Short-Run and Long-Run Factor Structure in Equity Options

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Comments are greatly appreciated.

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Abstract

This paper investigates theoretically and empirically the twin discourse of equity and index option prices. At the index level, we adopt a two-factor stochastic volatility model, where aggregate market volatility is decomposed into a persistent factor and a transient factor. We introduce a pricing kernel that links the physical and risk neutral distributions. Using simultaneous data from the S&P 500 index and options markets, we find negative prices for both variance factors implying that investors are willing to pay for insurance against increases in volatility risk, even if those increases have little persistence. We also obtain negative correlations between shocks to the market returns and each volatility factor, where correlation is less significant in transient factor and therefore has a less significant effect on the index skewness. At the firm level, we develop an equity option valuation model that is linked to the market model to capture the two-factor structure. Our proposed factor structure and closed-form option pricing equations yield separate expressions for the exposure of equity options to both volatilities and overall market returns. These expressions allow a portfolio manager to hedge her portfolio’s exposure to the underlying risk factors. In cross-sectional analysis our model predicts that firms with higher transient beta have a steeper term structure of implied volatility and a steeper implied volatility moneyness slope. Our model also predicts that variance risk premium has a more significant effect on the equity option skew when the transient beta is higher. On the empirical front, for the firms listed on the Dow Jones index, our model provides a good fit to the observed equity option prices.

JEL Classification: G10; G12; G13
Keywords: two-factor stochastic volatility; equity options; factor models; joint estimations; implied volatility;
1 Introduction

The dynamics of index return volatility and their role in pricing options have had a long history following the classic early works by Wiggins [1987] and Heston [1993], that recognized the volatility's stochastic nature and managed to derive closed form expressions for the resulting European options. Related early contributions were also by Duan [1995] and Heston and Nandi [2000] under GARCH return dynamics, with option prices derived either by numerical methods or with closed form expressions. More recent studies, however, have pointed out that a single factor stochastic volatility (SV) or GARCH is not sufficient to represent both the underlying ($P$) and the risk neutral ($Q$) measures of the joint dynamics of returns and variances for the key S&P 500 index and its options.\footnote{See, for instance, Bollerslev and Zhou [2002], Alizadeh et al. [2002], and Chernov et al. [2003] for the $P$-returns and Bates [2000], Christoffersen et al. [2008], and Christoffersen et al. [2009] for the option-based $Q$-distribution.} In particular, these studies show that one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility, and that two volatility factors (one with persistent dynamics and one with transient dynamics) are needed to explain return volatility dynamics; similar considerations apply also to option-based risk neutral returns.

This paper start with examining index option pricing under two SV factors, building up on an idea originally formulated by Xu and Taylor [1994] in foreign exchange options, and further elaborated by Engle and Lee [1999], who decompose the aggregate market volatility into a more persistent volatility component, which has nearly a unit root, and a transitory volatility component, which has more rapid time decay. We present an affine two-factor SV process for the underlying index returns and introduce an admissible pricing kernel to find the risk-neutral returns dynamics and to price European options.\footnote{Note that the extracted risk-neutral dynamics are not restricted to the introduced admissible pricing kernel, where investors variance risk preference is distinguished from equity risk preference.} As in the one-factor volatility of Christoffersen et al. [2013], we also introduce an associated component volatility model (two-factor GARCH model) and derive the corresponding pricing kernel linking the $P$- and $Q$-distributions.\footnote{The proof is available from the author upon request.}

We investigate empirically the pricing performance of our two-factor SV model in S&P 500 options by estimating the joint dynamics of returns and variances under the $P$ and $Q$ measures.\footnote{The importance of joint estimation of the structural parameters of the underlying returns and volatility dynamics has been addressed in Bates [1996], Chernov and Ghysels [2000], Pan [2002], Eraker [2004], and Broadie et al. [2007] among others.} We filter two vectors of daily spot variances using the Particle Filter (PF) method.\footnote{For the application of PF in estimating the model parameters see Gordon et al. [1993], Johannes et al. [2009], Johannes and Polson [2009], Christoffersen et al. [2010], and Boloorforoosh [2014].} We follow the conventional filtration procedure of similar studies but provide a novel and methodologically important solution for the challenging issue of how to separate the two variances’ paths. We then obtain a consistent set of structural parameters for the two-factor SV model that simultaneously captures the information contents of the time-series of index
returns and the cross-sections of options prices.\textsuperscript{6,7} Our study is the first one to estimate simultaneously the $P$- and $Q$-parameters from underlying index return and option price data where the underlying index follows an affine two-factor SV process. Consistent with previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility factors is highly persistent (long-run volatility component) while the immediate impact of volatility shocks on the short-run volatility component is bigger but short-lived. We observe a big difference between the volatility of volatility in the first and second variance components. We also find negative prices for both variance components, which is consistent with Adrian and Rosenberg [2008].

Next, we turn to individual equities and extend our two factor SV models to the equilibrium pricing of equity options. We show that the existence of multiple volatility factors in index returns has significant implications for equity returns and the associated equity option prices. Within the asset pricing models, when market volatility is stochastic, the classical Intertemporal CAPM (ICAPM) model of Merton [1973] and Merton [1980] implies that in the presence of two state variables, namely return and volatility, the excess returns on the market portfolio should also be related to the volatility of the market. In other words, the asset risk premiums are not only determined by the covariation of asset returns with the market returns, but also by its covariation with the state variables that govern the market volatility.\textsuperscript{8} For this reason we explicitly model the factor structure in equity options when the market model follows a two-factor SV.

We extend the one-volatility-factor model in Christoffersen et al. [2015] and assume that individual equity returns are related to the market index with two distinct systematic components, as well as an idiosyncratic component which is stochastic and follows the standard square root process. Hence, the equity option prices are related to the market index with two distinct betas, one of which captures the short-run variation in market returns and the other one captures its long-run counterpart. We obtain a closed-form option pricing equation for individual equity options as the proposed model belongs to the affine class of models. Our estimation shows that the two betas have quite different values: in our sample of 27 firms, all belonging to the Dow Jones index, the short-run beta has values ranging from 1.01 to 1.35, while the long run beta is about half the value, range from 0.34 to 0.68. Our empirical investigation of this model using individual equity option prices finds support for the proposed factor structure in equity returns.

Our models’ framework is especially important for a portfolio manager who hedges her portfolio’s exposure to the systematic risk factors in the portfolio of stocks and options. Our

\textsuperscript{6}According to Christoffersen et al. [2009, Section 6], “an integrated analysis of multifactor models using option data as well as underlying returns out to be done.”

\textsuperscript{7}Although the main drawback in this efficient joint estimation procedure is its heavy computational burden, (See Pan [2002] and Eraker [2004]), we managed to keep a large time-series of returns and the entire cross-section of daily option prices over the same time span.

\textsuperscript{8}See Ang et al. [2006], who show that the show that the aggregate market volatility is a significant cross-sectional asset pricing factor, and Adrian and Rosenberg [2008], who find that both volatility components have negative and highly significant prices of risk.
proposed factor structure and closed-form option pricing equations make this analysis readily available and yields similar closed-form expressions for the exposure of equity options to the short-run and long-run market volatility in addition to its exposure to the overall market returns. We then provide a closed-form expression for the expected equity option returns with respect to their sensitivity to the level of market index and market variances. Therefore, we are able to disentangle the effect of market risk premium from those of volatilities premiums on the expected equity option returns.

The proposed factor structure has a number of important cross-sectional implications for equity options. Consistent with the findings of Duan and Wei [2009], our model confirms that firms with higher average betas have higher implied volatilities. It also predicts that firms with higher average betas have steeper term structures of implied volatility. In particular, we observe that the term-structure of the equity implied volatility is more sensitive to the short-term beta while the impact of the other beta on the term structures of implied volatility is marginal. Consistent with the finding in Christoffersen et al. [2015], our proposed model also predicts that the implied volatility moneyness slope is steeper for the firms with the higher average betas. On top of that, our factor structure decomposes the effect of short-run and long-run betas on the moneyness slope of the implied volatility of equity options and predicts that the long-run beta has little effect on the moneyness slope. Finally, we find that the effect of the market variances risk premiums on the equity option skew depends on the equity betas. As we expect, the variance risk premium has more significant effect on the implied volatility of equity options when the beta is higher. In particular, we observe that while the short-run beta plays a major role on the relation between market variances risk premiums and the slope of equity implied volatility curve, the long-run beta has marginal effect.

Our proposed factor structure in equity options is motivated by the extensive empirical evidence that supports the presence of two volatility components, with different sensitivities to the volatility shocks and different mean reversions, in the dynamics of the market returns, with one component highly persistent and the other highly mean-reverting. In the $P$-distribution domain the relative performance of the two-factor SV structure compared to its one-factor counterpart in capturing the dynamics of the exchange rates and equity returns has been examined in several studies. These studies document that two volatility factors (one with persistent dynamics and one with transient dynamics) are needed to explain volatility dynamics, since one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility. For instance, Chernov et al. [2003] suggest that the addition of a second volatility factor breaks the link between tail thickness and volatility persistence. They show that the second SV factor in affine models leads to a significant improvement relative to a single SV models in capturing the return dynamics. They also

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9Adrian and Rosenberg [2008] show that the equilibrium pricing kernel depends on both the short- and long-run volatility components as well as the excess market return.

10See, for instance, Bollerslev and Zhou [2002], Alizadeh et al. [2002], and Chernov et al. [2003].

11There is also widespread evidence that multifactor are needed to model the term structure of the interest rate. See Dai and Singleton [2000, 2002] among others.
find that when the second volatility factor is allowed to have its own correlation with returns, the correlation parameters can take on both positive and negative values, contrary to the findings in single factor volatility models, where the correlation parameter is always found to be negative.

Similar considerations also hold for the \( Q \)-distribution. Although an extensive empirical literature in the options markets has documented that SV models are helpful in the modeling of the volatility smirk by incorporating a leverage effect\(^{12}\) and in the modeling of the volatility term structure effect by incorporating mean reversion in variance dynamics, these empirical studies observe that the shape of the volatility smirk can be either flat or steep at a given volatility level but cannot accommodate both at the same time for a given parameterization.\(^{13}\) This so called structural problem in one factor SV models is more restraining especially when estimating the model parameters using multiple cross-sections of options data. Such a restriction is mostly related to the fact that in one-factor SV models the correlation between stock returns and variance is constant across all cross-sections of option contracts regardless of the level and shape of the volatility.

Similar inconsistencies in the joint estimation of the SV model are illustrated by Broadie et al. [2007]. They note the failure of SV model to reconcile the \( P \)- and \( Q \)-estimates of certain structural parameters of the model, namely the correlation coefficient and volatility of volatility, and conclude that the SV model is basically misspecified. They also show that the joint restrictions on the returns and volatility dynamics under the \( P \) and \( Q \) measures leads to the poor performance of the SV model, which is measured by the high level of IVRMSE. They indicate that due to the joint-restriction, SV model cannot generate sufficient amounts of conditional skewness and kurtosis.

Multiple SV models, on the other hand, can better capture the time-varying nature of the smirk as the correlation between stock returns and total volatility is stochastic.\(^{14}\) Such models, therefore, have more flexibility to fit the term structure of the volatility and to control the level and the slope of volatility smirk in cross-sections of option prices.\(^{15}\) Moreover, the conditional skewness and kurtosis are more flexible for given levels of conditional variance. Our own empirical results confirm that these important characteristics lead to superior performance of multifactor SV models compared to their single factor counterpart.

Although our study is not the first one to examine multifactor SV and GARCH models, it is the only one to present consistent \( P \)- and \( Q \)-parameter estimates both theoretically and empirically. For instance Bates [2000] examined a multifactor specification in option pricing by relying on the \( Q \)-distribution only. Christoffersen et al. [2008] introduced a two-component GARCH model, which can generate more flexible skewness and volatility of volatility dynamics in capturing the dynamics of the S&P 500 index returns and in pricing

\(^{12}\)See, among others, Bakshi et al. [1997], Bates [2000], and Jones [2003].

\(^{13}\)See Derman [1999].

\(^{14}\)Christoffersen et al. [2009, Equation 15] show that the correlation between returns and total volatility in a two-factor SV model is stochastic.

\(^{15}\)See, for instance, Egloff et al. [2010] and Mencía and Sentana [2013].
European S&P 500 call options. That study documents that the empirical performance of this two-factor model is significantly better than that of the benchmark GARCH(1,1) model, both in-sample and out-of-sample, and also finds that the proposed volatility component specification could better capture the volatility term structure. Nonetheless, the absence of an explicit pricing kernel linking the $P$- and $Q$-distributions in that study necessitated either the use of an arbitrary price of volatility risk or the estimation of the risk neutral parameters by relying on the $Q$-distribution only.

Multiple variance factors in option pricing were further explored by Christoffersen et al. [2009], who investigated the relative significance of the first four principal components of Black-Scholes implied variances on a sample of S&P 500 index options from 1990 until 2005 and found that the first two principal components together explain more than 95% of the variation in the implied variances.\footnote{16}{See Christoffersen et al. [2009, Figure 2, Table 2].} They also showed that two-factor SV models can generate stochastic correlation between total instantaneous volatility and stock returns. In our own empirical results we show that this feature enables our two-factor model to better capture the conditional skewness and kurtosis and reduces significantly the vega-weighted root mean squared error of option pricing compared to the one-factor models when we impose the joint-estimation restrictions.\footnote{17}{Note that, using the option prices only, Christoffersen et al. [2009, Section 3.1] show that two-factor SV model improves over its one-factor benchmark model both in-sample and out-of-sample.} \footnote{18}{The vega weighted option pricing errors serve as an approximation to the implied volatility errors. See for example Carr and Wu [2007] and Christoffersen et al. [2009].}

The paper proceeds as follows. Section 2 presents the theoretical model for pricing index options and equity options. In Section 3 we discuss the implications of the model. Section 4 contains the description of the data sets. In Section 5 we discuss the estimation methodologies for both index and equity options. Then we present the estimation results and investigate the performance of the models. Section 6 concludes. The appendix provides the proofs of the theoretical results.

\section{Model Setup}

We start by a multiple-factor stochastic volatility dynamics that governs the market index returns under the $P$-distributions and then introduce a pricing kernel that links the $P$-dynamics to their risk-neutral counterparts by imposing appropriate martingale’s restrictions on pricing kernel. We complete the the index model by deriving a closed-form pricing equation for index options. We then describe the dynamics of individual equity returns under both $P$ and $Q$ measures, with the equity return dynamics and index return dynamics linked by a factor model. Last, we develop a closed-form valuation equation for options on individual equities.

