T-2}^2} E_{T-2} \left[ \hat{m}_{2,T-2}'' \right]$$

$$\frac{\Delta \hat{P}_{1,T-2}^2}{\hat{P}_{1,T-2}^2} = \omega_{2,T-2} - \frac{\Delta^2_{\Omega_1,T-1}}{\hat{P}_{1,T-2}^2} E_{T-2} \left[ \hat{m}_{2,T-2}'' \right] \left( \frac{\hat{P}_{1,T-1} + d_{1,T-1}}{\hat{P}_{1,T-2}} - \frac{1}{P_{0,T-2}} \right)$$

$$+ 2 \hat{m}_{2,T-2}' \frac{\hat{P}_{1,T-1}'}{\hat{P}_{1,T-2}} + \hat{m}_{2,T-2}'' \frac{\hat{P}_{1,T-1}''}{\hat{P}_{1,T-2}} ,$$

and it depends on the state of the economy, dividend to price ratio of the asset, in addition to its dependence on the uninformed investor’s share of aggregate consumption and the amount of imprecision, which we have already discussed.

The state of the economy affects the magnitude of mispricing though its effect on the severity of constraint, which affects the sensitivity of payoffs to the wealth distribution, and its effect on the sensitivity of the uninformed investor’s SDF to the wealth distribution. The sensitivity of the risky asset’s payoffs is related to the state of the economy through the dependence of its price on on $\Omega_{1,T-1}$

$$\frac{\hat{P}_{1,T-1}'}{\hat{P}_{1,T-1}} = -\gamma \frac{f_\omega'}{f_\omega} = -\gamma \max \left( 1, \frac{1}{\Omega_{1,T-1}} \right)$$

$$\frac{\hat{P}_{1,T-1}''}{\hat{P}_{1,T-1}} = \gamma (\gamma + 1) \left( \frac{f_\omega'}{f_\omega} \right)^2 = \gamma (\gamma + 1) \max \left( 1, \frac{1}{\Omega_{1,T-1}^2} \right).$$

Given that $\omega_{T-1}^{\min} = \phi^{\min} E_{T-1} \left[ \frac{c_{T-1}^{\min}}{d_T} \right]$ evolves counter-cyclically, i.e. $\omega_{T-1}^{\min}$ is higher during periods of lower aggregate dividend and lower during periods of higher aggregate dividends, $\frac{1}{\Omega_{1,T-1}} = \frac{1 - \omega_{T-1}^{\min}}{1 - \omega_{T-1}^{\min}}$ also evolves counter-cyclically. Thus, $\frac{\hat{P}_{1,T-1}'}{\hat{P}_{1,T-1}}$ and $\frac{\hat{P}_{1,T-1}''}{\hat{P}_{1,T-1}}$ increase during states with lower aggregate dividend. The sensitivity of the SDF and its derivatives to the state of the economy can be approximately
expressed in terms of dividend growth rates (see Appendix B.10 for details)

\[ E_{T-2} [\hat{m}_2] = \beta E_{T-2} [(1 + g)^{-\gamma}] \]
\[ E_{T-2} [\hat{m}'_2] = \beta \gamma E_{T-2} [(1 + g)^{-2\gamma}] \]
\[ E_{T-2} [\hat{m}''_2] = \beta \gamma^2 E_{T-2} [(1 + g)^{-3\gamma}] \]

In states with lower aggregate dividend growth rates (recessions), SDF and its derivatives will be higher, amplifying the mispricing in both assets. Moreover, since \( \hat{m}''_2 \) is more sensitive to the changes in growth rate than \( \hat{m}'_2 \) and \( \hat{m}_2 \), the price of the riskfree asset, and hence its demand, will be affected more in recessions, compared to that of the risky asset. Since \( \frac{\Delta \hat{P}''_{0,T-2}}{\hat{P}''_{0,T-2}} > 0 \), the uninformed investor’s demand for the riskfree asset will always increase in the recessions. If \( \frac{\Delta \hat{P}'_{1,T-2}}{\hat{P}'_{1,T-2}} - \frac{\Delta \hat{P}''_{0,T-2}}{\hat{P}''_{0,T-2}} < 0 \), then the uninformed investor’s demand for the risky asset will decrease in a recession. Together, this increase in the demand for the riskfree asset and decrease in the demand for the risky asset may create a ‘flight to quality’ type effect.

The dividend to price ratio, \( \frac{d_{1,T-2}}{\hat{P}_{1,T-2}} \), of the risky asset affects the magnitude of relative mispricing between the riskfree and risky asset, through its effect on \( \frac{\hat{P}'_{1,T-1}}{\hat{P}_{1,T-2}} \) and \( \frac{\hat{P}''_{1,T-1}}{\hat{P}_{1,T-2}} \), which are given by

\[
\frac{\hat{P}'_{1,T-1}}{\hat{P}_{1,T-2}} = -\gamma \frac{\hat{P}_{1,T-1} f_\omega}{\hat{P}_{1,T-2} f_\omega} = \frac{(-\gamma) \hat{P}_{1,T-1}}{E_{T-2} \left[ \hat{m}_2 \left( \frac{\hat{P}_{1,T-1} + d_{1,T-1}}{\hat{P}_{1,T-1}} \right) \right]} \frac{f_\omega}{f_\omega}
\]
\[
\frac{\hat{P}''_{1,T-1}}{\hat{P}_{1,T-2}} = \gamma (\gamma + 1) \frac{\hat{P}_{1,T-1}}{\hat{P}_{1,T-2}} \left( \frac{f_\omega}{f_\omega} \right)^2 = \frac{\gamma (\gamma + 1) \hat{P}_{1,T-1}}{E_{T-2} \left[ \hat{m}_2 \left( \frac{\hat{P}_{1,T-1} + d_{1,T-1}}{\hat{P}_{1,T-1}} \right) \right]} \left( \frac{f_\omega}{f_\omega} \right)^2.
\]

The magnitude of mispricing for assets with higher dividend to price ratios will be lower. If \( \frac{\Delta \hat{P}_{1,T-2}}{\hat{P}_{1,T-2}} - \frac{\Delta \hat{P}''_{0,T-2}}{\hat{P}''_{0,T-2}} \) is positive (negative), risky assets will be over-priced (under-priced), but assets with higher dividend to price ratios will be less over-priced (under-priced). The price to dividend ratios are likely to differ for assets with different growth opportunities, with higher price to dividend ratios for assets with higher growth opportunities, and different durations, with higher price to dividend ratios for assets with longer durations. Thus, assets with different
growth opportunities and durations may earn different expected returns. This can create a cross-sectional difference in expected returns between value and growth stocks, which typically differ in terms of growth opportunities. If the imprecision about the wealth distribution makes the risky asset more attractive relative to the riskfree asset, i.e. \[ \frac{\Delta \hat{P}_{n,T-2}}{\hat{P}_{n,T-2}} - \frac{\Delta \hat{P}_{0,T-2}}{\hat{P}_{0,T-2}} > 0, \] then assets with higher price to dividend ratios will earn lower expected returns, as their prices will benefit more from the imprecision about the wealth distribution.

4.2.2 Excess Variance in Long-Term Assets

Giglio and Kelly (2015) find that long-maturity assets exhibit significantly more volatility than short-maturity assets, which violates the internal consistency conditions imposed by standard asset pricing models. Here, we show how imprecise information about the wealth distribution may cause long-maturity assets to exhibit excess-volatility relative to short-maturity assets. Because the payoffs of longer term assets have a higher price to dividend ratio, these assets will be more sensitive to the uncertainty about the wealth distribution. As a result, their realised prices may exhibit higher variance relative to short-maturity assets.

To see this, consider two nodes, \( u \) and \( d \), at time \( T - 2 \). Denote the price of a long-term (short-term) asset, which matures at time \( T (T-1) \), by \( \hat{P}_{L,T-2} \) (\( \hat{P}_{S,T-2} \)). The realised variance of these two assets at time \( T - 2 \) can be written as

\[
Var \left( \hat{P}_{n,T-2} \right) = E \left[ \left( \hat{P}_{n,T-2} - E \left[ \hat{P}_{n,T-2} \right] \right)^2 \right]
= Var \left( \hat{P}_{n,T-2} \right) + 2Cov \left( \hat{P}_{n,T-2}, \Delta \hat{P}_{n,T-2} \right) + Var \left( \Delta \hat{P}_{n,T-2} \right).
\]

(61)

In the absence of any imprecision about the wealth distribution, the difference in variances of long- and short-maturity assets is

\[
\Delta \hat{P}_{L,S} \ddot{\sigma}^2 = Var \left( \hat{P}_{L,T-2} \right) - Var \left( \hat{P}_{S,T-2} \right)
\]

(62)
In the presence of uncertainty about the wealth distribution, the difference in variances of long- and short-maturity assets incurs two additional terms

\[
\Delta_{L,S}\sigma^2 = \text{Var}\left(\hat{P}_{L,T-2}\right) - \text{Var}\left(\hat{P}_{S,T-2}\right) \\
+ \text{Var}\left(\Delta\hat{P}_{L,T-2}\right) - \text{Var}\left(\Delta\hat{P}_{S,T-2}\right) \\
+ 2\text{Cov}\left(\hat{P}_{n,T-2}, \Delta\hat{P}_{n,T-2}\right) - 2\text{Cov}\left(\hat{P}_{n,T-2}, \Delta\hat{P}_{n,T-2}\right).
\]

Thus, the change in excess volatility of long-maturity assets relative to short-maturity assets is given by (see Appendix for details)

\[
\Delta_{L,S}\sigma^2 - \Delta_{L,S}\hat{\sigma}^2 = \hat{\omega}_{2,T-2} \frac{\Delta^2\hat{\Omega}}{2} \left[ \text{Var}\left(E_{T-2} \left[ 2\hat{m}_2'P'_{L,T-1} + \hat{m}_2P''_{L,T-1}\right]\right) \\
+ \text{Cov}\left(E_{T-2} \left[(\hat{m}_2'' + \hat{m}_2)X_{L,T-1}\right], E_{T-2} \left[2\hat{m}_2'P'_{L,T-1} + \hat{m}_2P''_{L,T-1}\right]\right) \right].
\]

The variance is always non-negative, and its contribution is always to increase the volatility of long-term assets relative to short-term assets. Moreover, since \(P'_{L,T-1}\) and \(P''_{L,T-1}\) are proportional to prices (as shown in Equation (60)), the covariance term is also likely to be positive, further increasing the relative excess volatility of long-term assets.

4.2.3 Equilibrium with a Derivative

Now we explore equilibrium allocations and prices when investors have access to a linear derivative.

Equilibrium prices are given by

\[
\Delta\hat{P}_{n,T-2} = \hat{\omega}_{2,T-2} \frac{\Delta^2\hat{\Omega}}{2} \left( 1 + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \frac{\Delta^2\hat{\Omega}}{2} \hat{P}_{\gamma+1} \frac{\partial^2 \left( \hat{P}_{\gamma+1} \right)^{-1}}{\partial \hat{\Omega}^2} \right) \frac{\partial^2 \hat{P}_{n,T-2}}{\partial \hat{\Omega}^2},
\]

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and the optimal allocations are given by

$$
\Delta \theta_{m,T-2} = -\frac{c_{2,T-2}}{\gamma} B_m \left[ \frac{\Delta P_2^2}{P_0} - \frac{(\Delta P_2^1)^2}{P_0} \left( \frac{\Delta P_2^1}{P_1} - \frac{\Delta P_2^2}{P_0} \right) + \frac{\Delta P_2^1}{P_1} \frac{\Delta P_2^2}{P_0} - \frac{\Delta P_2^2}{P_1} \right],
$$

where we have suppressed the subscript \( \gamma+1 \) on prices, \( \hat{P}_{ij,\gamma+1} \).

