The Implications of Tail Dependency for Counterparty Credit Risk Pricing∗

Abstract

This paper investigates the counterparty credit risk of interest rate swaps positions using the credit valuation adjustment (CVA) measure, and examines the potential dependency relationships between the probability of default (PD) and the exposure at default (EAD). We empirically tested, using interest rate swaption implied market volatilities, three tail dependency models: a Basel Committee (BCBS, 2011b) independent model, a Gaussian copula dependent model, and a Wrong Way Risk (WWR) with copula dependency approach. The results show that the CVA underestimation when using a Gaussian copula for modelling the dependence of PD and EAD is about 51%–362% compared to using WWR, and the underestimation between using the standardised Basel independent model and using the Gaussian copula is about 527%–1609%, including the period of the 2007/2008 crisis. These results have important implications for regulators, financial institutions, and credit risk managers when calculating counterparty risk.

Keywords: Credit risk, Counterparty Credit Risk, Credit Value Adjustment, Dependency of credit risk components, Pricing swaps.

JEL Classifications: G10; G13; G33.

1. Introduction

Banking regulation has advanced in establishing both control procedures and capital requirements, bringing different impacts on financial institutions. The financial crisis in 2007/2008 highlighted the need for improvements in credit risk measurement and management. Counterparty credit risk has become the focus of attention of managers and regulators, since many losses incurred during the 2007/2008 crisis resulted from events associated with the deteriorating credit quality of market participants (BCBS, 2011b). Although credit risk has already been the subject of numerous studies, an important element that the crisis has brought to attention involves pricing models of portfolios that have positions in derivatives. In this context, not only can actual events of default cause substantial devaluations in financial institutions’ positions, but the deterioration of credit quality, of both banks and clients, can also affect the value of portfolios exposed to counterparty credit risk.1

∗This document was a collaborative effort.

1In fact, a study by the BCBS on the 2007/2008 crisis indicated that approximately 2/3 of credit losses came from the marked-to-market devaluations of positions exposed to counterparty credit risk and only 1/3 of losses came from the actual default
The Basel Committee on Banking Supervision (BCBS) defines the CVA as the difference between the value of a portfolio with derivatives considering, respectively, the absence and the presence of credit risk of the counterparty (BCBS, 2011a). Importantly, capital requirements under the CVA are linked to portfolios of derivatives, distinguishing them from the traditional credit capital requirements associated with Expected Losses (EL) and Unexpected Losses (UL). In fact, according to BCBS (2006) and BCBS (2011b), typical credit portfolios – for example, loans and financing – are analysed in the guidelines of Basel II, while portfolios exposed to counterparty credit risk in derivative transactions are analysed in Basel III.

In this research, our contribution to the literature is twofold. First, we calculate the CVA of interest rate swaps positions before, during, and after the 2007/2008 financial crisis. We establish a general pricing of CVA based on the discussion in Cherubini (2013), which allows us to study the elements of the model proposed by Sørensen and Bollier (1994) to price swaps exposed to counterparty credit risk. Later, we test a WWR copula-based model applied to the calculation of the CVA of swaps. Additionally, we calculate a CVA with Gaussian copula dependence for comparison purposes.

Our study uses risk-neutral implied volatilities extracted from the interest rate swaption market from the period of the 10 May 2005 to 1 August 2013 for the pricing of the three CVA dependence models. Although we did consider the credit quality of the counterparties, our research aims to analyse the impact of modelling the wrong way risk (WWR) in the underestimation of counterparty credit risk between two large institutions acting as dealer and customer. Finally, we conduct sensitivity analyses to allow comparison of our results between traditional simple models based on independent risk components – such as, for example, the one suggested by the Basel Committee – and the dependency model based on copulas.

The counterparty credit risk literature is extensive. Relevance of counterparty risk in liquidity and its relationship with market rates and central clearing are studies, e.g., by Heider et al. (2015) and Loon and Zhong (2014). General approaches for assessing the counterparty credit risk of derivatives portfolios include Hull (1989), Duffie and Huang (1996), and Jarrow and Yu (2001). Bomfim (2003) was the first study to assess the effects of the financial crisis over the counterparty credit risk in interest rate swaps. Bomfim (2003) found that there existed no difference between synthetic interest rate swaps constructed with Eurodollar future prices by a no-arbitrage relationship over the real interest rate swaps prices, indicating that there was of counterparties (BCBS, 2009). As a result, banking regulation has been directing its efforts to improve capital requirements, more specifically, by measuring the credit valuation adjustment (CVA – Credit Valuation Adjustment), which is a metric to estimate marked to market losses due to the exposure to counterparty credit risk.