We assume the following two-factor stochastic volatility process governing the dynamics of
the market index returns and variance under the physical distributions.

\[
dS_t/S_t = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{\bar{v}_{1,t}}d\tilde{z}_{1,t} + \sqrt{\bar{v}_{2,t}}d\tilde{z}_{2,t} \\
dv_{1,t} = \kappa_1 (\theta_1 - v_{1,t})dt + \sigma_1 \sqrt{\bar{v}_{1,t}}dw_{1,t} \\
dv_{2,t} = \kappa_2 (\theta_2 - v_{2,t})dt + \sigma_2 \sqrt{\bar{v}_{2,t}}dw_{2,t}
\]  

where, as in Christoffersen et al. [2009] we assume the stochastic structure (2).

\[
\begin{align*}
dw_{1,t} \cdot dz_{1,t} &= \rho_1 dt, \quad -1 \leq \rho_1 \leq 1 \\
dw_{2,t} \cdot dz_{2,t} &= \rho_2 dt, \quad -1 \leq \rho_2 \leq 1 \\
dw_{1,t} \cdot dw_{2,t} &= 0 \\
\rho_1^2 + \rho_2^2 &\leq 1
\end{align*}
\]

As in Heston [1993], \(\theta_1\) and \(\theta_2\) are unconditional average variances, \(\kappa_1\) and \(\kappa_2\) capture the speed of mean reversion for each volatility factor, and \(\sigma_1\) and \(\sigma_2\) measure the volatility of variances. The market equity risk premiums are denoted by \(\mu_1 v_{1,t}\) and \(\mu_2 v_{2,t}\). Following Bollerslev and Zhou [2006] we expect that \(\mu_1\) and \(\mu_2\) measure the long-run and short-run “continuous-time” volatility feedback effects or risk-return trade-offs. The instantaneous correlation between shocks to the market returns and shocks to the long-run variance is described by \(\rho_1\) and the instantaneous correlations between market returns and short-run variance is given by \(\rho_2\). According to Bollerslev and Zhou [2006], we expect that \(\rho_1\) and \(\rho_2\) account for the long-run and short-run “continuous-time” leverage (asymmetry) effect.

Note that (2) implies that the total return variance and the total correlation between returns and variances are as follows.

\[
\begin{align*}
\text{Var}_t[dS_t/S_t] &= v_{1,t}dt + v_{2,t}dt = v_t dt \\
\text{Corr}_t[dS_t/S_t, dV_t] &= \frac{\rho_1 \sigma_1 v_{1,t} + \rho_2 \sigma_2 v_{2,t}}{\sqrt{\sigma_1^2 v_{1,t} + \sigma_2^2 v_{2,t}} \sqrt{\bar{v}_{1,t} + \bar{v}_{2,t}}} dt
\end{align*}
\]  

We may then prove the following result.

**Proposition 1.** The market index has the following dynamics under the risk-neutral measure:

\[
\begin{align*}
dS_t/S_t &= r dt + \sqrt{\bar{v}_{1,t}}d\tilde{z}_{1,t} + \sqrt{\bar{v}_{2,t}}d\tilde{z}_{2,t} , \\
dv_{1,t} &= \bar{\kappa}_1 (\bar{\theta}_1 - v_{1,t})dt + \sigma_1 \sqrt{\bar{v}_{1,t}}\tilde{w}_{1,t} , \\
dv_{2,t} &= \bar{\kappa}_2 (\bar{\theta}_2 - v_{2,t})dt + \sigma_2 \sqrt{\bar{v}_{2,t}}\tilde{w}_{2,t} ,
\end{align*}
\]

where, \(\bar{\kappa}_1 = \kappa_1 + \lambda_1\), \(\bar{\kappa}_2 = \kappa_2 + \lambda_2\), \(\bar{\theta}_1 = \frac{\kappa_1 \theta_1}{\kappa_1 + \lambda_1}\), and \(\bar{\theta}_2 = \frac{\kappa_2 \theta_2}{\kappa_2 + \lambda_2}\), and where \(\lambda_1\) and \(\lambda_2\) are the prices of the variances risk factors as in the single factor Heston [1993] model.

One admissible pricing kernel that links the physical dynamics in (1) to the risk-neutral dynamics in (4) takes the following exponential affine form.
\[
\frac{M_t}{M_0} = \left(\frac{S_t}{S_0}\right)^\phi \exp \left[ \delta t + \eta_1 \int_0^t v_1,s ds + \eta_2 \int_0^t v_2,s ds + \zeta_1 (v_{1,t} - v_{1,0}) + \zeta_2 (v_{2,t} - v_{2,0}) \right],
\]

where, \(\{\delta, \eta_1, \eta_2\}\) governs the time-preferences, while \(\{\phi, \zeta_1, \zeta_2\}\) governs the respected risk aversion to the index and variances risk factors as in Christoffersen et al. [2013].

Proof. See Appendix A.

We note that equation (5) is one way of “completing the market” by linking \(P\)- to \(Q\)-dynamics. The proposed transformation between these two dynamics is not restricted to the introduced pricing kernel. An alternative way of getting the same prices for equity and variances risk factors shown in Appendix A without explicit assumptions about the investor’s variance preferences is by assuming the following change-of-measure we get the same results.

\[
\frac{dQ}{dP}(t) = \exp \left[ \int_0^t \gamma_u' dW_u - \frac{1}{2} \int_0^t \gamma_u' d\langle W,W \rangle_u \right],
\]

where \(\gamma_u \equiv [\psi_{1,u}, \psi_{2,u}, \psi_{3,u}, \psi_{4,u}, \psi_{1,u}' \psi_{2,u}] \forall i\) is the vector of market prices of risk, \(W_u \equiv [z_{1,u}, z_{2,u}, w_{1,u}, w_{2,u}, z_{1,u}', z_{2,u}'] \forall i\) is a \((2(n + 2)) \times 1\) vector of innovations to returns and volatilities, and \(\langle \cdot, \cdot \rangle \) is the covariance operator.\(^{19}\)

Christoffersen et al. [2009] estimate annual structural parameters in (A.4) using only S&P 500 index options data and find that this model provides more flexible modeling of the volatility term structure by capturing the independent fluctuations in the level and the slope of the volatility smirk over time. This flexibility is a direct consequence of the assumed two-factor volatility model where the factors have distinct correlations with market returns, while the weights of factors vary over time. Following the observations of Bates [1996] and Broadie et al. [2007], it is of particular interest to investigate the pricing performance of multiple-factor model when we place joint restrictions on the returns and variances dynamics under \(P\) and \(Q\) measures. Therefore, we estimate a consistent set of structural parameters for multiple SV model that captures the information contents of the time-series of index prices and the cross-sections of options prices.\(^{20}\)

To embed the options market data into the estimation of structural parameters, we determine a closed-from expression for the price of the European call options, with strike price \(K\) and time to maturity \(\tau\), by inverting the conditional characteristic function of the log spot index prices, \(x_t = \ln(S_t)\).

\[
C_t(S_t, K, v_{1,t}, v_{2,t}, \tau) = S_t P_1 - K e^{-r\tau} P_2,
\]

where,\(^{19}\) The proof, similar to Christoffersen et al. [2015] is available from the authors upon request.\(^{20}\) According to Christoffersen et al. [2009, Section 6], “an integrated analysis of multifactor models using option data as well as underlying returns out to be done”.

\(^{19}\) The proof, similar to Christoffersen et al. [2015] is available from the authors upon request.

\(^{20}\) According to Christoffersen et al. [2009, Section 6], “an integrated analysis of multifactor models using option data as well as underlying returns out to be done”.

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and where the conditional risk-neutral characteristic function of the natural logarithm of the stock prices at expiration, \( x_{t+\tau} \), is

\[
\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) \equiv E_t^Q[\exp(i\phi x_{t+\tau}) \mid x_t].
\]

Since the two-factor SV model in (4) is an affine process, following Heston [1993], the conditional risk-neutral characteristic function in (8) has the following affine exponential form.\(^{21}\)

\[
\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) = \exp \left[ i\phi x_t + i\phi r \tau + A_1(\tau, \phi) + A_2(\tau, \phi) + B_1(\tau, \phi)v_{1,t} + B_2(\tau, \phi)v_{2,t} \right],
\]

where\(^{22}\)

\[
A_1(\tau, \phi) = \frac{\tilde{\kappa}_1 \tilde{\theta}_1}{\sigma_1^2} \left[ (\tilde{\kappa}_1 - \rho_1 \sigma_1 i\phi - d_1) \tau - 2 \ln \left( \frac{1 - c_1 e^{-d_1 \tau}}{1 - c_1} \right) \right],
\]

\[
A_2(\tau, \phi) = \frac{\tilde{\kappa}_2 \tilde{\theta}_2}{\sigma_2^2} \left[ (\tilde{\kappa}_2 - \rho_2 \sigma_2 i\phi - d_2) \tau - 2 \ln \left( \frac{1 - c_2 e^{-d_2 \tau}}{1 - c_2} \right) \right],
\]

\[
B_1 = \frac{\tilde{\kappa}_1 - \rho_1 \sigma_1 i\phi - d_1}{\sigma_1^2} \left[ \frac{1 - e^{-d_1 \tau}}{1 - c_1 e^{-d_1 \tau}} \right], \quad B_2 = \frac{\tilde{\kappa}_2 - \rho_2 \sigma_2 i\phi - d_2}{\sigma_2^2} \left[ \frac{1 - e^{-d_2 \tau}}{1 - c_2 e^{-d_2 \tau}} \right],
\]

\[
c_1 = \frac{\tilde{\kappa}_1 - \rho_1 \sigma_1 i\phi - d_1}{\tilde{\kappa}_1 - \rho_1 \sigma_1 i\phi + d_1}, \quad c_2 = \frac{\tilde{\kappa}_2 - \rho_2 \sigma_2 i\phi - d_2}{\tilde{\kappa}_2 - \rho_2 \sigma_2 i\phi + d_2},
\]

\[
d_1 = \sqrt{(\tilde{\kappa}_1 - \rho_1 \sigma_1 i\phi)^2 + \sigma_1^2 \phi(\phi + i)}, \quad d_2 = \sqrt{(\tilde{\kappa}_2 - \rho_2 \sigma_2 i\phi)^2 + \sigma_2^2 \phi(\phi + i)}.
\]

Next, we turn to individual equities and their options. We assume that equity returns are related to the market risk factors as well as to an idiosyncratic risk factor. The idiosyncratic risk factor \( \xi_t^i \), follows a standard square-root process but is not a priced factor.\(^{23}\) For each

\(^{21}\)Note that the conditional risk-neutral characteristic function of the natural logarithm of returns, \( x_{t+\tau} - x_t = \ln(S_{t+\tau}/S_t) \), can be defined with the same expression as (9) but without the first component, \( i\phi x_t \).

\(^{22}\)Following Duffie et al. [2000], the coefficients \( A_1, A_2, B_1, \) and \( B_2 \) are the solutions of a system of Riccati equations subject to appropriate boundary conditions. Here for the ease of computation we modify these solutions based on the little Heston trap formulation of Albrecher et al. [2006].

\(^{23}\)Recently, Boloorforoosh [2014] study a single-factor SV model where idiosyncratic volatility is priced. We discuss the implications of priced idiosyncratic volatility in our model in the following sections.
individual equity idiosyncratic return innovations and variances’ innovations are correlated with the coefficient of $\rho^i$. This factor derives the continuous-time leverage (asymmetry) effect in the dynamics of individual equity returns. Following the empirical evidence we expect that the observed asymmetry should be weaker but still present in the case of individual equities.\footnote{See Andersen et al. [2001].}

$$
\begin{align*}
\frac{dS_t^i}{S_t^i} &= \mu^i dt + \beta^1_1(\mu_1 v_{1,t} dt + \sqrt{v_{1,t}} dz_{1,t}) + \beta^1_2(\mu_2 v_{2,t} dt + \sqrt{v_{2,t}} dz_{2,t}) + \sqrt{\xi_t^i} dz^i_t \\
\frac{d\xi_t^i}{\xi_t^i} &= \kappa^i(\theta^i - \xi_t^i) dt + \sigma^i \sqrt{\xi_t^i} dw^i_t
\end{align*}
$$

As a result of the proposed specification in (11), the total instantaneous spot variance for equity $i$ at time $t$ under $P$ is given by $v_t^i = (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_t^i$.

In order to price options on individual equities, we define the process that governs the risk-neutral dynamics of individual equity returns in the following proposition. Following the conventional assumption in stochastic volatility models, we assume that the prices of market volatility risk factors are proportional to the spot volatilities but that idiosyncratic volatility risk is not priced. We can simply extend our model by including the priced idiosyncratic volatility factor, where idiosyncratic volatility risk is also proportional to the spot idiosyncratic volatility.

**Proposition 2.** Given the dynamics of the individual equity returns under $P$-measure in (11), the following dynamics govern its $Q$-measure counterpart.

$$
\begin{align*}
\frac{dS_t^i}{S_t^i} &= r dt + \beta^1_1 \sqrt{\nu_{1,t}} d\tilde{z}_{1,t} + \beta^1_2 \sqrt{\nu_{2,t}} d\tilde{z}_{2,t} + \sqrt{\xi_t^i} d\tilde{z}_t^i \\
\frac{d\xi_t^i}{\xi_t^i} &= \kappa^i(\theta^i - \xi_t^i) dt + \sigma^i \sqrt{\xi_t^i} d\tilde{w}_t^i
\end{align*}
$$

**Proof.** See Appendix B. \qed

As the dynamics of individual equity prices are affine, the conditional risk-neutral characteristic function of the natural logarithm of the equity prices is derived analytically in the following proposition. Accordingly, we compute the closed-from pricing equation of the European equity call options with strike price $K$ and time to maturity $\tau$. See also Appendix C.

**Proposition 3.** Given the dynamics of the individual equity returns under the $Q$-measure in (12), the risk-neutral conditional characteristic function of the natural logarithm of the individual equity price, $x_{t+\tau}^i = \ln(S_{t+\tau}^i)$, is:
\[
\tilde{f}(x^i_t, v_{1,t}, v_{2,t}, \xi^i_t, \beta_1^i, \beta_2^i, \tau, \phi) = E^Q \left[ \exp\left(i \phi x^i_{t+1} \right) \mid x^i_t \right] \\
= \exp\left[ i \phi x^i_t + i \phi \tau - A_1(\tau, \phi) - A_2(\tau, \phi) - B(\tau, \phi) \right. \\
\left. + C_1(\tau, \phi) v_{1,t} + C_2(\tau, \phi) v_{2,t} + D(\tau, \phi) \xi^i_t \right],
\]

where, the expressions for \( A_1(\tau, \phi) \), \( A_2(\tau, \phi) \), \( B(\tau, \phi) \), \( C_1(\tau, \phi) \), \( C_2(\tau, \phi) \), and \( D(\tau, \phi) \) are provided within the proof.

\[\text{Proof.} \text{ See Appendix C.} \]

3 Model Properties and Implications

We want to explore in this section both theoretically and numerically some of the properties of the proposed factor structure in the dynamics of equity returns. In particular, we examine the relative importance of our short-run and long-run factor structure on the sensitivity of the equity option returns with respect to the level of the market index and to each component of the market variance, as well as on the expected returns of equity options. We close this section by exploring a number of important cross-sectional implications of our two-factor structure for equity options.