It can be seen that

- \( \Delta \hat{\theta}^2_{m,T-2} \) is no longer zero, i.e. the optimal demand for the derivative asset is non-zero;
- \( \Delta \hat{P}_{2,T-2} - \Delta \hat{P}_{1,T-2} \) is non-zero, i.e. the difference in the prices of the risky asset and the derivative is no longer equal to the corresponding difference in the perfect information case, creating relative mispricing between assets;
- \( \Delta \hat{P}_{0,T-2} \) and \( \Delta \hat{P}_{1,T-2} \) are not the same as they were in the absence of the derivative, i.e. the prices of the riskfree and risky assets are affected by the presence of the derivative, due to the \( \hat{P}_{\gamma+1} \cdot \frac{\partial^2 (\hat{P}_{\gamma+1})^{-1}}{\partial \Omega^2} \) term.

To understand the optimal demand for the derivative, we look at how the allocation to the derivative, \( \Delta \hat{\theta}^2_{m,T-2} \), affects the exposure of the uninformed investor’s consumption at time \( T-1 \) to \( \Omega_{1,T-1} \)

$$
\Delta \hat{c}_{2,T-2} = \Delta \hat{\theta}^2_{2,T-2} + \Delta \hat{\theta}^1_{2,T-2} d_{1,T-1} + \left( \Delta \hat{\theta}^1_{2,T-2} \hat{P}_{1,T-1} + a_1 \Delta \hat{\theta}^2_{2,T-2} \hat{P}_{1,T-1} + a_0 \Delta \hat{\theta}^2_{2,T-2} \hat{P}_{0,T-1} \right) f_{\omega}^{-\gamma} (\Omega_{1,T-1}).
$$

The exposure to \( \Omega_{1,T-1} \) depends on term in brackets, as the other two terms are independent of \( \Omega_{1,T-1} \). Thus, the uninformed investor can control the exposure of its consumption to \( \Omega_{1,T-1} \) by changing its allocations to the derivative asset. The derivative asset can be used both for hedging and speculating on the excess uncertainty in prices due to uncertain \( \Omega_{1,T-1} \). A desire to hedge excess uncertainty in prices would cause the uninformed investor to reduce its exposure to \( \Omega_{1,T-1} \) at
time $T - 1$, and a desire to speculate on excess uncertainty would induce them to increase this exposure. The dependence of $\widehat{c}_{1,T-2}$ on $\Omega_{1,T-1}$ can be eliminated by setting

$$\Delta \widehat{b}_{2,T-2}^2 = -\Delta \widehat{b}_{2,T-2}^1 \frac{\widehat{P}_{1,T-1}}{a_0 \widehat{P}_{0,T-1} + a_1 \widehat{P}_{1,T-1}}.$$  

(64)

In a simple case where $a_0 = 0$ and $a_1 = 1$, this condition reduces to

$$\Delta \widehat{b}_{2,T-2}^2 = -\Delta \widehat{b}_{2,T-2}^1.$$  

(65)

After some rearrangement, $\Delta \widehat{b}_{2,T-2}^2$ in Equation (63) can be written as

$$\Delta \widehat{b}_{2,T-2}^2 = -\Delta \widehat{b}_{2,T-2}^1 + \frac{\gamma}{\delta_{2,T-2}} B_m \left[ \frac{\widehat{P}_0 \widehat{P}_1 \left( \widehat{P}_{12} - \widehat{P}_{22} \right) + \widehat{P}_1 \widehat{P}_2 \left( \widehat{P}_2 - \widehat{P}_1 \right)}{\left| \partial_\theta \widehat{P}_{\delta_{m,T-2}} \right|} \left( \frac{\Delta \widehat{P}_2}{\widehat{P}_1} - \frac{\Delta \widehat{P}_0}{\widehat{P}_0} \right) \right. \right.$$  

$$- \left. \frac{\widehat{P}_0 \widehat{P}_2 \left( \widehat{P}_{11} - \widehat{P}_{12} \right) + \widehat{P}_1 \widehat{P}_2 \left( \widehat{P}_2 - \widehat{P}_1 \right)}{\left| \partial_\theta \widehat{P}_{\delta_{m,T-2}} \right|} \left( \frac{\Delta \widehat{P}_2}{\widehat{P}_2} - \frac{\Delta \widehat{P}_1}{\widehat{P}_1} \right) \right].$$

Thus, the optimal allocation to the derivative consists of both a hedging component and a speculative component. The speculative component depends both on the relative mispricing between risky and riskfree assets as well as the relative mispricing between derivative and risky assets. The relative mispricing between risky and riskfree assets measures the combined effect of the sensitivity of asset’s beta and the risky asset’s payoff to the wealth distribution, which is higher during bad economic states. Hence, the speculative demand and the mispricing between derivative and fundamental asset will be higher when economic conditions are worse. The relative mispricing between derivative and risky assets depends on the convexity of the SDF and the deviation of the derivative’s payoff from the risky asset’s payoff. More ‘exotic’ derivatives, whose payoffs differ more significantly for the underlying asset’s payoff, will have a higher speculative demand, and will be more mispriced in equilibrium.

To understand the relative mispricing between primary and derivative assets, notice that the price of the derivative in the perfect information case can be written

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Given no change in dividends, and conditional expectations and covariances of future prices with dividends, the change in the price of the derivative upon a change in the observed time-$T - 2$ prices of the two primary assets is given by

$$\delta \hat{P}_{T-2} = E_{T-2} \left[ a_0 \hat{P}_{0,T-1} \right] + Cov_T \left( \frac{a_1 \hat{P}_{1,T-1}}{\hat{P}_{1,T-1} + d_{1,T-1}} \right).$$

Thus, in the perfect-information case a change of $\Delta \hat{P}_{0,T-2}$ and $\Delta \hat{P}_{1,T-2}$ in the prices of riskfree and risky assets, respectively, would imply a change of $\delta \hat{P}_{T-2}$ in the price of the linear derivative, which is not equal to the change in the equilibrium price of the derivative in the case of imperfect-information, $\Delta \hat{P}_{T-2}$. In other words, the mispricing in the derivative asset will not be completely explained by the observed mispricings in the primary assets, and, thus, from the perspective of the perfect-information model, the derivative asset will appear to be mispriced relative to the primary assets by an amount given by

$$\Delta \hat{P}_{T-2} - \delta \hat{P}_{T-2} = \varpi_{2,T-2} \frac{\Delta^2 \varpi_{1,T-1}}{2} \left( 1 + (\omega_{0,T-2} + \omega_{1,T-2}) \frac{\Delta^2 \varpi_{1,T-1}}{2} \hat{P}_{\gamma+1} \cdot \frac{\partial^2 \left( \hat{P}_{\gamma+1}^2 \right)^{-1}}{\partial \Omega^2_{1,T-1}} \right) \right)$$

$$\frac{\partial^2}{\partial \Omega^2_{1,T-1}} \left( Cov_T \left( \frac{\beta}{d_{1,T-2}} a_0 \hat{P}_{0,T-1} \right) + Cov_T \left( \frac{\beta}{d_{1,T-2}} \hat{X}_{1,T-1} \frac{a_1 \hat{P}_{1,T-1}}{\hat{X}_{1,T-1}} \right) \right).$$

Thus, the relative mispricing in the derivative depends on the covariance of the discounted payoffs of the two primary assets with their forecasted price/dividend ratios.
For a more general nonlinear derivative with a payoff function of the form
\[ f(d_0,T,d_1,T) = f_0(d_0,T) + f_1(d_1,T) \] (67)
the relative mispricing between the derivative and primary assets can be written as
\[
\Delta \hat{P}_{1',T-2} - \delta \hat{P}_{T-2} \propto \frac{\partial^2}{\partial \Omega_{1',T-1}^2} \left( Cov_{T-2} \left[ \beta \frac{d_{1,T-2}^\gamma}{d_{1,T-2}} f_0 \left( \hat{P}_{0,T-1} \right) \right] + Cov_{T-2} \left[ \beta \frac{d_{1,T-2}^\gamma}{d_{1,T-2}} \left( \tilde{X}_{1,T-1} \right), f_1 \left( \hat{P}_{1,T-1} \right) \right] \right),
\]
which depends on the functional form of the derivative’s payoff. Therefore, different derivatives may exhibit systematic mispricing relative to primary assets.

4.2.4 Implications for Arbitrage Strategies

The dependence of equilibrium prices on the uninformed investor’s imprecision about the wealth distribution has implications for the informed investors’ allocation decisions. Consider informed investors’ allocation decisions at time \( T - 3 \). If informed investors do not take into account the effect of this imprecision on time \( T - 2 \) prices, and make their allocation decisions as if the prices at time \( T - 2 \) are given by \( \hat{P}_{n,T-2} \), their price forecasts will be biased, which will lead to consumption losses at time \( T - 2 \). In order to avoid such losses, informed investors may wish to take \( \Delta \hat{P}_{n,T-2} \) into account. In this case, arbitrageurs will try to learn about the amount of imprecision, \( \Delta \Omega \), in the uninformed investor’s forecast through observed prices at time \( T - 3 \), and then use this estimate of \( \Delta \Omega \) to make their future price forecast. Given time \( T - 3 \) prices, an estimate of \( \Delta \Omega \) can be obtained as
\[
\frac{\Sigma^2_{\Omega}}{2} = \frac{P_{n,T-3} - \hat{P}_{n,T-3}}{\tilde{\omega}_{T-2} \frac{\partial^2 \hat{P}_{n,T-3}}{\partial \Omega_{T-2}^2}} = \frac{\Delta^2_{\Omega}}{2} \frac{\partial^2 \hat{P}_{n,T-3}}{\partial \Omega_{T-2}^2} = \frac{\Delta^2_{\Omega}}{2}.
\]
where $P_{n,T-3}$ is the observed price at time $T - 3$. Using this estimated $\Delta \Omega_{T-2}$ informed ($m = \{0, 1\}$) investors’ subjective price an asset at time $T - 3$ changes by

$$E_{T-3} \left[ \hat{m}_{T-3,T-2} \Delta \hat{P}_{n,T-2} \right] = E_{T-3} \left[ \hat{m}_{T-3,T-2} \Delta \hat{P}_{n,T-2} \right]$$

Thus the change in informed investors’ subjective prices of an asset will be approximately the discounted present value of $\Delta \hat{P}_{n,T-2}$, and informed investors will not trade against this mispricing. Therefore, mispricings within this magnitude may survive.