For the numerical simulations, we fix the structural parameters as follows, based on Christoffersen et al. [2009] for the market model and Christoffersen et al. [2015] for the equity model. Since we are interested in the effects of betas, we discuss these properties for different sets of betas while keeping the total unconditional risk-neutral equity variance evaluated at its mean reverting value constant, \( \bar{v}^i = (\beta_1^i)^2 \bar{\theta}_1 + (\beta_2^i)^2 \bar{\theta}_2 + \theta^i = 0.11 \). We also fix the total unconditional risk-neutral market variances to 0.05, with \( \bar{\theta}_1 = 0.006 \) and \( \bar{\theta}_2 = 0.044 \). Therefore, the unconditional idiosyncratic equity variance for every set of betas can be defined by \( \theta^i = \bar{v}^i - (\beta_1^i)^2 \bar{\theta}_1 - (\beta_2^i)^2 \bar{\theta}_2 \). The spot market variances are set equal to \( v_{1,t} = 0.012 \) and \( v_{2,t} = 0.048 \) and the total spot equity variance is \( v^i_t = 0.05 \). Consequently, we define the spot idiosyncratic variance for different sets of betas as \( \xi^i_t = v^i_t - (\beta_1^i)^2 v_{1,t} - (\beta_2^i)^2 v_{2,t} \). We choose the remaining structural parameters of the market and equity dynamics as follows: \( \{ \tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83 \} \) and \( \{ \tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0 \} \). We keep the risk-free rate as 4% per year and examine at-the-money equity options with 3 months to maturity. We test the model properties by assuming the ratio of spot index price over spot equity price equal to \( S^i_t/S_t = 0.1 \). These parameters values are chosen to highlight the advantages of our two-factor structure relative to the one-factor structure introduced in Christoffersen et al. [2015].

Our two-factor structure explicitly shows how changes in the level of the spot market index are translated into the equivalent changes in the equity option prices. Moreover, the proposed factor structure enables us to examine how equity option prices respond to the market
variances, both short-run and long-run. The following proposition establishes these relations and creates a basis for further sensitivity analysis.

**Proposition 4.** Given the closed-form equity option pricing expression in Proposition (3), the sensitivity of the prices of individual equity options, $C_i^t$, with respect to the level of the market index, $S_t$, is given by:

$$\frac{\partial C_i^t}{\partial S_t} = \frac{\partial C_i^t}{\partial S_t^i} S_t^i (\beta_1^i + \beta_2^i).$$

(14)

Moreover, since $v_i^t = (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_i^t$, the sensitivity of the price of individual equity options $C_i^t$ with respect to the each component of the market variance, $v_{1,t}$ and $v_{2,t}$, is:

$$\frac{\partial C_i^t}{\partial v_{1,t}} = \frac{\partial C_i^t}{\partial v_{1,t}^i} (\beta_1^i)^2,$$

$$\frac{\partial C_i^t}{\partial v_{2,t}} = \frac{\partial C_i^t}{\partial v_{2,t}^i} (\beta_2^i)^2.$$

(15)

**Proof.** See Appendix D.

We interpret the expression in equation (14) as the market delta for equity options and the expressions in (15) as the market vegas for equity options. Figure (1) shows the market sensitivity of the model-implied equity call option prices for the structural parameters values given above. We plot this sensitivity for different sets of betas to examine the relative importance of short-run and long-run factors on the market delta. This figure shows the substantial effect of betas on the market delta across the firms with different sets of betas. As expected, firms with higher average betas are more sensitive to the changes in the level of the market index. However, the effects of short-run and long-run betas on the market delta are mixed. Note that the top panel on the left-hand-side (LHS) provides the market delta following the calibration in one-factor model of Christoffersen et al. [2015].

[Figure (1) about here]

Figures (2) and (3) plot the sensitivity of the model-implied equity call option prices with respect to the long-run and short-run variances of the market (short-run market vega and long-run market vega) using the parameter values described above. As we expect, the higher the beta the higher the equivalent market vega. Although each beta has a dominant effect when we examine the market vega with respect to the respected variance factor, our plots show that the other beta also plays an important role in determining the market vega. This feature is more pronounced in Figure (3) where we plot the market vega with respect to the second volatility factor (short-run market vega). This figure shows how the first beta
(long-run beta) can affect the short-run market vega among different levels of moneyness. These distinctive properties of our model allow portfolio managers to better examine the exposure of their portfolios to the variations in market returns, a feature that is absent in the proposed single factor structure in Christoffersen et al. [2015].

Moreover, our two-factor structure and closed-form equity option pricing formula shed some light on the relation between the expected returns on individual equity options and the characteristics of the market returns and market variances as expressed in equation (16) below. This result allows us to disentangle the effect of the market risk premium from those of volatility premiums on the equity option returns. It also shows how equity betas play a direct role on the equity option returns. In particular, the second component in the right-hand-side (RHS) of equation (16), which is related to the market risk premium, affects the equity option returns by an adjustment factor which includes the betas and is related to the market delta of equity options. Moreover, the third component in RHS of (16), which is related to variances risk premiums, shows how equity betas affect the equity option returns through the total market vega of equity options. Note that $\partial C^i_t/\partial v_t$ measures the total market vega of equity options.

**Proposition 5.** Given the closed-form equity option pricing expression in equations (C.12)-(C.13) Appendix C, the dynamics of the market index (1), and the dynamics of the individual equity returns (11) the instantaneous expected excess returns on the individual equity call options under the physical measure can be characterized as follows.

\[
\frac{1}{dt}E^P_t\left[\frac{dC^i_t}{C^i_t} - rdt\right] = \left[\left(\mu^i - r\right)\frac{S^i_t}{C^i_t} \frac{\partial C^i_t}{\partial S^i_t}\right]
+ \left[\frac{\beta^i_1 \mu_1 v_{1,t} + \beta^i_2 \mu_2 v_{1,t} S_t}{C^i_t} + \beta^i_1 + \beta^i_2\right] \frac{\partial C^i_t}{\partial S^i_t}
+ \left[\frac{(\beta^i_1)^2 \lambda_1 v_{1,t} + (\beta^i_2)^2 \lambda_2 v_{2,t}}{(\beta^i_1)^2 + (\beta^i_2)^2} \frac{1}{C^i_t}\right] \frac{\partial C^i_t}{\partial v_t}
\]

**Proof.** See Appendix D.

Our proposed factor structure provides important cross-sectional implications for equity options. Christoffersen et al. [2015] documents that firms with higher beta have a steeper term structure of variance. However, our proposed factor structure moves further and provides a novel term structure effect. In particular, we show how the term structure of equity variance responds differently to the short-run and long-run variations in market returns. We use the same parameter values introduced at the beginning of this section and show how $\beta_1$ and $\beta_2$
have a separate and non-trivial effect on the implied volatility term structure of individual equity options. Figure (4) plots the model implied volatility for at-the-money equity call options with respect to time-to-maturity for different sets of betas. Consistent with the finding in Christoffersen et al. [2015] (the top LHS panel), the higher the average betas of equities the steeper the term structure of the implied volatility of equity options (the top RHS panel). On top of that, as Figure (4) shows, short-run and long-run betas have different contribution to the term structure of individual equity options. In particular, our model predicts that the term structure of implied volatility of equity options is more sensitive to the second beta (bottom LHS panel), which is related to the short-term variations in market returns, while the impact of the other beta on the term structure of implied volatility of equity options is marginal (the bottom right panel). Note that in all the graphs the total unconditional equity variance is fixed, \( \tilde{v}_i = (\beta_i^1)^2 \tilde{\theta}_1 + (\beta_i^2)^2 \tilde{\theta}_2 + \theta^i = 0.11 \).

Figure (5) plots the model implied volatility for 3 months at-the-money equity call options with respect to the moneyness \((S/K)\) for different sets of betas. Consistent with Christoffersen et al. [2015], reported in the top left panel, our model predicts that the higher the average betas of a firm, the steeper the implied volatility moneyness slope of equity options (the top RHS panel). More to the point, our factor structure decomposes the effect of short-run and long-run betas and predicts that the first beta (long-run) has little effect on the slope of implied volatility of equity options across moneyness (the bottom right panel) while the contribution of the second beta is significant in describing the moneyness slope of individual equity options (the bottom LHS panel). This interesting result links our findings to those of Bakshi et al. [2003] who discussed skewness of equity options. Note that in all the graphs the total unconditional equity variance is fixed \( \tilde{v}_i = (\beta_i^1)^2 \tilde{\theta}_1 + (\beta_i^2)^2 \tilde{\theta}_2 + \theta^i = 0.11 \) and the parameter values are as before.

We close this section by discussing the implication of two-factor structure on the relation between the market variances risk premiums and the equity option skew. Figure (6) plots the difference between the model implied volatility for 3 months at-the-money equity call options with respect to the moneyness \((S/K)\) for different sets of betas. The implied volatility difference is calculated when \( \lambda_1 = \lambda_2 = -0.5 \) and when \( \lambda_1 = \lambda_2 = 0 \). As we expect, the variances risk premiums have more significant effect on the implied volatility of equity options when the average betas is higher (the top RHS panel). In particular, we observe that the short-run beta has more significant role on the equity option skew (the bottom LHS panel) compare to the long-run beta (the bottom right panel). Note that for all the graphs the total unconditional equity variance is fixed \( \tilde{v}_i = (\beta_i^1)^2 \tilde{\theta}_1 + (\beta_i^2)^2 \tilde{\theta}_2 + \theta^i = 0.11 \). Note also that the top left panel shows the effect of market variance risk premium on the equity option skew (slope of IV curve) following the calibration in one-factor model of Christoffersen et al. [2015].
4 Data

We obtain daily prices of S&P 500 index options from the OptionMetrics volatility surface data set, which is based on the midpoint of bid-ask quotes. Our sample of S&P 500 index options is from January 4, 1996 through December 29, 2011. We follow the data cleaning routine commonly used in the empirical option pricing literature: we remove options with implied volatility less than 5% and greater than 150%; we also follow the filtering rules in Bakshi et al. [1997] to remove options that violate various no-arbitrage conditions. We focus on out-of-the-money (OTM) options with maturity up to and including one-year and with 10% moneyness (spot price over strike price).25,26 As in-the-money (ITM) S&P 500 call options are less liquid than OTM call options, we use OTM S&P 500 put options, which are more liquid, and convert them into ITM call options. After cleaning, we use 345,710 S&P 500 index option quotes together with daily underlying returns to filter daily spot variances and to estimate a set of structural parameters.

For individual equities, we choose all the firms listed in the Dow Jones Industrial Average index and collect the individual equities options data from OptionMetrics as well.27 We keep all options with 10% moneyness and with maturity up to and including 1 year. Similarly, due to the liquidity issues, we rely on OTM put options and then use put-call-parity to get ITM call options. Note that options on individual equities are American, the price of which could be affected by early exercise premium. To prevent any bias in the estimation of the structural parameters of equities and daily spot idiosyncratic volatility, the loss function needs to be defined based on the implied volatility as implied volatilities and deltas for the equity options reported in OptionMetrics are computed by Cox et al. [1979] binomial tree model. Otherwise, if the loss function is based on mean-squared errors of option prices, we either need to restrict our attention to equity call options that are less sensitive to early exercise premium or have to adjust the equity prices by taking into account all future dividends. Due to computational burden of this adjustment and considering the closed-from European option pricing equation in Proposition (3), we focus on equity call option.28

25 This range of moneyness implies that we keep OTM call options with moneyness less than 1.1 and OTM put options with moneyness greater than 0.9.
26 As discussed in previous section, multiple-factor SV models could better capture the slope and the level of smirk compare to single-factor SV models. Therefore, unlike similar analysis, we undertake a more extensive calibration exercise by incorporating the information content of options on longer maturity horizons and wider moneyness ranges. For instance, Ait-Sahalia and Kimmel [2007, Section 7] only include short-maturity at-the-money S&P 500 Index Options; Eraker [2004] use 3,270 call options contracts recorded over 1,006 trading days; Jones [2003] models are estimated using a sample of 3537 S&P 100 index options from January 1986 to June 2000.
27 Note that we drop the Bank of America, the Kraft Foods Incorporation, and the Travelers Companies Incorporation.
28 Bakshi et al. [2003] show that even though the implied volatility for American options is usually smaller
Table (1) presents the summary of the data sets that we used to filter daily spot market variances and daily spot idiosyncratic variance, and to estimate the structural parameters. This table contains the number of available call and put option contracts for each firm. We also report the average number of days-to-maturity for available calls and puts in our sample. In total, we have used 4,241,990 individual equity call options and 3,209,990 individual equity put options, where the average days-to-maturity of the option contracts is 135 days. On average, for every firm we use 275,999 option contracts to estimate its structural parameters and to filter its vector of daily spot idiosyncratic variance.

The data for daily equity prices and equity returns, daily index level and index returns, and for the dividend yields are from CRSP. In our analysis we first adjust daily equities prices and index level with dividend yields and then compute the option prices using the dividends adjusted returns. Risk-free interest rates for all maturities are estimated by linear interpolation between the closest zero-coupon rates of the Zero Coupon Yield Curve from OptionMetrics.

5 Estimation Methodology

To estimate the two-factor stochastic volatility model of the index we follow the literature on the estimation of stochastic volatility models, where the main challenge is the estimation of unobserved latent volatility. There are several approaches to stochastic volatility model estimation. Our own approach combines the information from underlying market and options market to impose consistency between structural parameters under $P$ and $Q$ distributions and uses likelihood functions that contain a return component and an option component, as in Santa-Clara and Yan [2010] and Christoffersen et al. [2013].

Our estimation methodology is twofold. At the market level, we do a joint-estimation by filtering the two vectors of daily spot variances, $\{v_{1,t}, v_{2,t}\}$, and then estimate a set of structural parameters of the dynamics of index returns and variances, including the market price of each variance component, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$.

At the individual equity level, we follow the two-step iterative procedure. The first step estimates the instantaneous idiosyncratic variances, $\{\xi^i_t\}$, for each firm $i$ at each date $t$, by minimizing the weighted mean squared option pricing errors on that day, given the set of structural parameters, $\Theta^i \equiv \{\kappa^i, \theta^i, \sigma^i, \rho^i, \beta^i\}$, the dynamics of market index and the filtered than the Black-Scholes implied volatility, the difference is negligible especially for OTM options. They confirm that the bias of adopting Black-Scholes implied volatility is small enough and could be ignored.

Consistency can also be imposed through moment-based and simulation-based methods; see Ait-Sahalia and Kimmel [2007], Eraker [2004], Jones [2003], Chernov and Glysels [2000], and Pan [2002]. Other approaches use only option-based data to estimate only the Q distribution; Bakshi et al. [1997], Bates [2000], Huang and Wu [2004], and Christoffersen et al. [2009].
spot variances. The second step is to estimate the structural parameters, for each firm \( i \), by minimizing the sum of the daily mean-squared option pricing errors given the filtered spot idiosyncratic variances, the dynamics of market index and the filtered spot variances. The procedure iterates between these two steps to minimize the pricing error.\(^{30}\)

### 5.1 Estimation of the Index Model

We start with the estimation of the market model. In order to have reliable prices of variance risk factors, we need a consistent set of structural parameters between the \( P \) and \( Q \) distributions. We obtain them by a two-component likelihood function, one for the index market and the other for the options market. Since the market variances are unobserved state variables, we first extract daily instantaneous variances using the Particle Filter (PF) method. This optimal filtering methodology provides a tool for learning about unobserved shocks and states from discretely observed prices generated by continuous-time models.\(^{31}\)

Although we generally follow the conventional filtration procedure in the literature, we provide a novel approach to the challenge of filtering the two separate variance paths. Our proposed solution is not trivial and to the best of our knowledge is novel and constitutes a methodological contribution to the option pricing literature.