In contrast, if the observed mispricing at time $T - 3$ was driven by uninformed investors’ imperfect information about dividends at time $T - 2$, asset prices at time $T - 2$ would be unaffected, and so would be informed investors’ subjective prices at time $T - 3$, meaning that the informed investors’ subjective prices depend on the source of mispricing. In a setting where uninformed investors have imprecise information about both the wealth distribution and dividends, arbitrageurs may not be able to distinguish the source of mispricing from observed prices. That is, given an observed $\Delta \hat{P}_{n,T-3}$, arbitrageurs may attribute the mispricing to uninformed investors imprecision about the wealth distribution, which will persist in future periods, and may not trade against it, even if the mispricing actually arises from uninformed investors’ imprecision about the dividends, and arbitrageurs could have profited by trading against it. Thus, only the mispricings that exceed the range that can be explained by uninformed investors’ imprecision about the wealth distribution may be arbitraged away. Given that $\Delta \hat{P}_{n,T-2}$ varies
with the state of the economy, asset prices may exhibit more or less informational efficiency under different economic conditions.\footnote{A comprehensive analysis of the interaction between information about cashflows and the wealth distribution, or more generally different types of information, is beyond the scope of this paper, and is left for future work.}

These results can be useful for devising optimal arbitrage strategies when arbitrageurs cannot precisely know the source of mispricing, and may suffer losses if they trade against mispricings that arise from rational trading strategies of other investors, and are likely to persist over time. The crash of Long-Term Capital Management (LTCM) in 1998 can be seen as one example of such an event (Edwards (1999)), where an increase in mispricing for a sufficiently long period of time caused losses large enough to bring about the demise of LTCM. Our results can shed light on the evolution of one such source of mispricing, which may help arbitrageurs predict how mispricings may evolve, and trade accordingly.

5 Extensions

In this section, we consider the extensions of our model to a setting with large number of investors, and a setting with a large number of periods with the possibility of learning about the wealth distribution. We show that the results obtained in the three-investor model can be generalised to a large number of investors in a straightforward manner. While a full treatment of learning about the wealth distribution remains beyond the scope of this paper, we argue why imprecision about the wealth distribution may survive even in a model with many dates.

5.1 Extension to Many Investors

Consider a large number of investors, $M$, some of which face a funding-ratio constraint. For each constrained investor, the optimal allocations to the risky asset
are given by

\[ \theta^1_{i,T-1} = \begin{cases} 
\omega^{\min}_{i,T-1}, & \text{if } \omega_{1,T-1} < \omega^{\min}_{i,T-1} \\
\omega_{i,T-1}, & \text{if } \omega_{1,T-1} \geq \omega^{\min}_{i,T-1}
\end{cases} \]  

(68)

where \( \omega^{\min}_{i,T-1} = \phi_i^{\min} E_{T-1}[\frac{c_{i,T}}{\hat{d}_{i,T}}] \). If there are \( N_c \) investors for which the funding-ratio constraint is binding at time \( T - 1 \), then the aggregate demand for the risky asset of these constrained investors at time \( T - 1 \) is

\[ \sum_{i}^{N_c} \omega_{i,T-1}^{\min} = N_c \omega_T^{\min} \]  

(69)

where

\[ \omega_T^{\min} = \frac{1}{N_c} \sum_{i}^{N_c} \omega_{i,T-1}^{\min}. \]

And the allocations of investors which are either unconstrained or for which the constraint is not binding at time \( T - 1 \) are

\[ \hat{\theta}_{i,T-1}^1 = \hat{\omega}_{i,T-1} \left( \frac{1 - N_c \omega_T^{\min}}{1 - N_c \omega_{c,T-1}} \right), \]  

(70)

where

\[ \omega_{c,T-1} = \frac{1}{N_c} \sum_{i}^{N_c} \hat{\omega}_{i,T-1}, \]

denotes the average consumption share of a constrained investor. Thus, in the case of perfect information, the solution for a large number of investors can be obtained by replacing \( \Omega_{1,T-1} \) by \( \Omega_{T-1} \), which is given by

\[ \Omega_{T-1} = \frac{1 - N_c \omega_T^{\min}}{1 - N_c \omega_{c,T-1}}. \]  

(71)

\( \Omega_{T-1} \) can be interpreted as a measure of the average severity of the constraint in the economy. Hence, asset prices at time \( T - 1 \) depend not on the consumption-share
and constraint parameters of a single investor, but on average of these quantities across all constrained investors. Therefore, no individual investor will be able to forecast prices accurately, if they do not have precise information about the average severity of the constraint.

With imprecise information, equilibrium allocations and prices can be solved in the same manner (see Appendix B.12) as shown in the three-investor model. Therefore, equilibrium allocations can be written as

\[
\Delta \theta_{j,T-2} = \left( \theta_0 \hat{P}_{T-2}^j \right)^{-1} \cdot \sum_{i=0}^{M} \omega_i \left( \Delta \hat{P}_{T-2}^i - \Delta \hat{P}_{T-2}^j \right),
\]

(72)

where \( j \in \{0, \cdots, M\} \). Thus, the changes in each investor’s allocations to every asset are determined by a weighted average of the differences in investors’ subjective prices relative to each other.

The change in equilibrium price is given by

\[
\Delta \hat{P}_{T-2} = \sum_{i=0}^{M} \omega_i \Delta \hat{P}_{T-2}^i
\]

Thus, the change in the equilibrium prices is determined by a weighted average of the changes in subjective prices of individual investors, \( \Delta \hat{P}_{T-2}^i \), weighted by their consumptions shares, \( \omega_i \). Hence, the results of the three-investor model of Section 4 can be generalised to a large number of investors, by interpreting \( \Delta \hat{P}_{T-2} \) as the weighted average of subjective price changes across all investors.

5.2 Extension to a Large Number of Sources of Heterogeneity and Assets

In this section, we provide a qualitative sketch of the implications of uncertainty about the wealth distribution when there are multiple sources of heterogeneity, and a large number of risky assets. With multiple sources of heterogeneity, the wealth distribution across different investment objectives can be summarised by a set
of variables, each summarising the share of wealth for a given type of investment objective. In our notation, instead of a single variable, \( \Omega_t \), summarising the wealth distribution, we will have multiple variables, \( \{ \Omega_{t,l} , l \in [1, \cdots , L] \} \), where \( L \) is the total number of sources of heterogeneity, or the number of distinct types of investment objectives in the economy. For example, in the case of funding-ratio and liquidity-ratio constraints, we will have \( L = 3 \), and the effect of the wealth distribution can be summarised by three variables, \( \{ \Omega_{t,l} , l \in [1, \cdots , L] \} \), which will determine the effect of the two individual constraints and their interaction on asset prices.

With multiple risky assets having different exogenous characteristics, equilibrium prices of different assets, \( P_{n,t} \), may in general depend differently on different sources of heterogeneity, \( \Omega_{t,l} \). That is, the sensitivity of an asset’s price to a particular source of heterogeneity, \( \frac{\partial P_{n,t}}{\partial \Omega_{l,t+1}} , a \in [1, 2, \cdots] \), may differ across assets. In other words, these different sources of heterogeneity, denoted by \( \Omega_{t,l} \), can be interpreted as different risk factors, to which different assets may have different exposures.

An investor’s subjective price for an asset can then be written as

\[
\hat{P}_{n,t}^m = \hat{P}_{n,t} + \frac{1}{2} \sum_l \Delta^2_{l,t+1} \chi_{n,l,t},
\]

where

\[
\chi_{n,l,t} = \frac{\partial^2 \hat{P}_{n,t}}{\partial \Omega^2_{l,t+1}} = \frac{\partial^2 \hat{m}_{m,t+1}}{\partial \Omega^2_{l,t+1}} \hat{P}_{n,t} + 2 \frac{\partial \hat{m}_{m,t+1}}{\partial \Omega_{l,t+1}} \frac{\partial \hat{P}_{n,t+1}}{\partial \Omega_{l,t+1}} + \hat{m}_{m,t+1} \frac{\partial^2 \hat{P}_{n,t+1}}{\partial \Omega^2_{l,t+1}},
\]

measures the sensitivity of \( m^{th} \) investor’s subjective price of \( n^{th} \) asset to \( l^{th} \) source of heterogeneity. And as we can see from Equation 72, each investor’s allocations to different assets depend on their subjective prices for those assets, Equation 73 can be used to obtain some insights about optimal allocation strategies.

For given sensitivities of asset prices to the wealth distribution, \( \frac{\partial P_{n,t+1}}{\partial \Omega_{l,t+1}} , a \in [1, 2, \cdots] \), investors’ subjective asset prices, and hence their allocations, will differ due to differences in the sensitivities of their discount factors to the wealth
distribution, $\frac{\partial \hat{m}_{mt+1}}{\partial \Omega^a_{lt+1}}$, $a \in [1, 2, \cdots]$, as discount factors are equal in equilibrium, $\hat{m}_{mt+1} = \hat{m}_{mt_{+1}}$. The sensitivities of the stochastic discount factor to shocks to the wealth distribution can be written as

\[
\frac{\partial \hat{m}_{mt+1}}{\partial \Omega^a_{lt+1}} = -\frac{\gamma}{c_{mt+1}} \left( \sum_n \frac{\partial}{\partial \Omega^a_{lt+1}} \left[ \left( \frac{\theta^a_{mt} - \theta^a_{mt+1}}{P_{mt} + 1} \right) P_{nt+1} \right] \right)
\]

\[
\frac{\partial^2 \hat{m}_{mt+1}}{\partial \Omega^a_{lt+1}^2} = -\frac{\gamma}{c_{mt+1}} \sum_n \left[ \frac{\partial^2}{\partial \Omega^a_{lt+1}^2} \left[ \left( \frac{\theta^a_{mt} - \theta^a_{mt+1}}{P_{mt} + 1} \right) P_{nt+1} \right] \right] - \frac{\gamma}{c_{mt+1}} \left( \frac{\partial}{\partial \Omega^a_{lt+1}} \left[ \left( \frac{\theta^a_{mt} - \theta^a_{mt+1}}{P_{mt} + 1} \right) P_{nt+1} \right] \right)^2
\]

and depends on the level of investor’s consumption, $c_{mt+1}$, and the sensitivities (slope and convexity) of the value of rebalancing, $\sum_n \left( \frac{\theta^a_{mt} - \theta^a_{mt+1}}{P_{mt} + 1} \right) P_{nt+1}$, to the wealth distribution.

Different investors with different rebalancing needs will have different sensitivities of their rebalancing value to different components of the wealth distribution, and will need to optimally adjust their dynamic allocation strategies accordingly. If the value of rebalancing varies positively (negatively) with the price of an asset, i.e. the cost of rebalancing increases (decreases) when the price goes up, then the contribution of $\frac{\partial \hat{m}_{mt+1}}{\partial \Omega^a_{lt+1}} \frac{\partial \hat{P}_{mt+1}}{\partial \hat{m}_{lt+1}}$ term will be negative (positive), and the investor will optimally decrease (increase) its allocation to such assets. When an investor intends to liquidate some of its portfolio to finance consumption in the next period, the value of rebalancing will vary positively with the price of rebalanced portfolio. As a result, a decrease in the price of the portfolio will lower the amount of consumption financed from rebalancing. For example, consider a mutual fund that holds a similar portfolio as other mutual funds. If the price of the portfolio goes down in case of a shock to the average liquidity-ratio of mutual funds, driven by unexpected outflows e.g., the portfolio value of this mutual fund will also go gown, decreasing the fund’s ability to finance its payouts. This risk can become even more severe if the aggregate liquidity-ratio shock is positively correlated with the mutual fund’s idiosyncratic liquidity shock. In this case, the mutual fund’s demand for rebalancing will further increase upon the liquidity shock in order to satisfy required outflows, causing it to liquidate more of its portfolio when the prices are depressed. This is directly related to the risk of fire-sales arising from
shocks to investors’ wealths. In contrast, a mutual fund that wishes to increase its asset holdings instead of liquidating it, and whose idiosyncratic liquidity shocks are negatively correlated with the average liquidity shock, may optimally increase its allocation to assets that are more sensitive to shocks to the average liquidity-ratio, as this can allow the fund to increase its asset allocations at depressed prices.