To define the return-based likelihood function, we apply Ito’s lemma to equation (1) as follows.

\[
\begin{align*}
    &\quad d\ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t}, \\
    dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t}, \\
    dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t},
\end{align*}
\]

(17)

where, \( \mu \equiv r + \mu_1v_{1,t} + \mu_2v_{2,t} \). Equation (17) models the relation between observed index prices and unobserved variances at time \( t+1 \) conditional on the time \( t \) variances. To filter spot market variances we discretize it using the Euler scheme.\(^{32}\)

\[
\begin{align*}
    &\quad \ln(S_{t+\Delta t}) - \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))\Delta t + \sqrt{v_{1,t}}z_{1,t+\Delta t} + \sqrt{v_{2,t}}z_{2,t+\Delta t}, \\
    v_{1,t+\Delta t} &= \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}}w_{1,t+\Delta t}, \\
    v_{2,t+\Delta t} &= \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}}w_{2,t+\Delta t}.
\end{align*}
\]

(18)

Brownian shocks \( z_{1,t+\Delta t}, z_{2,t+\Delta t}, w_{1,t+\Delta t}, \) and \( w_{2,t+\Delta t} \) are normal random variables with mean zero and variance \( \Delta t \). From (18) we use the observed daily index log-prices \((\ln(S_t), \ln(S_{t+\Delta t}))\)
to first filter the daily return’s shocks \((z_{1,t+\Delta t}, z_{2,t+\Delta t})\) and then, using the filtered shocks in returns, we filter daily spot variances \((v_{1,t+\Delta t}, v_{2,t+\Delta t})\). We are able to filter the summation of return shocks and cannot separate the daily observed shocks into two components, \(z_{1,t+\Delta t}\) and \(z_{2,t+\Delta t}\). To overcome this issue we rewrite the underlying dynamics as (19), given that the return shocks are uncorrelated.

\[
\begin{align*}
    d\ln(S_t) &= (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t} + v_{2,t}}dz_t , \\
    dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} , \\
    dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} ,
\end{align*}
\]

(19)

with the correlation structure:

\[
\begin{align*}
    dw_{1,t} \cdot dz_t &= \rho_1 dt, \quad -1 \leq \rho_1 \leq +1 , \\
    dw_{2,t} \cdot dz_t &= \rho_2 dt, \quad -1 \leq \rho_2 \leq +1 , \\
    dw_{1,t} \cdot dw_{2,t} &= 0 .
\end{align*}
\]

(20)

We then decompose the variances shocks into orthogonal components in (21) and discretize the model in (19) using the Euler scheme in (22).

\[
\begin{align*}
    dw_{1,t} &= \rho_1 dz_t + \sqrt{1 - \rho_1^2} dB_{1,t} , \\
    dw_{2,t} &= \rho_2 dz_t - \frac{\rho_1\rho_2}{\sqrt{1 - \rho_1^2}} dB_{2,t} + \sqrt{1 - \rho_1^2 - \rho_2^2} dB_{2,t} + \sqrt{1 - \rho_1^2} dB_{1,t} , \\
    \langle dB_{1,t} , dB_{2,t} \rangle &= 0 .
\end{align*}
\]

(21)

\[
\begin{align*}
    \ln(S_{t+\Delta t}) - \ln(S_t) &= (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))\Delta t + \sqrt{v_{1,t} + v_{2,t}}z_{t+\Delta t} , \\
    v_{1,t+\Delta t} &= \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}}w_{1,t+\Delta t} , \\
    v_{2,t+\Delta t} &= \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}}w_{2,t+\Delta t} .
\end{align*}
\]

(22)

where, \(z_{t+\Delta t}\), \(w_{1,t+\Delta t}\), and \(w_{2,t+\Delta t}\) are all \(N(0, \Delta t)\). Now, using daily index log-returns, we proceed to filter the spot variances from the discretized model in (22) given the correlation structure in (21).

We follow Pitt [2002] and adopt a particular implementation of the PF, which is referred to as the sampling-importance-resampling (SIR) PF. Through this PF method, we approximate the true density of the variances \(v_{1,t+\Delta t}\) and \(v_{2,t+\Delta t}\) using two sets of particles that are updated through equations (22). Given two sets of particles, \(N\) particles of \(\{v_{1,t}^j\}_{j=1}^N\) and \(N\) particles of \(\{v_{2,t}^j\}_{j=1}^N\), the empirical distributions of \(v_{1,t}\) and \(v_{2,t}\), and the simulated return shocks, we

---

33 Note that in (21) the quadratic variations of the transformed process should remain the same as \(\sqrt{\Delta t}\).

34 See Pitt [2002], Christoffersen et al. [2010], and Boloorforoosh [2014] for a detailed description of the PF algorithm.
use the empirical distributions of \(v_{1,t+\Delta t}\) and \(v_{2,t+\Delta t}\) and generate two sets of particles, \(N\) particles \(\{v_{1,t+\Delta t}^j\}_{j=1}^N\) and another \(N\) particles \(\{v_{2,t+\Delta t}^j\}_{j=1}^N\) at any time \(t+\Delta t\).

Given the initial value of structural parameters and current particles \(\{v_{1,t}^j, v_{2,t}^j\}\), on every day \(t\) and for every particle \(j = 1, 2, ..., N\), we simulate return shocks according to (23) and then simulate the volatility shocks according to (24). Note that \(\epsilon_{1,t+\Delta t}^j\) and \(\epsilon_{2,t+\Delta t}^j\) are independent normal random variables with mean zero and variance \(\Delta t\).

\[ z_{t+\Delta t}^j = \left[ \ln(S_{t+\Delta t}/S_t) - (\mu - \frac{1}{2}(v_{1,t}^j + v_{2,t}^j))\Delta t \right] / \sqrt{v_{1,t}^j + v_{2,t}^j} \quad (23) \]

\[ w_{1,t+\Delta t}^j = \rho_1 z_{t+\Delta t}^j + \sqrt{1 - \rho_1^2} \epsilon_{1,t+\Delta t}^j \]

\[ w_{2,t+\Delta t}^j = \rho_2 z_{t+\Delta t}^j - \frac{\rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} \epsilon_{1,t+\Delta t}^j + \frac{1 - \rho_1^2 - \rho_2^2}{\sqrt{1 - \rho_1^2}} \epsilon_{2,t+\Delta t}^j \quad (24) \]

Then, using the simulated return’s shocks, \(\{z_{t+\Delta t}^j\}_{j=1}^N\), and simulated shocks to variances, \(\{w_{1,t+\Delta t}^j\}_{j=1}^N\) and \(\{w_{2,t+\Delta t}^j\}_{j=1}^N\), for every day \(t\), we calculate two new sets of particles, \(\{\bar{v}_{1,t+\Delta t}^j\}\) and \(\{\bar{v}_{2,t+\Delta t}^j\}\), according to (25).

\[ \bar{v}_{1,t+\Delta t}^j = v_{1,t}^j + \kappa_1 (\theta_1 - v_{1,t}) \Delta t + \sigma_1 \sqrt{v_{1,t} \omega_{1,t+\Delta t}} \]

\[ \bar{v}_{2,t+\Delta t}^j = v_{2,t}^j + \kappa_2 (\theta_2 - v_{2,t}) \Delta t + \sigma_2 \sqrt{v_{2,t} \omega_{2,t+\Delta t}} \quad (25) \]

In this sampling step we generate \(N\) possible daily values for the first variance component, \(v_{1,t+\Delta t}\), and another \(N\) possible daily values for the second variance component, \(v_{2,t+\Delta t}\). In the next step, we evaluate the importance of the sampled particles by assigning appropriate weights to the simulated daily particles, using a multivariate normal distribution. Intuitively, these weights, \(\tilde{W}_{t+\Delta t}^j\), are the likelihood that the next day return at \(t + 2\Delta t\) is generated by this set of particles. Then, the probability of each particle can be defined by normalizing the weights within each day.

\[ (r_{t+2\Delta t} | \{\bar{v}_{1,t+\Delta t}, \bar{v}_{2,t+\Delta t}\}) \sim N[ (\mu - \frac{1}{2}(\bar{v}_{1,t+\Delta t} + \bar{v}_{2,t+\Delta t})) \Delta t, (\bar{v}_{1,t+\Delta t} + \bar{v}_{2,t+\Delta t}) \Delta t ] \]

\[ \tilde{W}_{t+\Delta t}^j = \frac{1}{\sqrt{2\pi(\bar{v}_{1,t+\Delta t}^j + \bar{v}_{2,t+\Delta t}^j)\Delta t}} \cdot \exp \left( -\frac{1}{2} \left( \ln(S_{t+\Delta t}^j) - (\mu - \frac{1}{2}(\bar{v}_{1,t+\Delta t}^j + \bar{v}_{2,t+\Delta t}^j))\Delta t \right)^2 \right) \quad (26) \]

\[ \tilde{W}_{t+\Delta t}^j = \frac{\tilde{W}_{t+\Delta t}^j}{\sum_{j=1}^N \tilde{W}_{t+\Delta t}^j} \quad (27) \]
Note that the simplification in (19) imposes a restriction on the weights of daily particles. Therefore, we assign the importance probability to the summation of return’s shocks. However, the estimation results show that the path of filtered spot variances of the index (short-run spot variances and long-run spot variances) in multiple-factor SV model is not sensitive to this assumption as the filtered paths with particle filter under the $Q$ measure move closely with the spot variances estimated with two-step iterative procedure under the $Q$ measure. Note that the particle’s weights in (27) are the basis of our likelihood function under the $P$ distribution.

In the last step, resampling, we find the empirical distribution of smoothly resampled particles. Following the Pitt [2002] algorithm, we draw smoothed particles by assigning uniform distributions to the raw particles for each variance component. We recursively use these smoothly resampled particles to simulate the next period particles until we have the empirical distributions of each variance component over the entire sample.

Using the weights in equation (27), we define the return-based likelihood function as follows.

$$\ln L^R \propto \sum_{t=1}^{T} \ln \left( \frac{1}{N} \sum_{j=1}^{N} \tilde{W}_j^t(\Theta) \right)$$

(28)

Note that this return-based likelihood function is a function of the structural parameters of the market model under $P$ measure, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_1, \rho_1, \rho_2\}$. Note also that the filtered daily spot variances $v_{1,t}^P$ and $v_{2,t}^P$ can be defined as the average of the smoothly resampled particles.

$$\hat{v}_{1,t}^P = \frac{1}{N} \sum_{j=1}^{N} v_{1,t}^j, \quad \hat{v}_{2,t}^P = \frac{1}{N} \sum_{j=1}^{N} v_{2,t}^j$$

(29)

Next, we define the option-based likelihood function. In order to fully specify the market dynamics under the $Q$ measure, we need to estimate a set of structural parameters for the market model under $Q$ measure, $\hat{\Theta} \equiv \{\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\sigma}_1, \tilde{\sigma}_1, \tilde{\rho}_1, \tilde{\rho}_2, \lambda_1, \lambda_2\}$, and two vectors of daily spot variances. For the vectors of daily spot variances, we filter the unobserved spot variances under the $Q$ measure using the PF method. We follow the same procedure as described in (23)-(27) for the market variances under $P$ measure, but this time we use the set of structural parameters under $Q$, $\{\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\sigma}_1, \tilde{\sigma}_1, \tilde{\rho}_1, \tilde{\rho}_2, \lambda_1, \lambda_2\}$. Note that $\tilde{\kappa}_i = \kappa_i + \lambda_i$ and $\tilde{\theta}_i = \frac{\kappa_i \theta_i}{\kappa_i + \lambda_i}$ for $i = \{1, 2\}$. Then, we obtain daily spot market variances under $Q$ measure as the average of the smoothly re-sampled particles for each component of market variance.

$$\hat{v}_{1,t}^Q = \frac{1}{N} \sum_{j=1}^{N} v_{1,t}^j, \quad \hat{v}_{2,t}^Q = \frac{1}{N} \sum_{j=1}^{N} v_{2,t}^j$$

(30)
We define the option-based likelihood function using a vega weighted loss function for the index options. The vega weighted option pricing errors serve as an approximation to the implied volatility errors, and since they do not require a numerical inversion of the Black and Scholes [1973] model, they are very helpful in large scale optimization problems such as ours.

\[ \eta_n = (C_n^O - C_n^M(\tilde{\Theta}, \tilde{\nu}_1^Q, \tilde{\nu}_2^Q, S_t, K, \tau))/Vega_n , \quad n = 1, \ldots, M \]  

(31)

In this options-based likelihood function, \( C_n^O \) is the observed price and \( C_n^M(\tilde{\Theta}, \tilde{\nu}_1^Q, \tilde{\nu}_2^Q, S_t, K, \tau) \) is the model price of index option \( n \), which is defined in (6). \( M \) is the total number of index options. \( Vega_n \) is the Black and Scholes [1973] option vega. Assuming that these disturbances are i.i.d. normal, the option-based likelihood can be obtained as follows.

\[ \ln L^O \propto -\frac{1}{2} \left( M \ln(2\pi) + \sum_{n=1}^{M} \left( \ln(s^2) + \frac{\eta_n^2}{s^2} \right) \right), \]  

(32)

Last, we combine the returns-based likelihood function (28) and the options-based likelihood function (32) to estimate the structural parameters of the market model \( \hat{\Theta} \) and \( \tilde{\Theta} \) as the solution to the following optimization problem:

\[ \max_{\Theta, \tilde{\Theta}} \left( \ln L^R + \ln L^O \right). \]  

(33)

### 5.2 Estimation of the Equities Model

We estimate a set of structural parameters, \( \tilde{\Theta}^i \equiv \{\mu^i, \kappa^i, \theta^i, \sigma^i, \rho^i, \lambda^i, \beta_1^i, \beta_2^i\} \), and a vector of daily spot idiosyncratic variances, \( \{\xi_t^i\} \) for each individual equity in our sample by the two-step iterative approach of Bates [2000] and Huang and Wu [2004]. In the first step, given a set of initial structural parameters for each equity, \( \tilde{\Theta}_0^i \), we estimate a vector of daily spot idiosyncratic variance conditional on a set of risk-neutral structural parameters of the market model, \( \hat{\Theta} \), and the filtered daily risk-neutral spot variances, \( \{\hat{\nu}_1^{Q,t}, \hat{\nu}_2^{Q,t}\} \). We define the loss function using a vega-weighted loss function, similar to the one we used in the estimation of the market model. Therefore, for every individual equity, the set of daily spot idiosyncratic variances, \( \hat{\xi}_t^i \), can be obtained as the solution to the following optimization problem.

\[ \hat{\xi}_t^i = \arg \min_{\xi_t^i} \sum_{n=1}^{M_t^i} \frac{(C_{n,t}^i - C_{n,t}^{i,M}(\Hat{\Theta}_0^i, \tilde{\nu}_1^{Q,t}, \tilde{\nu}_2^{Q,t}, \xi_t^i))^2}{(Vega_{n,t}^i)^2}, \quad t = 1, \ldots, T, \]  

(34)

35See for example Carr and Wu [2007] and Christoffersen et al. [2009].

36Note that we replace \( s^2 \) by its sample analog \( \hat{s}^2 = \frac{1}{M} \sum_{n=1}^{M} \eta_n^2 \).
where $M^i_t$ is the total number of available option contracts for the equity $i$ on day $t$, $C^{i,O}_{n,t}$ is the observed price of equity option $n$ for stock $i$ on day $t$, $C^{i,M}_{n,t}$ is the model price for the same option, and $Vega^i_{n,t}$ is the Black-Scholes option vega for the same option contract.

For every individual equity we repeat the optimization in (34) every day to estimate a vector of spot idiosyncratic variances, $\tilde{\xi}^i_t$. For a set of spot idiosyncratic variances obtained in the first step, we then solve the following optimization problem over the entire sample to characterize a set of structural parameters, $\tilde{\Theta}^i$, under the risk neutral measure for every individual equity.

$$\hat{\tilde{\Theta}}^i = \arg \min_{\xi^i_t} \sum_{n=1}^{M^i_t} (C^{i,O}_{n,t} - C^{i,M}_{n,t}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}^i, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \hat{\xi}^i_t))^2 / (Vega^i_{n,t})^2 ,$$

(35)

where $M^i_t \equiv \sum_{t=1}^{T} M^i_t$ is the total number of available option contracts for equity $i$. For every equity, the procedure iterates between the optimizations in (34) and (35) until the change in the RMSE of the estimation in the second step will no longer be significant. Note that in every new iteration we use the estimation results in (35) as an initial set of the structural parameters, $\tilde{\Theta}^i_0 = \hat{\tilde{\Theta}}^i$.

6 Parameter Estimation Results

This section reports the filtered daily spot variances and structural parameters estimates for the S&P 500 Index. The index data set spans the period from January 4, 1996 to December 29, 2011. We use information from both returns and options markets. We set the market risk premium, $\mu$, equal to the sample average daily return and use 10% OTM index options. We use put-call-parity to convert OTM puts into ITM call options. To provide a basis for further comparison and to examine the model fit under the joint-estimation, we also report the structural parameters of the market model, estimated only from option data.

We also report a set of structural parameters and a vector of daily spot idiosyncratic variances for 27 firms from the DJIA Index. These results are conditional on the spot market variances and structural parameters of the market model. The equities data sets are from June 1, 1996 to November 30, 2011. Note that we drop the first 5 months of each equity’s data set to prevent any estimation bias, as the filtered spot market variances are too noisy in the first couple of months of the estimation period. Note also that S&P 500 Index options are European while the individual equity options are American, the price of which could be affected by early exercise premium. To reduce the bias in the calculation of equity option prices using the closed-form pricing equation in Proposition (3) we focus on OTM
options.\textsuperscript{37,38}

Table (2) reports the structural parameters estimates that characterize the dynamics of the market returns and the market variances. Panel A and Panel B present the structural parameters of the S&P 500 index from joint estimation. As we put a joint restriction on the $P$ and $Q$ parameters, the speeds of mean reversion and the unconditional mean of the market variances under $P$ and $Q$ measures are linked through the market prices of the volatility risk factors, $\lambda_1$ and $\lambda_2$ (see Proposition (1)). Panel C presents the estimates of the structural parameters using only option prices. We report this result for the sake of comparison with previous studies.

[Table (2) about here]

As discussed before, the purpose of two-factor stochastic volatility model is to capture independent movements in the underlying returns and option prices. Consistent with previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility factors is highly persistent and the other one is highly mean-reverting. In our joint-estimation, we find that the first volatility component is slowly mean-reverting with $\kappa_1 = 1.4271$. This value corresponds to a daily variance persistence of $1 - 1.4271/365 = 0.9961$ which is very high and confirms that this factor is highly persistent. We observe that the point estimate for the rate of mean reversion in the second volatility component is higher and equal to $\kappa_2 = 3.5874$. This value corresponds to a daily variance persistence of $1 - 3.5874/365 = 0.9901$ and confirms that the second volatility component is highly auto-correlated.\textsuperscript{39} These results confirm that the long-run (trend) volatility component is more persistent than the short-run component, where the immediate impact of volatility shocks on the short-run volatility component is larger but short-lived.

We observe that the unconditional market variance of the first volatility component (the persistent volatility factor) is $\theta_1 = 0.0026$, which is much less than the unconditional market variance of the second volatility component, $\theta_2 = 0.0171$. These unconditional market variances are consistent with the average filtered spot index variances. Consistent with our intuition, we observe a big difference between the volatility of volatility in the first and second variance components. We find negative prices for both variance components, namely $\lambda_1 = -1.0798$ and $\lambda_2 = -1.0355$. To the best of our knowledge none of the previous studies\textsuperscript{37,38}

\textsuperscript{37}Bakshi et al. [2003] show that for S&P 100 OTM American options the early exercise premium is negligible. They estimate two separate implied volatilities: the implied volatility that equates the option price to the American price, and the implied volatility that equates the option price to the Black-Scholes price where the discounted dividends are subtracted from the spot price. They show that although the American option implied volatility is smaller than its Black-Scholes counterparts, the difference is negligible and within the bid-ask spread.

\textsuperscript{38}Christoffersen et al. [2015] show that the early exercise premium is negligible for call options. We also estimate the equity model just by incorporating the equity call options and find that the structural parameters are similar to the case where we used OTM put and call options. This result is available from the author upon request.

\textsuperscript{39}Note that these parameter values are under the $P$ measure.
of two-factor stochastic volatility models reports the prices of the variances risk factors as they either focused on the options market data or the underlying returns data. The negative prices for both variance factors are consistent with asset pricing studies where the short-run and the long-run volatility components are priced cross-sectional asset pricing factors. Adrian and Rosenberg [2008] find that prices of risk of both short-run and long-run volatility components are negative and highly significant, which implies that investors are willing to pay for an insurance against an increase in volatility risk, even if that increase has little persistence.

Our joint estimation results show that the correlation between the shocks to the market returns and the shocks to the first component of the market variance is $\rho_1 = -0.6918$. The correlation between the shocks to the market returns and the shocks to the second component of the market variance is $\rho_2 = -0.2173$.\(^{40}\) The average correlations in Christoffersen et al. [2009, Table 3] are $\rho_1 = -0.96$ for the long-run component and $\rho_2 = -0.83$ for the short-run volatility component.\(^{41}\) Bates [2000] also reports the structural parameters estimates of a two-factor SV model using 1988-1993 S&P 500 futures option prices. He obtains one set of structural parameters over the entire sample where $\rho_1 = -0.78$ and $\rho_2 = -0.38$. To provide a basis for comparison, we also report one set of structural parameters, estimated only from the option prices over the same sample period, where we find $\rho_1 = -0.91$ and $\rho_2 = -0.49$. In all these studies the point estimates for the correlations are negative for both short-run and long-run factors. By contrast, Chernov et al. [2003] discuss that when a second volatility factor is allowed to have its own correlation with returns, the correlation parameters can take on both positive and negative values, contrary to the findings in single factor volatility models, where the correlation parameter is always found to be negative. They estimate the structural parameters in a two-factor SV model using the Dow Jones daily returns over the period of 1953-1999 and report $\rho_1 = -0.41$ and $\rho_2 = 0.9$. There are potential explanations for differences between the reported estimates of the correlation coefficients in these studies and our own study, not in the least the very different data set and the very different time span. However, in all these studies, we observe that the short-run factor has lower (in absolute value) level of correlation compared to the long-run (highly persistence) factor. In other words, the continuous time leverage (asymmetry) effect is less significant in the transitory component of the market volatility and so has mainly a temporary effect.

Broadie et al. [2007] refer to the inconsistency between the option-based estimates of certain structural parameters in SV model and the parameters estimates from underlying time-series of returns and indicate that the SV model is basically misspecified. In particular, they state that the point estimates of the correlation coefficient and volatility of volatility are incompatible under the $P$ and $Q$ measures. They also show that the joint restrictions on the returns and volatility dynamics under the $P$ and $Q$ measures lead to the poor performance of the stochastic volatility model. They measure this poor performance by the high level of

\(^{40}\)These two correlations, $\rho_1$ and $\rho_2$, can be seen as the short-run and long-run continuous time leverage (asymmetry) effect that derive the skewness and kurtosis of the returns dynamics.

\(^{41}\)Christoffersen et al. [2009] use data on European S&P 500 call options quotes over the period 1990-2004. Note that they estimate a separate set of structural parameters for every year in their sample.
RMSE of the SV model and indicate that due to the joint-restriction, a stochastic volatility model cannot generate sufficient amounts of conditional skewness and kurtosis. This drawback in standard one factor SV models is mainly attributed to the fact that the estimated conditional higher moments are highly correlated with the estimated conditional variance. By contrast, as discussed in Christoffersen et al. [2009, Section 3.1], a two-factor stochastic volatility model can generate stochastic correlation between volatility and stock returns. We show that this feature enables the two-factor model to better capture the conditional skewness and kurtosis and significantly reduces the Vega-weighted RMSE of option pricing when we impose the joint-restrictions.\footnote{Using the option prices only, Christoffersen et al. [2009, Section 3.1] show that under two-factor SV model improves on the benchmark SV model both in-sample and out-of-sample.} We observe that in our two-factor SV model, the Vega-weighted RMSE is 2.08\% for the joint-estimation and 1.86\% for option-based estimation. These option pricing errors are calculated by the following expression (36) and confirm the superior performance of two-factor model.

\[
\text{Vega RMSE} \equiv \sqrt{\frac{1}{N} \sum_{n,t} (C_{n,t}^O - C_{n,t}^M(\hat{\Theta}, \hat{v}_{Q1,t}, \hat{v}_{Q2,t}))^2 / (\text{Vega}_{n,t})^2},
\]

where, \(C_{n,t}^O\) is the observed price of index option \(n\) on day \(t\), \(C_{n,t}^M\) is the model price for the same index option and on the same day, and \(\text{Vega}_{n,t}\) is the Black-Scholes option Vega for the same option contract and on the same day.

Table (3) reports the structural parameter estimates that characterize the dynamics of the individual equity returns and spot idiosyncratic variance under the \(Q\) measure. This table also reports the estimates of individual equity betas for 27 firms in our sample. Note that these parameter values are conditional on the structural parameters of the market index, \(\hat{\Theta}\), and two vectors of the market sport variances, \(\hat{v}_{Q1,t}\) and \(\hat{v}_{Q2,t}\). The speed of mean reversion for risk-neutral idiosyncratic variance range from \(\kappa_i = 0.3920\) for Coca Cola to \(\kappa_i = 1.7078\) for 3M. This range of \(\kappa_i\) implies that most of the firms in our sample have highly persistent idiosyncratic variance.

The unconditional risk neutral idiosyncratic variance in our sample starts from \(\theta_i = 0.0093\) for General Electric and increases up to \(\theta_i = 0.0756\) for Hewlett-Packard. The point estimates for the volatility of the idiosyncratic variance ranges from \(\sigma_i = 0.0670\) for General Electric to \(\sigma_i = 0.3967\) for Hewlett-Packard. For all the firms in our sample, the average point estimates for the volatility of the idiosyncratic variance is 0.1823. The correlation between shocks to equity returns and shocks to the idiosyncratic variance, capturing the continuous time leverage (asymmetry) effect, range from \(\rho_i = -0.99\) for JP Morgan to \(\rho_i = 0.512\) for Verizon.

The betas estimates seem reasonable, however, to the best of our knowledge, this is the first paper that discusses two-betas for equity options and so there is no benchmark. The first beta ranges from \(\beta_1^i = 0.3430\) for American Express to \(\beta_1^i = 0.6798\) for IBM. The second beta
starts from $\beta_1^2 = 1.0125$ for Procter & Gamble and increases to $\beta_1^2 = 1.3466$ for JP Morgan. As we discussed in Section 3, the proposed two-factor structure has important implications for the delta, vegas, and expected returns of equity options. We also show how this two-factor structure affects the term structure of the model-implied volatility of equity options and the moneyness slopes of the model implied volatility of equity options. We are going to analyze the impact of the proposed factor structure on the spot volatility of equity options. We also want to extend our results and estimate the conditional equity betas by fixing the structural parameters in the market model and equity model and estimating the daily conditional betas and daily spot idiosyncratic variances.

We close this section by presenting the distributional properties of the filtered spot idiosyncratic variance for 27 firms in our sample. Table (4) reports the mean, median, standard deviation, and the maximum of the filtered spot idiosyncratic variances for every firm conditional on the structural parameters of the S&P 500 index and the two vectors of filtered market spot variances.

7 Concluding Remarks

In this paper we investigate a two-factor stochastic volatility model where the volatility factors capture the long- and short-term variations in market returns. We start with an appropriate change of measure or an admissible pricing kernel that links the proposed market dynamics under $P$ and $Q$ measures. As the proposed two-factor specification is affine, we obtain a closed-from pricing expression for European call options. Then, we jointly estimate a set of structural parameters and filter two vectors of the instantaneous market variances using the information contents of the S&P 500 index returns and index option prices. We show that the proposed decomposition of volatility can be characterized by different sensitivity of the variance components to the volatility shocks and different persistence in variance components. Consistent with the previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility factors is highly persistent (long-run) and the other one is highly mean-reverting (short-run), where the immediate impact of volatility shocks on the short-run volatility component is bigger but short-lived. We obtain negative prices for both volatility risk factors. These negative prices are consistent with the findings in asset pricing studies where the short-run and the long-run volatility components are priced factors. The negative prices of both volatility risk factors imply that investors are willing to pay for insurance against an increase in volatility risk, even if such increases have little persistence. We obtain negative correlations between the shocks to the market returns and the shocks to the short- and long-run components of the market variance. In particular, we observe that the short-run correlation is smaller than the long-run correlation. In other words, the continuous time leverage (asymmetry) effect
is less significant in the transitory component of the market volatility and so is mainly a temporary behavior.

Motivated by the principal component analysis of equity options on the firms listed on the Dow Jones Industrial Average index, we extend the model in Christoffersen et al. [2015] and assume that individual equity returns are related to the market index with two distinct systematic components, as well as an idiosyncratic component, which is stochastic and follows the standard square root process. Therefore, the equity option prices are related to two distinct betas, with one of them capturing the short-run variations in market returns and the other one capturing its long-run counterpart. We obtain a closed-form option pricing equation for individual equity options and we estimate a set of structural parameters and filter a vector of instantaneous idiosyncratic variance that together characterize the dynamics of the individual equity under the risk-neutral measure. We also report the estimates of individual equity betas for 27 firms in our sample.

Our equity option pricing model describes the impact of these two betas on the price, sensitivity, and expected returns of individual equity options and, indeed, supports a two factor structure in equity option returns. We also provide tools that allow a portfolio manager to hedge her portfolio exposure to the level of market index (market delta), and to the short- and long-run variations in market index (short- and long-run market vegas).

Last but not least, the proposed two factor structure has important cross-sectional implications for equity options. Consistent with the findings of Duan and Wei [2009], our model confirms that firms with higher average betas have higher implied volatility, a steeper term structure of implied volatility, and a steeper implied volatility moneyness slopes. We also observe that the variance risk premium has a more significant effect on the implied volatility of equity options when the beta is higher. In particular, our model predicts two novel effects. First, the term structure of implied volatility is more sensitive to the short-term beta while the impact of the other beta on the term structure of implied volatility is marginal. Second, the beta related to the long-run market volatility has little effect on the moneyness slope of implied volatility of equity options, while the contribution of the other beta is significant in describing that slope.
Appendix

A Proof of Proposition 1

We impose the condition that the product of any traded asset and the pricing kernel under physical measure is a martingale. We also impose the condition that the discounted price of any traded asset under risk neutral measure is also a martingale. We show that the two-factor stochastic volatility process under physical measure in (1) are linked to its risk-neutral counterpart in (4) by the unique arbitrage free pricing kernel introduced in (5) and deduce restrictions on the time-preference parameters, $\{\delta, \eta_1, \eta_2\}$, risk-aversion (equity aversion) parameter, $\phi$, and variance preference parameters (variance aversion), $\{\zeta_1, \zeta_2\}$. We close this proof by showing how physical Wiener processes $\{z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}\}$ are linked to risk neutral Wiener processes $\{\tilde{z}_{1,t}, \tilde{z}_{2,t}, \tilde{w}_{1,t}, \tilde{w}_{2,t}\}$ by equity premium $\{\mu_1, \mu_2\}$ and variance premium $\{\lambda_1, \lambda_2\}$ parameters.