The effect of the other term, $\frac{\partial^2 \tilde{m}_{m,i+1}}{\partial \Delta_{l,i+1}^2}$, is also related to the value of rebalancing, and depends on the convexity of the value of rebalancing, and can be analysed similarly. If the value of rebalancing exhibits positive convexity, then the shocks to the wealth distribution will on average benefit the investor, increasing its consumption, and decreasing its subjective SDF. Such an investor would optimally decrease its allocation to the riskfree asset, and may be willing to invest more in the risky assets.

In addition to the effects on the sensitivities of stochastic discount factor discussed above, the shocks to wealth distribution may affect different investors differently due to differences in their information about different components of the wealth distribution. That is, using our notation, if $\Delta^2_{l,m}$ denotes $m^{th}$ investor’s uncertainty about the $l^{th}$ component of the wealth distribution, then $\Delta^2_{l,m}$ may differ systematically across different segments of investors. For example, if mutual funds facing liquidity-ratio constraints have better information about the liquidity-ratios of the mutual fund industry, and the nature and dynamics of liquidity shocks, they may be able to forecast shocks to aggregate liquidity-ratios better than other investors. As a result, assets more sensitive to liquidity-ratio shocks may appear more or less attractive to mutual funds compared to other investors outside of the mutual fund industry, such as pension funds. Thus, our approach can provide a general framework to understand optimal dynamic allocation strategies in a setting with investor segmentation, where each investor faces risks arising from shocks to other investors’ wealth. A full analysis of these effects would require us to incorporate multiple sources of heterogeneity in our benchmark model, which we leave for future work.
5.3 Extension to a Model with Learning

One main shortcoming of our approach can be that in a model with a large number of periods, investors may be able to learn about the wealth distribution over time, and, hence, the effect of imprecision about the wealth distribution may go away. This is plausible if all investors have power utility and differ only due in their risk aversion or time-discount factor. In such a setting, all investors hold the market portfolio, and the wealth distribution across agents change only due to innovations to the aggregate dividend. This makes the changes in wealth distribution predictable, and facilitates learning in a Bayesian manner.

However, for more general forms of heterogeneity, for instance heterogeneity in constraints, beliefs, etc., investors optimally hold underdiversified portfolios. As a result, prices of different assets may depend differently on the wealth distribution, and the exact functional relation may not be known to all investors, and the wealth distribution will evolve not only with innovations to the aggregate dividend, but also with innovations to dividends of individual assets. This can considerably complicate the task of learning about the wealth distribution. In the language of state-space models, if we treat the wealth distribution across investment objectives as the latent state vector, and prices of traded assets as the observable vector,

\[
P_t = \Psi_t (\Omega_t) + u_t \\
\Omega_t = \Phi_t (\Omega_{t-1}) + v_t, \tag{75}
\]

where \(\Psi_t\) is the pricing model that takes into account the effect of wealth distribution on prices and \(\Phi_t\) is the state-transition function that determines the evolution of the wealth distribution, then both the exact form of observation and state-space equations may be unknown to investors. In addition, in general investors may have incomplete information about both dividends and the wealth distribution, requiring investors to simultaneously estimate both the time-varying functions and the latent state vector, possibly creating identification problems.
6 Concluding Remarks

In this paper, we have tried to explore the qualitative effects of unforecastable demand shocks. We have argued that, in an equilibrium model, the demand shocks can be represented as shocks to the wealth distribution across an unchanging set of all possible investment objectives in the economy. We see that even if individual investors cannot forecast the wealth distribution accurately, as long as the average forecast (across investors) is accurate, and individual investors ignore the bias in their forecasts in making their optimal allocation decision (or the bias is negligible), the equilibrium prices will remain unaffected. However, if the welfare impact of bias in individual investors’ forecast is significant, which is plausible given that a significant fraction of empirically observed fluctuations in prices is driven by demand shocks, then investors may wish to incorporate the uncertainty in their forecasts of the wealth distribution.

Incorporating uncertainty about the wealth distribution makes markets incomplete from the perspective of an individual investor even if there are sufficient assets to hedge cashflow risk, creating room for derivatives in investors’ optimal allocation decisions. Hence, derivatives become non-redundant and are mispriced relative to the expected cost of creating a synthetic derivative.

The mispricings in each asset can be related to the sensitivities of the asset’s payoff and the stochastic discount factor to the wealth distribution. As these sensitivities may evolve with the state of the economy, these pricing anomalies may become more or less strong under different economic conditions. For derivative assets, misprings depend also on the payoff function of the derivative, and can be expressed in terms of a derivative’s ‘greeks’, and derivatives with more nonlinear payoffs may in general be more more mispriced. Thus, our model’s ability to relate anomalous prices to economic conditions and assets’ payoff functions may provide some unique testable predictions of our model.

If information about cashflows and the wealth distribution is scattered across investors, then for any observed mispricing, each individual investor may not be able to decide the source of the mispricing. Even when the mispricings arise due to
some investors’ inaccurate information about cashflows, those with superior information about cashflows may refrain from trading against such mispricing as long as the mispricing can be explained by the wealth distribution, which they may not accurately know.

In addition to pricing anomalies, our model can also be extended to gain intuition for optimal dynamic allocation strategies in a setting with a large number of sources of heterogeneity, where risks arise from shocks to other investors’ wealths, such as the risk of fire-sales. We see that these risks affect investors through their rebalancing needs, and investors with different rebalancing needs may optimally choose different dynamic allocation strategies.
Appendices

A The Effect of Wealth Distribution on Asset Prices

Here we qualitatively describe how the wealth distribution affects asset prices and optimal allocation decisions. The simplest setting in which the wealth distribution can have an effect is a three-date setting with heterogeneous investors. If all investors have identical investment objectives, the wealth distribution does not affect asset prices. In this case, any change in the wealth distribution across investors merely reshuffles wealth among identical agents, which does not have any effect on equilibrium asset prices. Therefore, we consider investors with heterogeneous constraints.

First, we discuss asset prices at time $T - 1$. These prices are determined by discounted value of the assets’ total payoff at time $T$. Because ex-dividend prices of all assets are zero at time $T$, the total payoff for each risky asset at time $T$ is determined by its dividend. The discount rate is determined by investors’ optimal consumption at time $T$. Given $T - 1$ allocations, each investor’s optimal consumption in each state of the world at time $T$ is given by the budget constraint

\[
c_m,T = \theta^0_{m,T-1} + \sum_{n=1}^{N} \theta^n_{m,T-1} d_n,T
\]

\[
\Rightarrow c_{m,T} = c_{m,T} (\theta_{m,T-1}, d_T). \tag{76}
\]

Since all investors are assumed to have perfect information about the dividend distribution, they know their optimal consumption, $c_{m,T}$, in each state of the world at time $T$, as a function of their allocations at time $T - 1$. 

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Thus, subjective time-$T - 1$ prices for all assets can be written as

\[ P_{n,T-1}^m(\omega_{m,T-1}, \theta_{m,T-1}, \lambda_{T-1}^m) = E_{m,T-1} \left[ SDF_{m,T-1,T}(c_{m,T-1}, \theta_{m,T-1}, \lambda_{T-1}^m) \times d_n \right] \]

\[ \Rightarrow \hat{P}_{n,T-1}^m = P_{n,T-1}^m(\omega_{m,T-1}, \theta_{m,T-1}, \lambda_{T-1}^m), \]

where \( \theta_{m,T-1} \) denotes \( m \text{'th} \) investor’s allocations to \( N + 1 \) assets, \( SDF_{m,T-1,T} \) is the subjective discount rate of \( m \text{'th} \) investor at time \( T - 1 \) for cashflows received at time \( T \), \( \lambda_{T-1}^m \) is the vector Lagrange-coefficients for constraints determine by constraint parameters, and \( \omega_{T-1} \) denotes the vector of consumption shares (wealth distribution) at time \( T - 1 \). In equilibrium, all agents agree on the prices of traded assets

\[ P_{n,T-1}(\omega_{T-1}, \theta_{T-1}, \lambda_{T-1}) = P_{n,T-1}^m(\omega_{m,T-1}, \theta_{m,T-1}, \lambda_{T-1}^m), \forall m, n, \]

where \( \theta_{T-1} \) denotes the matrix of \( M + 1 \) investors’ allocations to \( N + 1 \) assets.

Since prices are observable, each agent at time \( T - 1 \) can determine its optimal allocations to \( N + 1 \) assets as a function of traded prices, \( P_{T-1} \), its own consumption share, \( \omega_{m,T-1} \), and the shadow price of its constraint, \( \lambda_{T-1}^m \)

\[ P_{n,T-1}(\omega_{T-1}, \theta_{T-1}, \lambda_{T-1}) = E_{m,T-1} \left[ SDF_{m,T-1,T}(c_{m,T-1}, \theta_{m,T-1}, \lambda_{T-1}^m) \times d_n \right], \forall n \]

\[ \Rightarrow \theta_{m,T-1}^n = \theta_{m,T-1}^n(P_{T-1}, \omega_{m,T-1}, \lambda_{T-1}^m). \]  

(77)

Hence, investors’ optimal allocation decisions at time \( T - 1 \), and the corresponding consumptions at time \( T \), do not require the knowledge of wealth distribution at time \( T - 1 \), and, hence, are unaffected by any imprecision regarding the wealth distribution. Intuitively, the wealth distribution at time \( T - 1 \) only affects agents’ optimal allocations through time \( T - 1 \) prices, which are observable at time \( T - 1 \).

Hence, agents do not need to know the wealth distribution at time \( T - 1 \) to make their allocation decisions.
Given that optimal allocations at time $T - 1$ can be written as in Equation 77, equilibrium prices at time $T - 1$ can be written as

$$P_{n,T-1}(\omega_{T-1}, \theta_{T-1}, \lambda_{T-1}) = P_{n,T-1}(\omega_{T-1}, \Delta_{T-1}).$$

(78)

Thus, assuming the economy is populated by agents with all possible values of shadow prices of constraints, $\lambda_{T-1}$, the task of determining prices at time $T - 1$ is reduced to determining the wealth distribution, $\omega_{T-1}$, across these agents. Therefore, we omit the functional dependence on $\Delta_{T-1}$ in the remainder of this section.

Next, we consider asset prices at time $T - 2$, which are determined by the discounted values of time-$T - 1$ payoffs. Given $T - 2$ allocations, each investor’s optimal consumption in each state of the world at time $T - 1$ is given by the payoff of their investments at time $T - 2$

$$c_{m,T-1} + F_{m,T-1} = \theta_{m,T-2}^n + \sum_{n=1}^N \theta_{m,T-2}^n \left( d_{n,T-1} + P_{n,T-1}(\omega_{T-1}) \right),$$

(79)

which depends on the prices of risky assets, $P_{n,T-1}$, which is a function of the time-$T - 1$ wealth distribution, $\omega_{T-1}$. As a result, the optimal consumption at time $T - 1$ and optimal allocations at time $T - 2$ will also depend on the wealth distribution, $\omega_{T-1}$:

$$\{\theta_{m,T-2}^n, c_{m,T-1}\} \equiv \{\theta_{m,T-2}^n(\omega_{T-1}), c_{m,T-1}(\omega_{T-1})\}.$$  

(80)

As a result, subjective prices for risky assets at time $T - 2$ become a function of the wealth distribution at time $T - 1$ and investors’ imprecision regarding the wealth distribution

$$P_{m,T-2}^n(\omega_{m,T-2}, \theta_{m,T-2}, \omega_{T-1}, F_{m,T-2})$$

$$= E\left[ SDF_{m,T-2,T-1}(c_{m,T-1}, \theta_{m,T-2}, \lambda_{T-2}^m) \times (d_{n,T-1} + P_{n,T-1}(\omega_{T-1})) \mid F_{m,T-2} \right].$$
In equilibrium, different investors will have to agree on the prices of traded assets, and, thus, the equilibrium prices will be a function of each investor’s consumption share, and its information regarding the consumption distribution

\[ P_{n,T-2}(\omega_{T-2}, \theta_{T-2}, \lambda_{T-2}, \mathcal{F}_{T-2}) = P_{m,T-2}(\omega_{m,T-2}, \theta_{m,T-2}, \lambda_{m,T-2}, \omega_{T-1}, \mathcal{F}_{m,T-2}), \]

\( \forall m, n \), where \( \mathcal{F}_{T-2} \) denotes the vector of information sets for different investors.