Consider that index return under physical and risk-neutral measures follows the dynamics (A.1) and (A.2).

\[
\begin{align*}
    dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt + \sqrt{v_{1,t}} dz_{1,t} + \sqrt{v_{2,t}} dz_{2,t} \\
    dv_{1,t} &= \kappa_1 (\theta_1 - v_{1,t}) dt + \sigma_1 \sqrt{v_{1,t}} (\rho_1 dz_{1,t} + \sqrt{1 - \rho_1^2} dW_{1,t}) \\
    dv_{2,t} &= \kappa_2 (\theta_2 - v_{2,t}) dt + \sigma_2 \sqrt{v_{2,t}} (\rho_2 dz_{2,t} + \sqrt{1 - \rho_2^2} dW_{2,t}) \\
    dS_t/S_t &= r dt + \sqrt{v_{1,t}} d\tilde{z}_{1,t} + \sqrt{v_{2,t}} d\tilde{z}_{2,t} \\
    dv_{1,t} &= \tilde{\kappa}_1 (\tilde{\theta}_1 - v_{1,t}) dt + \sigma_1 \sqrt{v_{1,t}} (\rho_1 d\tilde{z}_{1,t} + \sqrt{1 - \rho_1^2} d\tilde{W}_{1,t}) \\
    dv_{2,t} &= \tilde{\kappa}_2 (\tilde{\theta}_2 - v_{2,t}) dt + \sigma_2 \sqrt{v_{2,t}} (\rho_2 d\tilde{z}_{2,t} + \sqrt{1 - \rho_2^2} d\tilde{W}_{2,t})
\end{align*}
\] (A.1) (A.2)

Then, following Christoffersen et al. [2013], we show that the pricing kernel links the physical and risk neutral measures has the following exponential affine form.

\[
\frac{M_t}{M_0} = \left(\frac{S_t}{S_0}\right)^{\phi} \exp \left[\delta t + \eta_1 \int_0^t v_{1,s} ds + \eta_2 \int_0^t v_{2,s} ds + \zeta_1 (v_{1,t} - v_{1,0}) + \zeta_2 (v_{2,t} - v_{2,0})\right] \tag{A.3}
\]

In the sprite of Cox et al. [1985] and Heston [1993] we assume that the market price of each variance risk factor is proportional to spot variance. Therefore, the risk neutral process in (A.2) can be defined as follows.
Replacing (A.5) and (A.1) into (A.6) we have:

\[
dS_t / S_t = rd t + \sqrt{v_{1,t}} \, d\tilde{z}_{1,t} + \sqrt{v_{2,t}} \, d\tilde{z}_{2,t}
\]

\[
dv_{1,t} = (\kappa_1 (\theta_1 - v_{1,t}) - \lambda_1 v_{1,t}) \, dt + \sigma_1 \sqrt{v_{1,t}} \, d\tilde{w}_{1,t}
\]

\[
dv_{2,t} = (\kappa_2 (\theta_2 - v_{2,t}) - \lambda_2 v_{2,t}) \, dt + \sigma_2 \sqrt{v_{2,t}} \, d\tilde{w}_{2,t}
\]

(A.4)

The log stock price process and log pricing kernel process have the following dynamics respectively.

\[
d(\log(S_t)) = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2} v_{1,t} - \frac{1}{2} v_{2,t}) \, dt + \sqrt{v_{1,t}} \, dz_{1,t} + \sqrt{v_{2,t}} \, dz_{2,t}
\]

(A.5)

\[
d(\log(M_t)) = \phi \cdot d(\log(S_t)) + (\delta + \eta_1 v_{1,t} + \eta_2 v_{2,t}) \, dt + \zeta_1 dv_{1,t} + \zeta_2 dv_{2,t}
\]

(A.6)

Replacing (A.5) and (A.1) into (A.6) we have:

\[
d(\log(M_t)) = \left[ \phi (r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2} v_{1,t} - \frac{1}{2} v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \right. \\
+ \left. \zeta_1 \kappa_1 (\theta_1 - v_{1,t}) + \zeta_2 \kappa_2 (\theta_2 - v_{2,t}) \right] dt \\
+ \left[ \phi \sqrt{v_{1,t}} + \zeta_1 \rho_1 \sigma_1 \sqrt{v_{1,t}} \right] dz_{1,t} + \left[ \phi \sqrt{v_{2,t}} + \zeta_2 \rho_2 \sigma_2 \sqrt{v_{2,t}} \right] dz_{2,t} \\
+ \left[ \zeta_1 \sigma_1 \sqrt{v_{1,t}} \sqrt{1 - \rho_1^2} \right] dB_{1,t} + \left[ \zeta_2 \sigma_2 \sqrt{v_{2,t}} \sqrt{1 - \rho_2^2} \right] dB_{2,t}.
\]

(A.7)

As \(dM_t / M_t = d(\log(M_t)) + \frac{1}{2} [d(\log(M_t))]^2\) we have

\[
dM_t / M_t = \left[ \phi (r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2} v_{1,t} - \frac{1}{2} v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \right. \\
+ \left. \zeta_1 \kappa_1 (\theta_1 - v_{1,t}) + \zeta_2 \kappa_2 (\theta_2 - v_{2,t}) + \frac{1}{2} \phi^2 (v_{1,t} + v_{2,t}) \right] \\
+ \left[ \phi \sqrt{v_{1,t}} + \zeta_1 \rho_1 \sigma_1 \sqrt{v_{1,t}} \right] dz_{1,t} + \left[ \phi \sqrt{v_{2,t}} + \zeta_2 \rho_2 \sigma_2 \sqrt{v_{2,t}} \right] dz_{2,t} \\
+ \left[ \zeta_1 \sigma_1 \sqrt{v_{1,t}} \sqrt{1 - \rho_1^2} \right] dB_{1,t} + \left[ \zeta_2 \sigma_2 \sqrt{v_{2,t}} \sqrt{1 - \rho_2^2} \right] dB_{2,t}.
\]

(A.8)

The first restriction on the pricing kernel is that the product of the money market account, \(B_t = B_0 \exp(rt)\), and the pricing kernel, \(M_t\), should be a martingale. Therefore, \(E[d(B_t \cdot M_t)] = 0\) or \(E[dM_t / M_t] = -rdt\).  

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\[ \phi(r + \mu_1v_{1,t} + \mu_2v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1v_{1,t} + \eta_2v_{2,t} + \zeta_1\kappa_1(\theta_1 - v_{1,t}) + \zeta_2\kappa_2(\theta_2 - v_{2,t}) \\
+ \frac{1}{2}\phi^2(v_{1,t} + v_{2,t}) + \phi(\zeta_1\rho_1\sigma_1v_{1,t} + \zeta_2\rho_2\sigma_2v_{2,t}) + \frac{1}{2}\zeta_1^2\sigma_1^2v_{1,t} + \frac{1}{2}\zeta_2^2\sigma_2^2v_{2,t}] dt = -r dt \]  
(A.9)

As (A.9) holds for \( v_{1,t} = v_{2,t} = 0 \),

\[ \delta = -r(\phi + 1) - \zeta_1\kappa_1\theta_1 - \zeta_2\kappa_2\theta_2. \]  
(A.10)

(A.9) also holds for \( v_{1,t} = v_{2,t} = \infty \).

\[ \eta_1 = -\phi\mu_1 + 1/2\phi + \zeta_1\kappa_1 - 1/2(\phi^2 + \zeta_1^2\sigma_1^2 + 2\phi\zeta_1\sigma_1\rho_1) \]  
\[ \eta_2 = -\phi\mu_2 + 1/2\phi + \zeta_2\kappa_2 - 1/2(\phi^2 + \zeta_2^2\sigma_2^2 + 2\phi\zeta_2\sigma_2\rho_2) \]  
(A.11)

The second restriction on the pricing kernel is based on the fact that \([S_t, M_t]\) is also a martingale. Therefore, \( E[d(S_t \cdot M_t)] = 0 \). As a result of this restriction we have

\[ v_{1,t}(\mu_1 + \phi + \zeta_1\sigma_1\rho_1) + v_{2,t}(\mu_2 + \phi + \zeta_2\sigma_2\rho_2) = 0, \]  
\[ \phi = \frac{-1}{v_{1,t} + v_{2,t}}[(\mu_1 + \zeta_1\sigma_1\rho_1)v_{1,t} + (\mu_2 + \zeta_2\sigma_2\rho_2)v_{2,t}]. \]  
(A.12)

If we impose the restriction that \( \mu_1 + \zeta_1\sigma_1\rho_1 \equiv \mu_2 + \zeta_2\sigma_2\rho_2 \), then (A.12) can be simplified as follows.

\[ \phi = -(\mu_1 + \zeta_1\sigma_1\rho_1) = -(\mu_2 + \zeta_2\sigma_2\rho_2) \]  
(A.13)

We impose the third restriction on pricing kernel so that for any asset \( U \equiv U(S, v_1, v_2, t) \), \([U(t), M_t]\) is also a martingale under \( P \)-distribution. Therefore, \( E[d(U \cdot M_t)] = E[dU.M_t + U.dM_t + dU.dM_t] = 0 \). Replacing \( M_t \) and \( dM_t \) into this equation we have the following restriction where \( U_S = \partial U(S, v_1, v_2, t)/\partial S \), \( U_{v_1} = \partial U(S, v_1, v_2, t)/\partial v_1 \), and \( U_{v_2} = \partial U(S, v_1, v_2, t)/\partial v_2 \).
\[-rU + U_t + U_S(r + \mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}} \kappa_1 (\theta_1 - v_{1,t}) + U_{v_{2,t}} \kappa_2 (\theta_2 - v_{2,t})
+ \frac{1}{2} U_{SS}(v_{1,t} + v_{2,t}) + \frac{1}{2} U_{v_{1,t}v_{1,t}} \sigma_1^2 v_{1,t} + \frac{1}{2} U_{v_{2,t}v_{2,t}} \sigma_2^2 v_{2,t} + U_{Sv_{1,t}} \rho_1 \sigma_1 v_{1,t} + U_{Sv_{2,t}} \rho_2 \sigma_2 v_{2,t}
+ (U_S \sqrt{v_{1,t}} + U_{v_{1,t}} \rho_1 \sigma_1 \sqrt{v_{1,t}})(\phi \sqrt{v_{1,t}} + \zeta_1 \rho_1 \sigma_1 \sqrt{v_{1,t}})
+ (U_S \sqrt{v_{2,t}} + U_{v_{2,t}} \rho_2 \sigma_2 \sqrt{v_{2,t}})(\phi \sqrt{v_{2,t}} + \zeta_2 \rho_2 \sigma_2 \sqrt{v_{2,t}})
+ U_{v_{1,t}} \zeta_1 \sigma_1^2 v_{1,t} (1 - \rho_1^2) + U_{v_{2,t}} \zeta_2 \sigma_2^2 v_{2,t} (1 - \rho_2^2) = 0\] (A.14)

The last restriction is based on the fact that discounted price process should be a martingale under risk neutral measure. Therefore, for any asset, $U(S, v_{1,t}, v_{2,t})$, whose payoff depends on the state variables \{S, v_{1,t}, v_{2,t}\}, $U/B_t$ is a $Q$-martingale. This restriction implies that $E^Q[d(U/B_t)] = 0$ or equivalently $E^Q[d(U(S, v_{1,t}, v_{2,t}))] = rU(S, v_{1,t}, v_{2,t})$.

\[U_t + rSU_t + U_{v_{1,t}} (\kappa_1 (\theta_1 - v_{1,t}) - \lambda_1 v_{1,t}) + U_{v_{2,t}} (\kappa_2 (\theta_2 - v_{2,t}) - \lambda_2 v_{2,t}) + \frac{1}{2} U_{SS}(v_{1,t} + v_{2,t})
+ \frac{1}{2} U_{v_{1,t}v_{1,t}} \sigma_1^2 v_{1,t} + \frac{1}{2} U_{v_{2,t}v_{2,t}} \sigma_2^2 v_{2,t} + U_{Sv_{1,t}} \rho_1 \sigma_1 v_{1,t} + U_{Sv_{2,t}} \rho_2 \sigma_2 v_{2,t} = rU.\] (A.15)

Replace (A.15) from the last restriction into (A.14) from the third restriction.

\[U_S(\mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}} \lambda_1 v_{1,t} + U_{v_{2,t}} \lambda_2 v_{2,t}
+ (U_S \sqrt{v_{1,t}} + U_{v_{1,t}} \rho_1 \sigma_1 \sqrt{v_{1,t}})(\phi \sqrt{v_{1,t}} + \zeta_1 \rho_1 \sigma_1 \sqrt{v_{1,t}})
+ (U_S \sqrt{v_{2,t}} + U_{v_{2,t}} \rho_2 \sigma_2 \sqrt{v_{2,t}})(\phi \sqrt{v_{2,t}} + \zeta_2 \rho_2 \sigma_2 \sqrt{v_{2,t}})
+ U_{v_{1,t}} \zeta_1 \sigma_1^2 v_{1,t} (1 - \rho_1^2) + U_{v_{2,t}} \zeta_2 \sigma_2^2 v_{2,t} (1 - \rho_2^2) = 0\]

\[U_S(\mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}} \lambda_1 v_{1,t} + U_{v_{2,t}} \lambda_2 v_{2,t}
+ U_S \phi v_{1,t} + U_S \zeta_1 \rho_1 \sigma_1 v_{1,t} + U_{v_{1,t}} \rho_1 \sigma_1 \phi v_{1,t} + U_{v_{1,t}} \zeta_1 \sigma_1^2 v_{1,t}
+ U_S \phi v_{2,t} + U_S \zeta_2 \rho_2 \sigma_2 v_{2,t} + U_{v_{2,t}} \rho_2 \sigma_2 \phi v_{2,t} + U_{v_{2,t}} \zeta_2 \sigma_2^2 v_{2,t} = 0\] (A.16)

From the second restriction in (A.12) we know that $\mu_1 v_{1,t} + \mu_2 v_{2,t} = -\phi v_{1,t} - \zeta_1 \rho_1 \sigma_1 v_{1,t} - \phi v_{2,t} - \zeta_2 \rho_2 \sigma_2 v_{2,t}$. Therefore, we can further simplify (A.16).

\[U_{v_{1,t}} (\rho_1 \sigma_1 \phi + \lambda_1 + \zeta_1 \sigma_1^2) v_{1,t} + U_{v_{2,t}} (\rho_2 \sigma_2 \phi + \lambda_2 + \zeta_2 \sigma_2^2) v_{2,t} = 0\] (A.17)

One admissible solution for (A.17) would be:
\[ \rho_1 \sigma_1 \phi + \lambda_1 + \zeta_1 \sigma_1^2 = 0 \] (A.18)
\[ \rho_2 \sigma_2 \phi + \lambda_2 + \zeta_2 \sigma_2^2 = 0 \]

If we combine restrictions in (A.18) with those introduced in (A.13) and replace them back into (A.13) we have \( \phi, \zeta_1, \) and \( \zeta_2. \)

\[ \zeta_1 = \frac{\rho_1 \sigma_1 \mu_1 - \lambda_1}{\sigma_1^2(1 - \rho_1^2)} \] (A.19)
\[ \zeta_2 = \frac{\rho_2 \sigma_2 \mu_2 - \lambda_2}{\sigma_2^2(1 - \rho_2^2)} \]
\[ \phi = -\mu_1 - \frac{\rho_1^2 \sigma_1^2 \mu_1 - \lambda_1 \rho_1 \sigma_1}{\sigma_1^2(1 - \rho_1^2)} = -\mu_2 - \frac{\rho_2^2 \sigma_2^2 \mu_2 - \lambda_2 \rho_2 \sigma_2}{\sigma_2^2(1 - \rho_2^2)} \] (A.20)

Therefore, an admissible pricing kernel linking the \( P \) and \( Q \) dynamics in (A.1) and (A.2) is as follows.

\[ \frac{dM_t}{M_t} = -r dt - \mu_1 \sqrt{v_{1,t}} d\tilde{z}_{1,t} - \mu_2 \sqrt{v_{2,t}} d\tilde{z}_{2,t} + \frac{\rho_1 \sigma_1 \mu_1 - \lambda_1}{\sigma_1^2(1 - \rho_1^2)} dB_{1,t} + \frac{\rho_2 \sigma_2 \mu_2 - \lambda_2}{\sigma_2^2(1 - \rho_2^2)} dB_{2,t} \] (A.21)

This is the pricing kernel introduced in Proposition 1.

Now, we show that how physical shocks are linked to risk neutral shocks through equity premium \( \{\mu_1, \mu_2\} \) and variance premium \( \{\lambda_1, \lambda_2\} \) parameters.

\[ d\tilde{z}_{1,t} = d\tilde{z}_{1,t} + (\psi_1 + \rho_1 \psi_3) dt \]
\[ d\tilde{z}_{2,t} = d\tilde{z}_{2,t} + (\psi_2 + \rho_2 \psi_4) dt \]
\[ dw_{1,t} = dw_{1,t} + (\psi_3 + \rho_1 \psi_1) dt \]
\[ dw_{2,t} = dw_{2,t} + (\psi_4 + \rho_2 \psi_2) dt \] (A.22)

Replace physical shocks in return dynamics (1) by risk neutral shocks introduced in (A.22).

\[ dS_t/S_t = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt + \sqrt{v_{1,t}} d\tilde{z}_{1,t} - (\psi_1 + \rho_1 \psi_3) \sqrt{v_{1,t}} dt + \sqrt{v_{2,t}} d\tilde{z}_{2,t} - (\psi_2 + \rho_2 \psi_4) \sqrt{v_{2,t}} dt \] (A.23)
As a result of risk neutralization in (A.23), the expected stock returns in (A.23) should be equal to the risk free rate of returns. Therefore, we have the following restriction.

\[(\mu_1 v_{1,t} + \mu_2 v_{2,t}) dt = (\psi_1 + \rho_1 \psi_3) \sqrt{v_{1,t}} dt + (\psi_2 + \rho_2 \psi_4) \sqrt{v_{2,t}} dt \quad (A.24)\]

One possible solution of (A.24) is as follows.

\[
\begin{align*}
\mu_1 \sqrt{v_{1,t}} &= \psi_1 + \rho_1 \psi_3 \\
\mu_2 \sqrt{v_{2,t}} &= \psi_2 + \rho_2 \psi_4 
\end{align*}
\quad (A.25)
\]

Similarly, we replace the proposed transformation in (A.22) into the dynamics of volatilities in (1).

\[
\begin{align*}
dv_{1,t} &= \kappa_1 (\theta_1 - v_{1,t}) dt + \sigma_1 \sqrt{v_{1,t}} d\tilde{w}_{1,t} - \sigma_1 \sqrt{v_{1,t}} (\psi_3 + \rho_1 \psi_1) dt \\
dv_{2,t} &= \kappa_2 (\theta_2 - v_{2,t}) dt + \sigma_2 \sqrt{v_{2,t}} d\tilde{w}_{2,t} - \sigma_2 \sqrt{v_{2,t}} (\psi_4 + \rho_2 \psi_2) dt
\end{align*}
\quad (A.26)
\]

The risk-neutral variance dynamics in (A.26) should be equivalent to those in (A.4), where the market price of variance risk factors is proportional to spot variance. Therefore, we have following restrictions:

\[
\begin{align*}
\sigma_1 \sqrt{v_{1,t}} (\psi_3 + \rho_1 \psi_1) &= \lambda_1 v_{1,t} \\
\sigma_2 \sqrt{v_{2,t}} (\psi_4 + \rho_2 \psi_2) &= \lambda_2 v_{2,t} 
\end{align*}
\quad (A.27)
\]

Combining the restrictions in (A.25) and (A.27), we have the following results, which link the physical distribution (1) to the risk neutral distribution (4).

\[
\begin{align*}
\psi_1 &= \frac{\sigma_1 \mu_1 - \rho_1 \lambda_1}{\sigma_1 (1 - \rho_1^2)} \sqrt{v_{1,t}} \\
\psi_2 &= \frac{\sigma_2 \mu_2 - \rho_2 \lambda_2}{\sigma_2 (1 - \rho_2^2)} \sqrt{v_{2,t}} \\
\psi_3 &= \frac{\lambda_1 - \rho_1 \sigma_1 \mu_1}{\sigma_1 (1 - \rho_1^2)} \sqrt{v_{1,t}} \\
\psi_4 &= \frac{\lambda_2 - \rho_2 \sigma_2 \mu_2}{\sigma_2 (1 - \rho_2^2)} \sqrt{v_{2,t}} 
\end{align*}
\quad (A.28)
\]
B Proof of Proposition 2

The price of individual equity risk factors can be defined jointly with those of index risk factors by imposing the no-arbitrage conditions on individual equity prices, $S^i_t$. Therefore, the discounted products of returns and pricing kernel is $P$-martingale. This proof is available from the author upon request. Here we provide an alternative (and shorter) version of the proof.

The dynamics of individual equity returns under physical measure is:

$$\frac{dS^i_t}{S^i_t} = \mu^i_t dt + \beta^1_i(\mu_1 v_1 dt + \sqrt{v_1} dz_{1,t}) + \beta^2_i(\mu_2 v_2 dt + \sqrt{v_2} dz_{2,t}) + \sqrt{\xi^i_t} dz^i_t,$$

$$d\xi^i_t = \kappa^i(\theta^i - \xi^i_t) dt + \sigma^i \sqrt{\xi^i_t} d\xi^i_t.$$  \hfill (B.1)

In order to convert the above $P$-dynamics to their $Q$-dynamics counterpart, we follow Grisanov’s theorem and transform shocks to individual equity returns and shocks to idiosyncratic volatility by introducing the relevant prices of risk factors in (B.2).

$$dz^i_t = dz^i_t + (\psi^1_i + \rho^i \psi^2_i) dt$$
$$d\tilde{w}_i^i = d\tilde{w}_i^i + (\psi^2_i + \rho^i \psi^1_i) dt$$  \hfill (B.2)

Similar transformations for the innovations in market returns can be defined by combining (A.22) and (A.25) as follows.

$$d\tilde{z}^1_t = d\tilde{z}^1_t + (\psi^1_1 + \rho^1 \psi^2_1) dt$$
$$d\tilde{z}^2_t = d\tilde{z}^2_t + (\psi^2_2 + \rho^2 \psi^1_1) dt$$  \hfill (B.3)

Replacing (B.2) and (B.3) into the physical return dynamics (B.1), we have the following restriction to get the risk-neutral return dynamics in (12).

$$\mu^i dt - \sqrt{\xi^i_t} (\psi^2_i + \rho^i \psi^1_i) dt = r dt$$  \hfill (B.4)

Then replace (B.2) into the physical idiosyncratic volatility dynamics (B.1).

$$d\xi^i_t = \kappa^i(\theta^i - \xi^i_t) dt + \sigma^i \sqrt{\xi^i_t} (d\tilde{w}_i^i - (\psi^2_i + \rho^i \psi^1_i) dt)$$

If idiosyncratic volatility is priced, its price should be proportional to idiosyncratic volatility and we have $\sigma^i \sqrt{\xi^i_t} (\psi^2_i + \rho^i \psi^1_i) = \lambda^i \xi^i_t$. However, in our setup, idiosyncratic volatility is not priced and $\lambda^i = 0$. Therefore, we have the following restriction.

$$\sigma^i \sqrt{\xi^i_t} (\psi^2_i + \rho^i \psi^1_i) = 0$$  \hfill (B.5)
Combining restrictions in (B.4) and (B.5) we have the following prices of risk factors.

\[
\psi_i^1 = \frac{\mu_i - r}{\sqrt{\xi_i(1 - \rho^2)}} \\
\psi_i^2 = -\frac{\mu_i - r}{\sqrt{\xi_i(1 - \rho^2)}} \rho^i
\]

(B.6)

Combining (B.6) with (B.3) delivers the risk-neutral dynamics for individual equity returns in Proposition 2.

C Proof of Proposition 3

Given the \( Q \) dynamics of index returns and individual equities returns in (4) and (12), applying Ito’s lemma on \( x_i^t \), delivers the following expression.

\[
x_{t+\tau}^i - x_i^t = r\tau - \frac{1}{2} \left[ \beta_1^2 v_{1,t,t+\tau} + \beta_2^2 v_{2,t,t+\tau} + \xi_i^t \right] \tau + \beta_1^i \int_t^{t+\tau} \sqrt{v_{1,u}} d\tilde{z}_{1,u} + \beta_2^i \int_t^{t+\tau} \sqrt{v_{2,u}} d\tilde{z}_{2,u} + \int_t^{t+\tau} \sqrt{\xi_i^t} d\tilde{z}_i^u
\]

(C.1)

For the ease of notations we define:

\[
\tilde{z}_{v_1,\tau} \equiv \int_t^{t+\tau} \sqrt{v_{1,u}} d\tilde{z}_{1,u}, \\
\tilde{z}_{v_2,\tau} \equiv \int_t^{t+\tau} \sqrt{v_{2,u}} d\tilde{z}_{2,u}, \\
\tilde{z}_{\xi,\tau} \equiv \int_t^{t+\tau} \sqrt{\xi_u} d\tilde{z}_i^u.
\]

By the definition of risk-neutral conditional characteristic function of log-returns in (13) we have:\(^{43}\)

\[
\tilde{f}^i(\tau, \phi) = E_t^Q \left[ \exp \left[ i\phi(r\tau - \frac{1}{2} (\beta_1^2 v_{1,t,t+\tau} + \beta_2^2 v_{2,t,t+\tau} + \xi_i^t) \tau + \beta_1^i \tilde{z}_{v_1,\tau} + \beta_2^i \tilde{z}_{v_2,\tau} + \tilde{z}_{\xi,\tau}) \right] \right].
\]

(C.2)

Define the stochastic exponential \( \tilde{\zeta}(\cdot) \) as follows.

\(^{43} \) For compactness, the dependence of risk-neutral conditional characteristic function to \( x_i^t, v_{1,t}, v_{2,t}, \xi_i^t, \beta_1^i, \) and \( \beta_2^i \) is suppressed in (C.2).
\[ \zeta(\int_0^t w_u'dW_u) \equiv \exp \left[ \int_0^t w_u'dW_u - \frac{1}{2} \int_0^t w_u'd\langle W, W' \rangle w_u \right] \]  
(C.3)

Therefore,

\[ \zeta(i\phi \beta_1^i \tilde{z}_{v_1, \tau}) = \exp \left[ i\phi \beta_1^i \tilde{z}_{v_1, \tau} - \frac{1}{2} (i\phi \beta_1^i)^2 \langle \tilde{z}_{v_1, \tau}, \tilde{z}_{v_1, \tau} \rangle \right] \]

\[ = \exp \left[ i\phi \beta_1^i \tilde{z}_{v_1, \tau} + \frac{1}{2} \phi^2 \beta_1^2 v_{1.t+\tau} \right]. \]

(C.4)

Similar to (C.4), define \( \zeta(i\phi \beta_2^i \tilde{z}_{v_2, \tau}) \) and \( \zeta(i\phi \tilde{z}_{\xi t}) \) and then combine these three stochastic exponential with (C.2) to get the following risk-neutral conditional characteristic function.

\[ \tilde{f}^i(\tau, \phi) = e^{i\phi r T} \tilde{E}_t^Q \left[ \zeta(i\phi \beta_1^i \tilde{z}_{v_1, \tau}) \zeta(i\phi \beta_2^i \tilde{z}_{v_2, \tau}) \zeta(i\phi \tilde{z}_{\xi t}) \exp \left[ - g_1 v_{1.t+\tau} - g_2 v_{2.t+\tau} - g_3 \xi_t^i \right] \right] \]

where, \( g_1 = \frac{1}{2} i\phi \beta_1^2 (1 - i\phi) \), \( g_2 = \frac{1}{2} i\phi \beta_2^2 (1 - i\phi) \), and \( g_3 = \frac{1}{2} i\phi (1 - i\phi) \). Following Carr and Wu [2004], we define a new change-of-measure from \( Q \)-measure to \( C \)-measure as follows.

\[ \frac{dC}{dQ}(t) \equiv \zeta(i\phi \beta_1^i \tilde{z}_{v_1, \tau}) \zeta(i\phi \beta_2^i \tilde{z}_{v_2, \tau}) \zeta(i\phi \tilde{z}_{\xi t}) \]  
(C.6)

The Radon-Nikodym derivatives of \( C \) with respect to \( Q \) in (C.6) allows to write (C.5) as

\[ \tilde{f}^i(\tau, \phi) = e^{i\phi r T} \tilde{E}_t^Q \left[ \frac{dC}{dQ}(T) \exp \left[ - g_1 v_{1.t+\tau} - g_2 v_{2.t+\tau} - g_3 \xi_t^i \right] \right] \]

\[ = e^{i\phi r T} \tilde{E}_t^C \left[ \exp \left[ - g_1 v_{1.t+\tau} - g_2 v_{2.t+\tau} - g_3 \xi_t^i \right] \right]. \]

(C.7)

Accordingly, we transform the risk-neutral shocks to index returns volatilities and to the idiosyncratic returns volatility to their \( C \)-measure counterparts by applying the extension of Grisanov’s theorem within the complex plane.

\[ dw_{1,t} = dw_{1,t}^C + (i\phi \rho_1 \beta_1^i \sqrt{v_{1.t}}) dt \]

\[ dw_{2,t} = dw_{2,t}^C + (i\phi \rho_2 \beta_2^i \sqrt{v_{2.t}}) dt \]

\[ dw_t^i = dw_t^i C + (i\phi \tilde{z}_{\xi t}) dt \]

As a results, the index volatilities dynamics and idiosyncratic volatility dynamics of individual equity under the \( C \)-measure are

\[ \text{(C.8)} \]