Thus, each investor’s optimal allocation decisions at time \( T - 2 \) can be written as

\[ \theta^m_{m,T-2} \equiv \theta^m_{m,T-2}(P_{T-2}, \omega_{T-1}, \lambda_{T-2}, \mathcal{F}_{m,T-2}), \]  

which depends on the wealth distribution at time \( T - 1, \omega_{T-1} \). Intuitively, asset prices at time \( T - 2 \) depend on the wealth distribution at time \( T - 2 \) both through asset prices at time \( T - 2 \), which are observable, and at time \( T - 1 \), which are unobservable at time \( T - 1 \). Thus, investors need to forecast the wealth distribution at time \( T - 1 \) to accurately forecast time-\( T - 1 \) prices, in order to make their optimal consumption and allocation decisions at time \( T - 2 \). Hence, their optimal decisions are affected by the imprecision in their knowledge of the wealth distribution. Therefore, investors with different information regarding the wealth distribution at time \( T - 1 \) will form different beliefs about the expected asset payoffs at time \( T - 1 \), and differ in their estimates of assets’ fundamental values at time \( T - 2 \). Investors who have more precise information of the wealth distribution at time \( T - 1 \) may be able to better forecast asset prices at time \( T - 1 \). These investors can then ‘steal’ consumption from other investors who have less precise information about the wealth distribution.
B  Proofs

B.1 Optimal Allocations at time \( T - 1 \)

When the constraint is not binding, the optimal allocations are given by

\[
\begin{align*}
\hat{\theta}^0_{m,T-1} &= 0 \\
\hat{\theta}^1_{m,T-1} &= \omega_{m,T-1}.
\end{align*}
\]  

(82)

This can be easily verified that this solution satisfies all the kernel and market-clearing conditions.

To obtain optimal allocations when the constraint is binding, first, we guess that the allocations to the riskfree asset at time \( T - 1 \) are zero for all investors

\[
\hat{\theta}^0_{m,T-1} = 0,
\]  

(83)

and, hence, the market clearing conditions for the riskfree asset are satisfied. Moreover, due to the market completeness, we have

\[
\begin{align*}
\left( \frac{\hat{c}_{m,T}}{\hat{c}_{m,T-1}} \right)^{-\gamma} &= \left( \frac{\hat{c}_{m',T}}{\hat{c}_{m',T-1}} \right)^{-\gamma} \\
\Rightarrow \left( \frac{\hat{\theta}^1_{m,T-1} d_{1,T}}{\hat{\omega}_{m,T-1} d_{1,T-1}} \right)^{-\gamma} &= \left( \frac{\hat{\theta}^1_{m',T-1} d_{1,T}}{\hat{\omega}_{m',T-1} d_{1,T-1}} \right)^{-\gamma}
\end{align*}
\]  

(84)

We conjecture that the constrained investor can maintain its funding-ratio requirement by holding a fraction, \( \omega^\text{min}_{T-1} = E_{T-1} [\omega^\text{min}_T] \), of the aggregate wealth, which can be achieved by holding \( \omega^\text{min}_{T-1} \) shares of the equity. Hence,

\[
\hat{\theta}^1_{m,T-1} = \omega^\text{min}_{T-1}.
\]  

(85)
Using the kernel condition for the riskfree asset for \( m = 0 \) and \( m = 2 \) investor, we can relate the risky asset allocations of the two investors

\[
E_{T-1} \left[ \left( \frac{\hat{\theta}_{1,T-1}^{1} d_{1,T}}{\hat{\omega}_{0,T-1} d_{1,T-1}} \right)^{-\gamma} \right] = E_{T-1} \left[ \left( \frac{\hat{\theta}_{2,T-1}^{2} d_{1,T}}{\hat{\omega}_{2,T-1} d_{1,T-1}} \right)^{-\gamma} \right]
\]

\Rightarrow \hat{\theta}_{2,T-1} = \frac{\hat{\omega}_{2,T-1}}{\hat{\omega}_{0,T-1}} \hat{\theta}_{0,T-1}

(86)

The market clearing condition for the risky asset allows us to solve for the risky asset allocations in terms of consumption shares

\[
\hat{\theta}_{0,T-1} \left( 1 + \frac{\hat{\omega}_{2,T-1}}{\hat{\omega}_{0,T-1}} \right) = 1 - \omega_{T-1}^{\min}
\]

\[
\hat{\theta}_{0,T-1} = \frac{\hat{\omega}_{0,T-1} \left( \frac{1 - \omega_{T-1}^{\min}}{1 - \omega_{1,T-1}} \right)}{1 - \omega_{1,T-1}}
\]

\[
\hat{\theta}_{2,T-1} = \frac{\hat{\omega}_{2,T-1} \left( \frac{1 - \omega_{T-1}^{\min}}{1 - \omega_{1,T-1}} \right)}{1 - \omega_{1,T-1}}
\]

The remaining two equations are the kernel conditions for riskfree and risky assets for \( m = 0 \) and \( m = 1 \) investors. For the riskfree asset, it is given by

\[
E_{T-1} \left[ \left( \frac{1 - \omega_{T-1}^{\min} d_{1,T}}{1 - \omega_{1,T-1} d_{1,T-1}} \right)^{-\gamma} \right] = E_{T-1} \left[ \left( \frac{\hat{\theta}_{1,T-1}^{1} d_{1,T-1}}{c_{1,T-1} - \lambda_{T-1}^{fr}} \right)^{-\gamma} \right]
\]

For the risky asset, it is given by

\[
E_{T-1} \left[ \left( \frac{1 - \omega_{T-1}^{\min} d_{1,T}}{1 - \omega_{1,T-1} d_{1,T-1}} \right)^{-\gamma} d_{1,T} \right] = E_{T-1} \left[ \left( \frac{\hat{\theta}_{1,T-1}^{1} d_{1,T-1}}{c_{1,T-1} - \lambda_{T-1}^{fr}} \right)^{-\gamma} \right]
\]

Both of these equations can be satisfied by setting

\[
\lambda_{T-1}^{fr} = (\omega_{1,T-1} d_{1,T-1})^{-\gamma} - (\omega_{T-1}^{\min} d_{1,T-1})^{-\gamma} \left( \frac{1 - \omega_{T-1}^{\min}}{1 - \omega_{1,T-1}} \right)^{-\gamma}
\]

Thus the assumed solution satisfies all equilibrium conditions.
B.2 Prices at Time $T - 2$

Prices at time $T - 2$ can be written as

$$P_{n,T-2} = \beta E_{T-2} \left[ \frac{\tilde{c}_{0,T-1} X_{n,T-1}}{c_{0,T-2}} \right]$$

$$= \beta E_{T-2} \left[ \frac{d_{1,T-1}^{-\gamma}}{d_{1,T-2}} X_{n,T-1} \right].$$

(87)

B.3 Allocations at Time $T - 2$

Optimal allocations at time $T - 2$ can be obtained by solving the budget constraint

$$c_{m,T-1,u} + F_{m,T-1,u} = \sum_n \theta^n_{m,T-2} X_{n,T-1,u}$$

$$c_{m,T-1,d} + F_{m,T-1,d} = \sum_n \theta^n_{m,T-2} X_{n,T-1,d}.$$

B.4 Proof of Proposition 4.3

The consumption at time $T - 1$ for $m^{th}$ investor can be written as

$$c_{m,T-1} = W_{m,T-1} - F_{m,T-1}$$

$$c_{m,T-1} = \theta^0_{m,T-2} + \theta^1_{m,T-2} X_{1,T-1} + \theta^{1'}_{m,T-2} P_{1,T-1}$$

$$- \theta^0_{m,T-1} P_{0,T-1} + \theta^1_{m,T-1} P_{1,T-1} - \theta^{1'}_{m,T-1} P_{1,T-1}$$

$$= \theta^0_{m,T-2} + \theta^1_{m,T-2} d_{1,T-1} + \left( \theta^1_{m,T-2} - \theta^{1'}_{m,T-1} \right) P_{1,T-1} + \left( \theta^{1'}_{m,T-2} - \theta^{1'}_{m,T-1} \right) P_{1,T-1}$$

$$= \theta^0_{m,T-2} + \theta^1_{m,T-2} d_{1,T-1} - \theta^{1'}_{m,T-1} \hat{P}_{1,T-1} f^1_{\omega}$$

$$+ \left[ a_0 \theta^{1'}_{m,T-2} \hat{P}_{0,T-1} + \left( \theta^1_{m,T-2} + a_1 \theta^{1'}_{m,T-2} \right) \hat{P}_{1,T-1} \right] f^{-\gamma}_{\omega}$$

$$= A_{m,0} + A_{m,1} f^{-\gamma}_{\omega} + A_{m,2} f^1_{\omega}$$

which allows us to obtain Equations 46 and 47.
B.5 Proof of Corollary 6

Defining

\[ m(\Omega_{1,T-1}) = \beta \left[ A_{2,0} + A_{2,1} f_{\omega}^{-\gamma} (\Omega_{1,T-1}) + A_{2,2} f_{\omega}^{1-\gamma} (\Omega_{1,T-1}) \right] \]

\[ I_n (\Omega_{1,T-1}) = m (\Omega_{1,T-1}) \hat{P}_{n,T-1} (\Omega_{1,T-1}), \]

The subjective prices for investor-2 can be written as

\[ P^2_{n,T-2} = E_{T-2} \left[ \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} g_2 (\Omega_{1,T-1}) I_n (\Omega_{1,T-1}) d\Omega_{1,T-1} \right]. \]

Expanding \( I_{n,T-1}(\Omega_{1,T-1}) \) around \( \bar{\Omega}_{1,T-1} \), we get

\[ \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} I_{n,T-1}(\Omega_{1,T-1}) d\Omega_{1,T-1} = I_{n,T-1}(\bar{\Omega}_{1,T-1}) \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} g_2 (\Omega_{1,T-1}) d\Omega_{1,T-1} \]

\[ + I'_{n,T-1}(\bar{\Omega}_{1,T-1}) \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} g_2 (\Omega_{1,T-1}) (\Omega_{1,T-1} - \bar{\Omega}_{1,T-1}) d\Omega_{1,T-1} \]

\[ + I''_{n,T-1}(\bar{\Omega}_{1,T-1}) \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} g_2 (\Omega_{1,T-1}) (\Omega_{1,T-1} - \bar{\Omega}_{1,T-1})^2 d\Omega_{1,T-1} \]

\[ + \cdots \]

\[ = I_{n,T-1}(\bar{\Omega}_{1,T-1}) + I''_{n,T-1}(\bar{\Omega}_{1,T-1}) \Delta_{\Omega_{1,T-1}}^2 + \cdots, \]

where

\[ \Delta_{\Omega_{1,T-1}}^2 \equiv \int_{\Omega_{1,T-1}^{\min}}^{\Omega_{1,T-1}^{\max}} g_2 (\Omega_{1,T-1}) (\Omega_{1,T-1} - \bar{\Omega}_{1,T-1})^2 d\Omega_{1,T-1}. \]  \hspace{1cm} (88)

\( I''_{n,T-1}(\bar{\Omega}_{1,T-1}) \) is given by

\[ I''_{n,T-1}(\bar{\Omega}_{1,T-1}) = m'(\bar{\Omega}_{1,T-1}) X_{n,T-1} + 2m' (\bar{\Omega}_{1,T-1}) X'_{n,T-1} + m(\bar{\Omega}_{1,T-1}) X''_{n,T-1}. \]
The subjective prices for investor-2 then become

\[ P_{n,T-2}^2 = E_{T-2} \left[ \frac{\Delta^2}{2} \left( \frac{m'}{m} + 2 \frac{m}{m} \frac{\bar{X}_{n,T-1}}{\bar{X}_{n,T-1}} \right) \right] + \cdots, \]

where \( m, \bar{X}_{n,T-1}, m', \bar{X}'_{n,T-1}, \cdots \) are all evaluated at \( \Omega_{1,T-1} \).

### B.6 Proof of Proposition 4.4

In equilibrium, investors equate their subjective prices for all traded assets. In the case of perfect information, this implies

\[ \hat{P}_{n,T-2}(\theta_{0,T-2}) = \hat{P}_{n,T-2}^0(\theta_{0,T-2}) = \hat{P}_{n,T-2}^1(\theta_{0,T-2}) = \hat{P}_{n,T-2}^2(\theta_{0,T-2}). \]  

(89)

In the case of imperfect information, the subjective prices of \( m = \{0, 1\} \) investors are unchanged

\[ \hat{P}_{n,T-2}^0(\theta_{0,T-2}) = \hat{P}_{n,T-2}^0(\theta_{0,T-2}), \]

\[ \hat{P}_{n,T-2}^1(\theta_{0,T-2}) = \hat{P}_{n,T-2}^1(\theta_{0,T-2}), \]  

(90)

and, using Corollary 6, the subjective prices of \( m = 2 \) investor can be written as

\[ \hat{P}_{n,T-2}^2(\theta_{2,T-2}) = \hat{P}_{n,T-2}^2(\theta_{2,T-2}) + \frac{\Delta^2}{2} \left( \frac{m'}{m} + 2 \frac{m}{m} \frac{\bar{X}_{n,T-1}}{\bar{X}_{n,T-1}} \right) \hat{\theta}_{2,T-2} \]

\[ = \hat{P}_{n,T-2}^2(\theta_{2,T-2}) + \frac{\Delta^2}{2} \left( \frac{m'}{m} + 2 \frac{m}{m} \frac{\bar{X}_{n,T-1}}{\bar{X}_{n,T-1}} \right) \hat{\theta}_{2,T-2} \]

\[ = \hat{P}_{n,T-2}^2(\theta_{2,T-2}) + \frac{\Delta^2}{2} \left( \frac{m'}{m} + 2 \frac{m}{m} \frac{\bar{X}_{n,T-1}}{\bar{X}_{n,T-1}} \right) \hat{\theta}_{2,T-2} \]

\[ = \hat{P}_{n,T-2}^2(\theta_{2,T-2}) + \frac{\Delta^2}{2} \left( \frac{m'}{m} + 2 \frac{m}{m} \frac{\bar{X}_{n,T-1}}{\bar{X}_{n,T-1}} \right) \hat{\theta}_{2,T-2} \]

\[ = \hat{P}_{n,T-2}^2(\theta_{2,T-2}) + \Delta^2 \hat{\theta}_{n,T-2} \]  

(91)

64
Using Equations 89, 90, and 91, we get

\[ \hat{P}_{n,T-2}(\hat{\theta}_{0,T-2}) = P_{n,T-2}^0(\hat{\theta}_{0,T-2}) = P_{n,T-2}^1(\hat{\theta}_{0,T-2}) = P_{n,T-2}^2(\hat{\theta}_{0,T-2}) - \Delta P_{n,T-2}^2, \]

which yields Equation (50).

**B.7 Proof of Corollary 7**

The derivative of \( m = 0 \) investor’s subjective price of \( n^{th} \) asset w.r.t to its allocations to \( j^{th} \) asset is given by

\[
\frac{\partial P_{n,T-2}^0(\hat{\theta}_{0,T-2})}{\partial \hat{\theta}_{0,T-2}^j} = \beta \frac{\partial}{\partial \hat{\theta}_{0,T-2}^j} P_{n,T-2} E_{T-2} \left[ \frac{\hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{0,T-2}^{-\gamma}} \right] \\
= \beta E_{T-2} \left[ (-\gamma) \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}} + \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}} \right] \\
= \beta E_{T-2} \left[ (-\gamma) \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}} + \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}} \right] \\
= \beta E_{T-2} \left[ (-\gamma) \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}} + \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}}{\hat{c}_{1,T-2} \hat{c}_{0,T-1}^{-\gamma} \hat{X}_{n,T-1}} \right] .
\]

The contribution of the \( \frac{\hat{c}_{1,T-1}^{-\gamma} \hat{P}_{n,T-1}^{-\gamma} \hat{X}_{j,T-1}}{\hat{c}_{1,T-1}^{-\gamma} \hat{P}_{n,T-1}^{-\gamma} \hat{X}_{n,T-1}} \) term is only non-zero when \( f_{\omega} \) is non-zero, which is the case when \( \omega_{T-1} < \omega_{T-1}^{\text{min}} \). In this case, if \( \omega_{T-1}^{\text{min}} < 1 \) is assumed
to be sufficiently small, then $\omega_{1,T-1} < 1$, and
\[
\frac{f'_\omega}{f_\omega} = \frac{1}{1 - \omega_{1,T-1}} = \frac{1 - \omega_{1,T-1}}{1 - \omega_{\min}^{1,T-1}} \approx 1
\]
\[
\frac{\hat{c}_{1,T-1}}{d_{1,T-1}} = \omega_{1,T-1} < 1
\]
\[
\frac{\hat{P}_{n,T-1}}{\hat{X}_{n,T-1}} = \frac{\widehat{P}_{n,T-1}}{\widehat{P}_{n,T-1} + d_{1,T-1}} < 1.
\]
Hence
\[
\frac{\hat{c}_{1,T-1} \hat{P}_{n,T-1}}{d_{1,T-1} \hat{X}_{n,T-1}} \frac{f'_\omega}{f_\omega} < < 1,
\]
and can be ignored. This approximation essentially implies that the change in the constrained investor’s allocations has a much larger effect on the investor’s stochastic discount factor compared to its effect on an asset’s payoff that the effect of a constrained investor’s change in allocations on an asset’s payoff is much smaller compared to its effect on the investor’s stochastic discount factor. As a result, the sensitivity of the constrained investor’s subjective price to its allocation decisions is determined by the sensitivity of the investor’s discount factor to its allocations. This approximation becomes more accurate in case of a large number of constrained investors, each one of whom would hold an even smaller share of consumption, decreasing the impact of its allocation decisions on future payoffs.

With this approximation, we can write
\[
\frac{\partial P_{n,T-2}^1}{\partial \theta_{1,T-2}^0} \approx \beta (-\gamma) E_{T-2} \left[ \frac{\hat{c}_{1,T-1}^{\gamma} \hat{X}_{n,T-1}^{\gamma} \hat{X}_{j,T-1}^{1}}{\hat{c}_{1,T-1}^{1}} \right]
\]
\[
= \beta \frac{\hat{c}_{0,T-2}^{\gamma} E_{T-2}}{-\gamma} \left[ \frac{\hat{c}_{1,T-1}^{\gamma} \hat{X}_{n,T-1}^{\gamma} \hat{X}_{j,T-1}^{1}}{\hat{c}_{1,T-2}^{1}} \right]
\]
\[
= \beta \frac{\hat{c}_{0,T-2}^{\gamma} E_{T-2}}{-\gamma} \left[ \frac{\hat{c}_{0,T-1}^{\gamma} \hat{X}_{n,T-1}^{\gamma} \hat{X}_{j,T-1}^{1}}{\hat{c}_{0,T-2}^{1}} \right]
\]
\[
= \frac{\hat{c}_{0,T-2}^{\gamma} \hat{P}_{0,n,T-2}^0}{\hat{c}_{1,T-2}^{1}} \frac{\partial P_{n,T-2}^1}{\partial \theta_{1,T-2}^0} \theta_{1,T-2}^0,
\] (92)
where we have used the fact that ratio of intertemporal consumptions, \( \frac{\hat{c}^{\gamma}_{0,T-1}}{\hat{c}^{\gamma}_{0,T-2}} \), is equal across investors, which follows from the equality of stochastic discount factors in perfect-information case

\[
\frac{\hat{c}^{\gamma}_{0,T-1}}{\hat{c}^{\gamma}_{0,T-2}} = \frac{\hat{c}^{\gamma}_{1,T-1}}{\hat{c}^{\gamma}_{1,T-2}} = \frac{\hat{c}^{\gamma}_{2,T-1}}{\hat{c}^{\gamma}_{2,T-2}} \Rightarrow \frac{\hat{c}_{0,T-1}}{\hat{c}_{0,T-2}} = \frac{\hat{c}_{1,T-1}}{\hat{c}_{1,T-2}} = \frac{\hat{c}_{2,T-1}}{\hat{c}_{2,T-2}}
\]  

(93)

**B.8 Proof of Proposition 4.5**

From the market clearing condition, we can write

\[
\Delta \hat{\theta}_{2,T-2} = -\Delta \hat{\theta}_{0,T-2} - \Delta \hat{\theta}_{1,T-2}.
\]  

(94)

The kernel condition for \( m = 0 \) and \( m = 2 \) investors then becomes

\[
\partial_{\theta} \hat{P}^0_{T-2} \cdot \Delta \hat{\theta}_{0,T-2} = -\left( \frac{\hat{w}_{0,T-2}}{\hat{w}_{2,T-2}} \partial_{\theta} \hat{P}^0_{T-2} + \partial_{\theta} \Delta \hat{P}^2_{T-2} \right) \cdot \left( \Delta \hat{\theta}_{0,T-2} + \Delta \hat{\theta}_{1,T-2} \right) + \Delta \hat{P}^2_{T-2}
\]  

(95)

Using the kernel condition for \( m = 0 \) and \( m = 1 \) investors, we get

\[
\Delta \hat{\theta}_{1,T-2} = \frac{\hat{w}_{1,T-2}}{\hat{w}_{0,T-2}} \left( \partial_{\theta} \Delta \hat{P}^0_{T-2} \right)^{-1} \cdot \partial_{\theta} \Delta \hat{P}^0_{T-2} \cdot \Delta \hat{\theta}_{0,T-2}
\]  

\[
= \frac{\hat{w}_{1,T-2}}{\hat{w}_{0,T-2}} \Delta \hat{\theta}_{0,T-2}.
\]  

(96)

Using Equation (96) in Equation (95), we get

\[
\left[ \partial_{\theta} \hat{P}^0_{T-2} + \frac{\hat{w}_{0,T-1} + \hat{w}_{1,T-1}}{\hat{w}_{0,T-1}} \left( \frac{\hat{w}_{0,T-2}}{\hat{w}_{2,T-2}} \partial_{\theta} \hat{P}^0_{T-2} + \partial_{\theta} \Delta \hat{P}^2_{T-2} \right) \right] \cdot \Delta \hat{\theta}_{0,T-2} = \Delta \hat{P}^2_{T-2}.
\]  