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\[ \text{As the Radon-Nikodym derivatives in (C.6) is defined based on the stochastic exponential } \zeta(\cdot), \text{ it is Martingale by definition.} \]
\[ dv_{1,t} = \kappa_1^C (\theta_1^C - v_{1,t}) dt + \sigma_1 \sqrt{v_{1,t}} dw_{1,t}^C, \]
\[ dv_{2,t} = \kappa_2^C (\theta_2^C - v_{2,t}) dt + \sigma_2 \sqrt{v_{2,t}} dw_{2,t}^C, \]
\[ d\xi_t^i = \kappa_i^C (\theta_i^C - \xi_t^i) dt + \sigma_i \sqrt{\xi_t^i} dw_t^i, \] (C.9)

where,

\[ \kappa_1^C = \tilde{\kappa}_1 - i\phi \rho_1 \beta_1 \sigma_1 \quad \theta_1^C = \frac{\tilde{\theta}_1}{\kappa_1^C}, \]
\[ \kappa_2^C = \tilde{\kappa}_2 - i\phi \rho_2 \beta_2 \sigma_2 \quad \theta_2^C = \frac{\tilde{\theta}_2}{\kappa_2^C}, \]
\[ \kappa_i^C = \kappa_i^i - i\phi \rho_i \sigma_i \quad \theta_i^C = \frac{\tilde{\theta}_i}{\kappa_i^C}. \]

Using the closed-form solution of the moment generating functions of \( E_t^C[\exp(-g_1 v_{1,t,t+r})] \), and \( E_t^C[\exp(-g_2 v_{2,t,t+r})] \), and \( E_t^C[\exp(-g_3 \xi_{t,t+r}^i)] \), the risk-neutral conditional characteristic function of log individual equity prices has the following affine form.

\[ \tilde{f}^i(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi) = \exp \left[ i\phi x_t^i + i\phi \tau^i - A_1(\tau, \phi) - A_2(\tau, \phi) - B(\tau, \phi) - C_1(\tau, \phi) v_{1,t} - C_2(\tau, \phi) v_{2,t} - D(\tau, \phi) \xi_t^i \right], \] (C.10)

\[ A_1(\tau, \phi) = \frac{\tilde{\kappa}_1 \tilde{\theta}_1}{\sigma_1^2} \left[ 2 \ln \left[ 1 - \frac{d_1 - \kappa_1^C}{2d_1} (1 - e^{-d_1 \tau}) \right] + (d_1 - \kappa_1^C) \tau \right], \]
\[ A_2(\tau, \phi) = \frac{\tilde{\kappa}_2 \tilde{\theta}_2}{\sigma_2^2} \left[ 2 \ln \left[ 1 - \frac{d_2 - \kappa_2^C}{2d_2} (1 - e^{-d_2 \tau}) \right] + (d_2 - \kappa_2^C) \tau \right], \]
\[ B(\tau, \phi) = \frac{\tilde{\kappa}_i \tilde{\theta}_i^i}{\sigma_i^2} \left[ 2 \ln \left[ 1 - \frac{d_i - \kappa_i^C}{2d_i} (1 - e^{-d_i \tau}) \right] + (d_i - \kappa_i^C) \tau \right], \]
\[ C_1(\tau, \phi) = \frac{2g_1 (1 - e^{-d_1 \tau})}{2d_1 - (d_1 - \kappa_1^C) (1 - e^{-d_1 \tau})}, \]
\[ C_2(\tau, \phi) = \frac{2g_2 (1 - e^{-d_2 \tau})}{2d_2 - (d_2 - \kappa_2^C) (1 - e^{-d_2 \tau})}, \]
\[ D(\tau, \phi) = \frac{2g_i (1 - e^{-d_i \tau})}{2d_i - (d_i - \kappa_i^C) (1 - e^{-d_i \tau})}, \] (C.11)

\[ d_1 = \sqrt{(\kappa_1^C)^2 + 2\sigma_1^2 g_1}, \]
\[ d_2 = \sqrt{(\kappa_2^C)^2 + 2\sigma_2^2 g_2}, \]
\[ d_i = \sqrt{(\kappa_i^C)^2 + 2\sigma_i^2 g_i}, \]
\[ g_1 = \frac{1}{2} i\phi \beta_1 \sigma_1 (1 - i\phi), \]
\[ g_2 = \frac{1}{2} i\phi \beta_2 \sigma_2 (1 - i\phi), \]
\[ g_i = \frac{1}{2} i\phi (1 - i\phi). \]
We determine the price of a European call option on an individual equity with the strike price $K$ and the time-to-maturity $\tau$ by inverting the risk-neutral conditional characteristic function of log-returns.\footnote{Note that the risk-neutral conditional characteristic function of the logarithm of individual equity returns, $x_{t+\tau}^i - x_t^i = \ln(S_{t+\tau}^i/S_t^i)$, can be defined with the same expression as \ref{eq:log_return_characteristic} but without the first component, $i\phi x_t^i$.}

\begin{equation}
C_t^i(S_t^i, K, \tau) = S_t^i P_1^i - Ke^{-r\tau} P_2^i,
\end{equation}

where,

\begin{align*}
P_1^i &= \frac{1}{2} + \frac{1}{\pi} \frac{1}{S_t^i e^{r\tau}} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi - i)}{i\phi} \right] d\phi, \\
P_2^i &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi)}{i\phi} \right] d\phi.
\end{align*}

\section*{Appendix D}

Proofs of Proposition (4) and Proposition (5) are available upon request.
References


Table 1: Data Sample Summary

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<tr>
<th>Company</th>
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<th>All Options</th>
<th>Avg DTM</th>
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<td>189,496</td>
<td>137,546</td>
<td>327,042</td>
<td>141</td>
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</table>

Note to Table: This table reports the number of available call and put option contracts for the index and for each firm in our sample with 10% moneyness and maturity up to and equal to 1 year during the sample period 1996-2012. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM) for available contract.
Table 2: Market Parameter Estimates

Panel A: Physical Parameter Estimates - Joint Estimation

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<th>$\kappa_1$</th>
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<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
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Panel B: Risk-neutral Parameter Estimates - Joint Estimation

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Panel C: Risk-neutral Parameter Estimates - Options Only (Q Only)

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Note to Table: This table reports the structural parameter estimates of the S&P 500 Index. Structural parameters under the physical measure are reported in Panel A and structural parameters under risk neutral measure are reported in Panel B. The reported results in Panel A and Panel B are estimated from the joint likelihood function of time series of the index returns and the cross-sections of option prices, where we include 10% OTM call and put options over the period 1996-2012. Panel C shows the structural parameters estimates using only option data, 10% OTM call and put options over the period 1996-2012.
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<th>$\rho$</th>
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</table>

Note to Table: This table reports the structural risk-neutral parameters estimates for individual equities conditional on the structural parameters of the S&P 500 index and the two vectors of filtered spot market variances. This table also reports the estimates of individual equity betas for 27 firms in our sample. The market parameters and variances are estimated from 10% OTM call and put options over the period 1996-2012. For individual equities, we use 10% OTM call and put options over the period 1996-2012, where we drop the first five months.
Table 4: Distributional Properties of Spot Idiosyncratic Volatility

<table>
<thead>
<tr>
<th>Company</th>
<th>Ticker</th>
<th>Mean</th>
<th>Std dev</th>
<th>Max</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa</td>
<td>AA</td>
<td>0.1259</td>
<td>0.1387</td>
<td>0.6879</td>
<td>0.0900</td>
</tr>
<tr>
<td>American Express</td>
<td>AXP</td>
<td>0.1068</td>
<td>0.1489</td>
<td>0.7138</td>
<td>0.0692</td>
</tr>
<tr>
<td>Boeing</td>
<td>BA</td>
<td>0.0633</td>
<td>0.0442</td>
<td>0.2484</td>
<td>0.0521</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>CAT</td>
<td>0.0783</td>
<td>0.0628</td>
<td>0.4395</td>
<td>0.0587</td>
</tr>
<tr>
<td>Cisco</td>
<td>CSCO</td>
<td>0.1497</td>
<td>0.1328</td>
<td>0.8274</td>
<td>0.0987</td>
</tr>
<tr>
<td>Chevron</td>
<td>CVX</td>
<td>0.0293</td>
<td>0.0267</td>
<td>0.2126</td>
<td>0.0260</td>
</tr>
<tr>
<td>Dupont</td>
<td>DD</td>
<td>0.0460</td>
<td>0.0476</td>
<td>0.2526</td>
<td>0.0292</td>
</tr>
<tr>
<td>Disney</td>
<td>DIS</td>
<td>0.0636</td>
<td>0.0515</td>
<td>0.2661</td>
<td>0.0460</td>
</tr>
<tr>
<td>General Electric</td>
<td>GE</td>
<td>0.0618</td>
<td>0.0938</td>
<td>0.6134</td>
<td>0.0413</td>
</tr>
<tr>
<td>Home Depot</td>
<td>HD</td>
<td>0.0741</td>
<td>0.0600</td>
<td>0.3230</td>
<td>0.0510</td>
</tr>
<tr>
<td>Hewlett-Packard</td>
<td>HPQ</td>
<td>0.1250</td>
<td>0.1231</td>
<td>0.4893</td>
<td>0.0903</td>
</tr>
<tr>
<td>IBM</td>
<td>IBM</td>
<td>0.0439</td>
<td>0.0482</td>
<td>0.2620</td>
<td>0.0260</td>
</tr>
<tr>
<td>Intel</td>
<td>INTC</td>
<td>0.1206</td>
<td>0.0882</td>
<td>0.6408</td>
<td>0.0927</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>JNJ</td>
<td>0.0225</td>
<td>0.0257</td>
<td>0.2340</td>
<td>0.0116</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>JPM</td>
<td>0.1070</td>
<td>0.1325</td>
<td>0.9138</td>
<td>0.0786</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>KO</td>
<td>0.0268</td>
<td>0.0308</td>
<td>0.1729</td>
<td>0.0133</td>
</tr>
<tr>
<td>McDonald’s</td>
<td>MCD</td>
<td>0.0389</td>
<td>0.0345</td>
<td>0.1638</td>
<td>0.0277</td>
</tr>
<tr>
<td>3M</td>
<td>MMM</td>
<td>0.0297</td>
<td>0.0304</td>
<td>0.1645</td>
<td>0.0180</td>
</tr>
<tr>
<td>Merck</td>
<td>MRK</td>
<td>0.0438</td>
<td>0.0367</td>
<td>0.2189</td>
<td>0.0358</td>
</tr>
<tr>
<td>Microsoft</td>
<td>MSFT</td>
<td>0.0749</td>
<td>0.0614</td>
<td>0.4605</td>
<td>0.0647</td>
</tr>
<tr>
<td>Pfizer</td>
<td>PFE</td>
<td>0.0490</td>
<td>0.0425</td>
<td>0.2021</td>
<td>0.0356</td>
</tr>
<tr>
<td>Procter &amp; Gamble</td>
<td>PG</td>
<td>0.0256</td>
<td>0.0326</td>
<td>0.2411</td>
<td>0.0103</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>T</td>
<td>0.0522</td>
<td>0.0532</td>
<td>0.5365</td>
<td>0.0359</td>
</tr>
<tr>
<td>United Technologies</td>
<td>UTX</td>
<td>0.0399</td>
<td>0.0374</td>
<td>0.2126</td>
<td>0.0258</td>
</tr>
<tr>
<td>Verizon</td>
<td>VZ</td>
<td>0.0428</td>
<td>0.0438</td>
<td>0.3520</td>
<td>0.0280</td>
</tr>
<tr>
<td>Walmart</td>
<td>WMT</td>
<td>0.0436</td>
<td>0.0550</td>
<td>0.2870</td>
<td>0.0193</td>
</tr>
<tr>
<td>Exxon Mobil</td>
<td>XOM</td>
<td>0.0234</td>
<td>0.0210</td>
<td>0.1556</td>
<td>0.0204</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the mean, median, standard deviation, and the maximum of the vector of spot idiosyncratic variance for every equity conditional on the structural parameters of the S&P 500 index and the two filtered spot market variances. The market parameters and variances are estimated from 10% OTM call and put options prices over the period 1996-2012. For individual equities, we use 10% OTM call and put options over the period 1996-2012, where we drop the first five months.
Figure 1: Market Delta of Equity Call Options

Note to Figure: This figure plots the sensitivity of the model-implied at-the-money equity call option prices with respect to the level of market index for different sets of betas. The top LHS panel shows this sensitivity following the calibration in in one-factor model of Christoffersen et al. [2015]. Note also that for all the graphs the total unconditional equity variance is fixed, \( \tilde{\nu} = (\beta_1)^2 \theta_1 + (\beta_2)^2 \theta_2 + \theta^2 = 0.11 \).
Figure 2: Market Vega of Equity Call Options (For the First Volatility Component)

Note to Figure: This figure plots the sensitivity of the model-implied at-the-money equity call option prices with respect to the first component of market variance for different sets of betas. The top LHS panel shows this sensitivity following the calibration in in one-factor model of Christoffersen et al. [2015]. Note also that for all the graphs the total unconditional equity variance is fixed, $\tilde{\sigma}^2 = (\beta_1)^2 \tilde{\theta}_1 + (\beta_2)^2 \tilde{\theta}_2 + \theta^i = 0.11$. 

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Note to Figure: This figure plots the sensitivity of the model-implied at-the-money equity call option prices with respect to the second component of market variance for different sets of betas. The top LHS panel shows this sensitivity following the calibration in in one-factor model of Christoffersen et al. [2015]. Note also that for all the graphs the total unconditional equity variance is fixed, \( \tilde{\sigma}^2 = (\beta_1^2 \tilde{\theta}_1 + (\beta_2^2 \tilde{\theta}_2 + \theta^2 = 0.11. \)
Note to Figure: This figure plots the model-implied volatility for at-the-money equity call options with respect to the time-to-maturity for different set of betas. The top LHS panel shows the term-structure effect following the calibration in in one-factor model of Christoffersen et al. [2015]. The rest of the panels show the term structure effect for the equity call options for different set of betas. The top LHS panel shows the term-structure effect following the calibration in one-factor model of Christoffersen et al. [2015]. The rest of the panels show the term structure effect for the equity call options for different set of betas. The top LHS panel shows the term-structure effect following the calibration in one-factor model of Christoffersen et al. [2015]. The rest of the panels show the term structure effect for the equity call options for different set of betas. 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The top LHS panel shows the term-structure effect following the calibration in one-factor model of Christoffersen et al. [2015]. The rest of the panels show the term structure effect for the equity call options for different set of betas.
Figure 5: Beta and Implied Volatility Across Moneyness

Note to Figure: This figure plots the model-implied volatility for 3 months at-the-money equity call options with respect to the moneyness (S/K) for different sets of betas. The top LHS panel shows the moneyness effect following the calibration in one-factor model of Christoffersen et al. [2015]. The rest of the panels show the effect of moneyness on the implied volatility of equity call options for different level of $\beta_1$ and $\beta_2$ when the market dynamics follow a multiple stochastic volatility model. Note that for all the graphs the total unconditional equity variance is fixed at $\tilde{\sigma}^2 = (\beta_1^2)\tilde{\theta}_1 + (\beta_2^2)\tilde{\theta}_2 + \theta^2 = 0.11$. 

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Figure 6: Impact of Market Variances Risk Premiums on the Equity Implied Volatility Smile

Note to Figure: This figure plots the difference between model implied volatility for 3 months at-the-money equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is calculated when $\lambda_1 = \lambda_2 = -0.5$ and when $\lambda_1 = \lambda_2 = 0$. The top LHS panel shows the effect of market variance risk premium on equity option skew (slope of IV curve) following the calibration in one-factor model of Christoffersen et al. [2015]. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{\sigma} = (\beta_1^2 \tilde{\theta}_1 + (\beta_2^2 \tilde{\theta}_2 + \theta^i = 0.11$. 

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