(97)
From Corollary 7, we know
\[
\frac{\hat{\omega}_{0,T-2}}{\hat{\omega}_{2,T-2}} \partial_\theta \hat{P}_{T-2}^0 = \partial_\theta \hat{P}_{T-2}^2. \tag{98}
\]

Therefore, Equation (97) can be written as
\[
\Delta \hat{\theta}_{0,T-2} = \hat{\omega}_{0,T-1} \left[ (\hat{\omega}_{0,T-1} + \hat{\omega}_{1,T-1} + \hat{\omega}_{2,T-1}) \partial_\theta \hat{P}_{T-2}^2 + (\hat{\omega}_{0,T-1} + \hat{\omega}_{1,T-1}) \partial_\theta \Delta \hat{P}_{T-2}^2 \right]^{-1} \cdot \Delta \hat{P}_{T-2}^2
\]
\[
= \hat{\omega}_{0,T-1} \left[ \partial_\theta \hat{P}_{T-2}^2 + (\hat{\omega}_{0,T-1} + \hat{\omega}_{1,T-1}) \partial_\theta \Delta \hat{P}_{T-2}^2 \right]^{-1} \cdot \Delta \hat{P}_{T-2}^2 \tag{99}
\]

Using Equation (99) in Equation (96), we get
\[
\Delta \hat{\theta}_{1,T-2} = \hat{\omega}_{1,T-1} \left[ \partial_\theta \hat{P}_{T-2}^2 + (\hat{\omega}_{0,T-1} + \hat{\omega}_{1,T-1}) \partial_\theta \Delta \hat{P}_{T-2}^2 \right]^{-1} \cdot \Delta \hat{P}_{T-2}^2, \tag{100}
\]

and using Equations 99 and 100 in Equation (94), we get
\[
\Delta \hat{\theta}_{2,T-2} = -(\hat{\omega}_{0,T-1} + \hat{\omega}_{1,T-1}) \left[ \partial_\theta \hat{P}_{T-2}^2 + (\hat{\omega}_{0,T-1} + \hat{\omega}_{1,T-1}) \partial_\theta \Delta \hat{P}_{T-2}^2 \right]^{-1} \cdot \Delta \hat{P}_{T-2}^2, \tag{101}
\]

**B.9 Proof of Corollary 8**

In the case of two assets, \( \partial_\theta \hat{\theta}_{m,T-2} \) can be evaluated as
\[
\partial_\theta \hat{\theta}_{m,T-2} = \left( \partial_\theta \hat{P}_{m,T-2}^\Delta \right)^{-1} = \frac{\text{Cofac}(\partial_\theta \hat{P}_{m,T-2}^\Delta)}{|\partial_\theta \hat{P}_{m,T-2}^\Delta|}
\]

where
\[
\partial_\theta \hat{P}_{m,T-2}^\Delta = \begin{bmatrix}
\partial \left( \hat{P}_{0,T-2}^2 + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \Delta \hat{P}_{0,T-2}^2 \right) / \partial \theta_{m,T-2} & \partial \left( \hat{P}_{0,T-2}^2 + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \Delta \hat{P}_{0,T-2}^2 \right) / \partial \theta_{m,T-2} \\
\partial \left( \hat{P}_{1,T-2}^2 + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \Delta \hat{P}_{1,T-2}^2 \right) / \partial \theta_{m,T-2} & \partial \left( \hat{P}_{1,T-2}^2 + (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2}) \Delta \hat{P}_{1,T-2}^2 \right) / \partial \theta_{m,T-2}
\end{bmatrix}
\]
Hence, the determinant of an asset that pays off \( \tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2} \Delta \tilde{P}_{n,T-2} \) can be written as

\[
\frac{\partial (\tilde{P}_{n,T-2}^2 + (\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2}) \Delta \tilde{P}_{n,T-2}^2)}{\partial \tilde{\theta}^2_{m,T-2}} = \frac{\partial (\tilde{P}_{n,T-2}^2)}{\partial \tilde{\theta}^2_{m,T-2}} + (\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2}) \frac{\partial (\Delta \tilde{P}_{n,T-2}^2)}{\partial \tilde{\theta}^2_{m,T-2}}
\]

\[
= - \frac{\gamma}{\bar{c}_{2,T-2}} E_{T-2} \left[ \tilde{m}_{2,\gamma-1} \tilde{P}_{n,T-1} \tilde{P}_{j,T-1} + \frac{\Delta^2_{b}(\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2})}{2} \frac{\partial^2}{\partial \Omega^2_{1,T-1}} \left( \tilde{m}_{2,\gamma-1} \tilde{P}_{n,T-1} \tilde{P}_{j,T-1} \right) \right]
\]

\[
= - \frac{\gamma}{\bar{c}_{2,T-2}} \tilde{P}_{nj,\gamma+1} \left[ \Delta^2_{b}(\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2}) \frac{\partial^2 (\tilde{P}_{nj,\gamma+1})}{\partial \Omega^2_{1,T-1}} \right] \tag{102}
\]

where \( \tilde{P}_{nj,\gamma+1} \) denotes the uninformed investor's \((m = 2)\) subjective price of an asset that pays off \( \tilde{P}_{n,T-1} \tilde{P}_{j,T-1} \) at time \( T - 1 \), computed using a risk aversion of \( \gamma + 1 \). It can be seen that

\[
\tilde{P}_{nj,\gamma+1} = \tilde{P}_{jn,\gamma+1} \tag{103}
\]

\[
\tilde{P}_{00,\gamma+1} = \tilde{P}_{0,\gamma+1} \tag{104}
\]

\[
\tilde{P}_{01,\gamma+1} = \tilde{P}_{1,\gamma+1} \tag{105}
\]

Thus, \( \partial \tilde{\theta}^2 \tilde{P}_{m,T-2}^2 \) can be written as

\[
\partial \tilde{\theta}^2 \tilde{P}_{m,T-2}^2 = - \frac{\gamma}{\bar{c}_{2,T-2}} \tilde{P}_{0,\gamma+1} \left[ \Delta^2_{b}(\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2}) \frac{\partial^2 (\tilde{P}_{0,\gamma+1})}{\partial \Omega^2_{1,T-1}} \tilde{P}_{0,\gamma+1} + \Delta^2_{b}(\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2}) \frac{\partial^2 (\tilde{P}_{0,\gamma+1})}{\partial \Omega^2_{1,T-1}} \tilde{P}_{1,\gamma+1} \right]
\]

\[
= - \frac{\gamma}{\bar{c}_{2,T-2}} \left[ 1 + (\tilde{\omega}_{0,T-2} + \tilde{\omega}_{1,T-2}) \frac{\Delta^2_{b}}{2} \frac{\partial^2}{\partial \Omega^2_{1,T-1}} \right] \tilde{P}_{\gamma+1}^2 \tag{106}
\]

where

\[
\tilde{P}_{\gamma+1}^2 = \begin{bmatrix} \tilde{P}_{0,\gamma+1}^2 & \tilde{P}_{0,\gamma+1} \tilde{P}_{1,\gamma+1}^2 & \tilde{P}_{1,\gamma+1} \tilde{P}_{1,\gamma+1} \end{bmatrix}
\]

Hence, the determinant of \( \partial \tilde{\theta}^2 \tilde{P}_{m,T-2}^2 \) can be written as

\[
|\partial \tilde{\theta}^2 \tilde{P}_{m,T-2}^2| = \frac{\gamma^2}{\bar{c}_{2,T-2}} |\tilde{P}_{\gamma+1}^2| = \frac{\gamma^2}{\bar{c}_{2,T-2}} \left( 1 + \frac{|\tilde{P}_{\gamma+1}^2|}{|\tilde{P}_{\gamma+1}^2|} \right) |\tilde{P}_{\gamma+1}^2|.
\]
The inverse of \( \partial_{\hat{p}} \hat{P}_{m,T-2} \) can then be written as

\[
\partial_{\hat{p}} \hat{P}_{m,T-2} = \frac{-\gamma}{c_{2,T-2}} \left[ \hat{P}_{11,\gamma+1} + \frac{\Delta_{\gamma}^2 (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2})}{2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \right] - \hat{P}_{11,\gamma+1} \frac{\Delta_{\gamma}^2 (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2})}{2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \\
= \frac{-\gamma}{c_{2,T-2}} \left[ \hat{P}_{11,\gamma+1} + \frac{\Delta_{\gamma}^2 (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2})}{2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \right] + \hat{P}_{11,\gamma+1} \frac{\Delta_{\gamma}^2 (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2})}{2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2}
\]

\[
= -\frac{\gamma}{c_{2,T-2}} \left[ \hat{P}_{11,\gamma+1} + \frac{\Delta_{\gamma}^2 (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2})}{2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2} \right] + \hat{P}_{11,\gamma+1} \frac{\Delta_{\gamma}^2 (\hat{\omega}_{0,T-2} + \hat{\omega}_{1,T-2})}{2} \frac{\partial^2 \hat{P}_{11,\gamma+1}}{\partial \hat{p}_{\gamma+1,1}^2}
\]

where we used the fact that subjective prices of investor-2 are equal to equilibrium prices, i.e. \( \hat{P}_{n,\gamma+1} = \hat{P}_{n,\gamma+1} \).

### B.10 Dependence of SDF and its Derivatives on the State of the Economy

In the case of perfect information, the SDF of the uninformed investor, and its derivatives, are given by

\[
\hat{m}_{2,\gamma} = \beta (1 + g_{T-2,T-1})^{-\gamma}
\]

\[
\hat{m}'_{2} = -\beta \gamma \hat{m}_{0,\gamma+1} \frac{1}{c_{2,T-2}} \frac{\partial \hat{c}_{2,T-2}}{\partial \hat{\Omega}_{1,T-1}}
\]

\[
\hat{m}''_{2} = -\beta \gamma \left[ \hat{m}'_{2,\gamma+1} \frac{1}{c_{2,T-2}} \frac{\partial \hat{c}_{2,T-2}}{\partial \hat{\Omega}_{1,T-1}} + \hat{m}_{2,\gamma+1} \frac{1}{c_{2,T-2}} \frac{\partial^2 \hat{c}_{2,T-2}}{\partial \hat{\Omega}_{1,T-1}^2} \right]
\]

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\[ \frac{\partial \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}} \] is given by

\[ \frac{\partial \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}} = \tilde{A}_{2,2} f_\omega - \gamma \left( \tilde{A}_{2,1} + \tilde{A}_{2,2} f_\omega \right) f_\omega^{-\gamma} \frac{f_\omega'}{f_\omega} \]

\[ = -\hat{P}_{1,T-1} \left[ \hat{\theta}_{1,T-2} + \gamma \left( \hat{\theta}_{2,T-2} - \hat{\theta}_{2,T-1} \right) \right] \]

\[ \approx -\hat{\theta}_{2,T-2} \hat{P}_{1,T-1} \]

\[ \approx -\hat{\omega}_{2,T-2} \hat{P}_{1,T-1} \]

(107)

because \( \hat{\theta}_{2,T-2} \approx \hat{\theta}_{1,T-1} \) and \( \hat{\theta}_{2,T-2} \approx \hat{\omega}_{2,T-2} \). \( \frac{1}{\tilde{c}_{2,T-2}} \frac{\partial \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}} \) is then given by

\[ \frac{1}{\tilde{c}_{2,T-2}} \frac{\partial \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}} \approx -\frac{\hat{P}_{1,T-1}}{d_{1,T-2}} \] (108)

Similarly, \( \frac{\partial^2 \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}^2} \) is given by

\[ \frac{\partial^2 \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}^2} = \tilde{A}_{2,1} \left[ \gamma (\gamma + 1) \left( \frac{f_\omega'}{f_\omega} \right)^2 - \gamma \frac{f_\omega''}{f_\omega} \right] f_\omega^{-\gamma} + \tilde{A}_{2,2} \left[ \gamma (\gamma - 1) \left( \frac{f_\omega'}{f_\omega} \right)^2 - (\gamma - 1) \frac{f_\omega''}{f_\omega} \right] f_\omega^{1-\gamma} \]

\[ \approx 2\gamma \hat{P}_{1,T-1} \hat{\theta}_{1,T-2} \]

\[ \approx 2\gamma \hat{P}_{1,T-1} \hat{\omega}_{2,T-2} \]

\[ \approx 2\gamma \frac{\hat{P}_{1,T-1}}{d_{1,T-2}} \hat{\tilde{c}}_{2,T-2} \] (109)

\[ \frac{1}{\tilde{c}_{2,T-2}} \frac{\partial^2 \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}^2} \] can be approximately written as

\[ \frac{1}{\tilde{c}_{2,T-2}} \frac{\partial^2 \tilde{c}_{2,T-2}}{\partial \Omega_{1,T-1}^2} \approx 2\gamma \frac{\hat{P}_{1,T-1}}{d_{1,T-2}} \]
\( P_{1,T-1}/d_{1,T-2} \) can be expressed in terms of dividend growth rates

\[
\frac{P_{1,T-1}}{d_{1,T-2}} = \frac{1}{d_{1,T-1}} \frac{E_{T-1}[d_{1,T-1}^{\gamma}]}{d_{1,T-1}^{\gamma}} = \frac{E_{T-1}[(1 + g_{T-2,T-1})^{1-\gamma}(1 + g_{T-1,T})^{1-\gamma}]}{(1 + g_{T-2,T-1})^{-\gamma}},
\]

where \( g_{t,t+1} \) denotes dividend growth rates from time \( t \) to \( t + 1 \). \( E_{T-2}\left[\frac{P_{1,T-1}}{d_{1,T-2}}\right] \) is then given by

\[
E_{T-2}\left[\frac{P_{1,T-1}}{d_{1,T-2}}\right] = E_{T-2} \left(1 + g_{T-2,T} \right) \left(\frac{(1 + g_{T-2,T-1})(1 + g_{T-1,T})}{1 + g_{T-2,T-1}}\right)^{-\gamma} \approx E_{T-2} \left[(1 + g)^{1-\gamma}\right],
\]

(110)

assuming that \( g_{T-2,T-1} \approx g_{T-1,T} \equiv g \).

Thus, \( E_{T-2}\left[\hat{m}_{2,\gamma}'\right] \) can be simplified to

\[
E_{T-2}\left[\hat{m}_{2,\gamma}'\right] \approx \beta \gamma E_{T-2}\left[\hat{m}_{2,\gamma+1}(1 + g)^{1-\gamma}\right] \approx \beta \gamma E_{T-2}\left[(1 + g)^{-2\gamma}\right].
\]

And \( \hat{m}_{2}' \) can be written as

\[
\hat{m}_{2,\gamma}'' = -\beta \gamma \left[ -\hat{m}_{2,\gamma+1}' \frac{\hat{P}_{1,T-1}}{d_{1,T-2}} + 2\gamma \hat{m}_{2,\gamma+1} \frac{\hat{P}_{1,T-1}}{d_{1,T-2}} \right] = -\beta \gamma \left[ -\gamma \hat{m}_{2,\gamma+2}' \left(\frac{\hat{P}_{1,T-1}}{d_{1,T-2}}\right)^2 + 2\gamma \hat{m}_{2,\gamma+1}' \frac{\hat{P}_{1,T-1}}{d_{1,T-2}} \right] = \beta \gamma^2 \hat{m}_{2,\gamma+2}' \left(\frac{\hat{P}_{1,T-1}}{d_{1,T-2}}\right)^2.
\]
because \( \left( \frac{\hat{P}_1}{d_{1,T-2}} \right)^2 > \frac{\hat{P}_1}{d_{1,T-2}} \). Thus, \( ET_{-2}[\hat{m}_{2,\gamma}'] \) can be simplified to

\[
ET_{-2}[\hat{m}_{2,\gamma}'] = \beta \gamma^2 ET_{-2}[(1 + g)^2 - 2]\]

\[
= \beta \gamma^2 ET_{-2}[(1 + g)^{-3\gamma}] .
\]

**B.11 Relation between Optimal Consumption and Initial Endowment**

In order to express deviation in prices, \( \Delta \hat{P}_{n,T-2} \), and allocations, \( \Delta \hat{\theta}_{m,T-2} \), in terms of the exogenously given initial endowments of the risky asset, \( \overline{\theta}_m \), we need to express the uninformed investor’s consumption share, \( \hat{\omega}_{2,T-2} \), in terms of its initial endowment, \( \overline{\theta}_2 \). This can be done using the budget constraint

\[
\hat{\omega}_{m,T-2} + \hat{F}_{m,T-2} = \overline{\theta}_m \left( \hat{P}_{1,T-2} + d_{1,T-2} \right).
\]

Using

\[
\hat{F}_{m,T-2} = \hat{\theta}_{m,T-2} P_{0,T-2} + \hat{\theta}_{m,T-2} P_{1,T-2},
\]

the budget constraint can be re-written as

\[
\hat{\omega}_{m,T-2} = \overline{\theta}_m - \hat{\theta}_{m,T-2} P_{0,T-2} d_{1,T-2} + \left( \overline{\theta}_m - \hat{\theta}_{m,T-2} \right) \frac{\hat{P}_{1,T-2}}{d_{1,T-2}} \]

\[
\Rightarrow \hat{\omega}_{m,T-2} = \overline{\theta}_m \frac{1 + \frac{\hat{P}_{1,T-2}}{d_{1,T-2}}}{1 + \frac{\hat{\theta}_{m,T-2} P_{0,T-2} d_{1,T-2}}{d_{m,T-2}} + \frac{\hat{\theta}_{m,T-2} P_{1,T-2}}{d_{m,T-2}} d_{1,T-2}}.
\]

For the uninformed investor, \( m = 2 \), \( \hat{\theta}_{m,T-2} \) is independent of \( \hat{\omega}_{2,T-2} \), and, hence, the consumption share can be written explicitly in terms of exogenous quantities
(initial endowment, and dividends) as

$$\hat{\omega}_{2,T-2} = \bar{\omega}_2 \frac{1 + \hat{P}_{1,T-2}}{d_{1,T-2}}.$$

B.12 Imprecise-Information Equilibrium with a Large Number of Investors

With imprecise information, the change in each investor’s subjective price of \(n\)th asset is given by

$$\Delta \hat{b}^{m}_{n,T-2} = \delta_{n,T-2}(\hat{\theta}_{m,T-2}) + \frac{\Delta^2}{2} \frac{\partial^2 \left( \tilde{P}_{m,n,T-2}(\hat{\theta}_{m,T-2}) \right)}{\partial \Omega_{T-1}^2},$$

(111)

where

$$\Delta^2 = \int_{\Omega_{T-1}^{\text{min}}}^{\Omega_{T-1}^{\text{max}}} g_m (\Omega_{T-1}) \left( \Omega_{T-1} - \bar{\Omega}_{m,T-1} \right)^2 d\Omega_{T-1}$$

(112)

is the variance of \(m\)th investor’s forecast of \(\Omega_{T-1}\), around its average forecast of \(\bar{\Omega}_{m,T-1}\).

Using first order Taylor expansion around optimal allocations in the case of perfect information, the kernel conditions can be written in a matrix form as

$$\Delta \hat{P}^{m}_{T-2} + \partial_{\theta} P^{m}_{T-2} \cdot \Delta \hat{\theta}_{m,T-2} = \Delta \hat{P}^{m'}_{T-2} + \partial_{\theta} P^{m'}_{T-2} \cdot \Delta \hat{\theta}_{m',T-2};$$

where \(m \neq m'\). Using

$$\partial_{\theta} P^{m}_{T-2} = \partial_{\theta} \left( \hat{P}^{m}_{T-2} + \Delta \hat{P}^{m}_{T-2} \right) \approx \partial_{\theta} \hat{P}^{m}_{T-2};$$

(113)

and

$$\partial_{\theta} \hat{P}^{m'}_{T-2} = \frac{\hat{\omega}_{m,T-2}}{\hat{\omega}_{m',T-2}} \partial_{\theta} \hat{P}^{m}_{T-2},$$

(114)
the kernel conditions can be written as

\[ \Delta \hat{P}^m_{T-2} + \partial_\theta \hat{P}^m_{T-2} \cdot \Delta \hat{\theta}_{m,T-2} = \Delta \hat{P}^m_{T-2} + \frac{\hat{\omega}_{m,T-2}}{\hat{\omega}_{m',T-2}} \partial_\theta \hat{P}^m_{T-2} \cdot \Delta \hat{\theta}_{m',T-2}. \]

Now, we start with kernel conditions for \( m = 0 \) and \( m = M \) investors

\[ \Delta \hat{P}^0_{T-2} + \partial_\theta \hat{P}^0_{T-2} \cdot \Delta \hat{\theta}_{0,T-2} = \Delta \hat{P}^M_{T-2} + \frac{\hat{\omega}_{0,T-2}}{\hat{\omega}_{M,T-2}} \partial_\theta \hat{P}^0_{T-2} \cdot \Delta \hat{\theta}_{M,T-2}. \]

Replace \( \Delta \hat{\theta}_{M,T-2} \) using market clearing conditions

\[ \Delta \hat{\theta}_{M,T-2} = - \left( \Delta \hat{\theta}_{0,T-2} + \cdots + \Delta \hat{\theta}_{M-1,T-2} \right), \]

and use kernel condition for \( m = 0 \) and \( m = j \) investors to obtain

\[ \Delta \hat{\theta}_{j,T-2} = \left( \frac{\hat{\omega}_{0,T-2}}{\hat{\omega}_{j,T-2}} \partial_\theta \hat{P}^0_{T-2} \right)^{-1} \left[ \partial_\theta \hat{P}^0_{T-2} \cdot \Delta \hat{\theta}_{0,T-2} - \left( \Delta \hat{P}^j_{T-2} - \Delta \hat{P}^0_{T-2} \right) \right]. \]  

(115)

This leads to

\[ \Delta \hat{\theta}_{0,T-2} = \left( \partial_\theta \hat{P}^0_{T-2} \right)^{-1} \cdot \sum_{i=0}^{M} \omega_i \left( \Delta \hat{P}^i_{T-2} - \Delta \hat{P}^0_{T-2} \right) \]  

(116)

Using Equation (116) in Equation (115), we obtain

\[ \Delta \hat{\theta}_{j,T-2} = \left( \partial_\theta \hat{P}^j_{T-2} \right)^{-1} \cdot \sum_{i=0}^{M} \omega_i \left( \Delta \hat{P}^i_{T-2} - \Delta \hat{P}^j_{T-2} \right). \]

The change in equilibrium prices is then given by

\[ \Delta \hat{P}_{T-2} = \partial_\theta \hat{P}_{T-2} \cdot \Delta \hat{\theta}_{j,T-2} \]

\[ = \sum_{i=0}^{M} \hat{\omega}_{i,T-2} \Delta \hat{P}^i_{T-2}. \]
References


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