2 According to the Bank for International Settlements (BIS), by the end of the second semester of 2015, interest rate swaps instruments accounted for 288,634 billion US dollars in notional amount from a total of 492,911 billion US dollars of the over-the-counter (OTC) global derivatives market, a share of 58.56% of the total OTC derivatives market.

3 We consider risk-neutral implied interest rate swaption volatilities, as there exists no market price for counterparty credit risk and, as Stulz (2004) and Bolton and Oehmke (2013) report, during times of distress financial institutions have misvalued OTC derivatives in favour of diminishing the collateral associated with the instruments held, which could mislead the study.

4 The main driver of the 2007/2008 financial crisis was the market and the funding liquidity that stemmed the cascade effect on counterparty defaults. Brunnermeier and Pedersen (2009) developed a model that associated market liquidity with the funding liquidity of an institution and Brunnermeier (2009) described the origin and the evolution of the crisis through these market liquidity effects.
no signalling effect before the crisis by the futures market. Counterparty credit risk models and studies on
credit default swaps (CDS)\(^5\) include Pan and Singleton (2008), Arora et al. (2012), and Bo and Capponi
(2015). Pan and Singleton (2008) developed and tested a new model for CDS terms structure that revealed
the fact that default and recovery information was included in the term structure. Arora et al. (2012) found
evidence of counterparty risk premiums priced in the CDS market and Bo and Capponi (2015) derived a new
closed-form expression for CVA valuation in presence of several defaults.

Although the BCBS has suggested standardised settings for the CVA, simplifying the analysis and
applying an exposure correction factor that reflects the spread used to discount a debt security with the
equivalent credit risk of the counterparty (BCBS, 2009), criticisms from practitioners and academics have
induced substantial changes in the criteria. Thus, the Basel Committee established a formula for the CVA
based on a more detailed exposure to counterparty credit, taking into account the specific elements associated
with credit risk components: probability of default (PD), loss given default (LGD) and exposure at default
(EAD) of portfolios that include derivatives.

However, despite the evolution of the model in relation to the initial proposal of an adjustment factor,
several areas of attention still exist, especially concerning the assumption of independence between the
components of credit risk. In the modelling established by the BCBS to calculate the CVA, credit risk
components are considered independent.

The dependency among these risk components may be relevant and, therefore, not considering possible
relationships can lead to inadequate measurement of risk. More specifically, the existence of the WWR can
affect risk assessment, leading to the underestimation of potential losses when using models that only take
into account independence among the credit risk components.

Our second contribution is that in light of the independent PD and EAD modelling for the CVA calculation
proposed by BCBS (2011b) in their standardised settings, we measured the magnitude of the underestimated
CVA of interest rate swaps positions in comparison with two alternative approaches: (i) a Gaussian copula
CVA dependence model and (ii) a WWR CVA dependence model.

One of the first studies to assess the dependency of credit variables in counterparty credit risk was Hull
and White (2001). This study considered a correlation between the different institutions in the valuation of
the CDS. More recently, Jorion and Zhang (2009) discovered an explanation for the “contagion” (default
correlation) effect of creditors with a large exposure. Jorion and Zhang (2009) found that the announcement
of the bankruptcy of borrowers was strongly correlated with the CDS spreads for the creditors. Lipton and
Sepp (2009) calculated the CVA via the Merton (1976) structural model for CDS, modelling the dependence
of two factors by a joint jump intensity process. Gregory (2009) applied an earlier version of the Cherubini
(2013) model of maximum bound copula WWR in Cherubini and Luciano (2002) to derive a general factor

\(^5\)Other studies of counterparty credit risk with other instruments include Hull and White (2006) for credit debt obligations
(CDO), Lo et al. (2013) for catastrophe equity put options, and Sakurai and Uchida (2014) for cross-currency swaps.
bivariate dependency model; nevertheless, the study was only theoretical, with no empirical findings. Eckert et al. (2016) provide a dependency model for the joint distribution of LGD, PD, and EAD; however, their model is based on the CreditMetrics normal distribution risk model and they do not provide empirical evidence.\textsuperscript{6}

The \textit{WWR} is the term commonly used to denote an unfavourable dependence between exposure to credit risk and the credit quality of the counterparty (BCBS, 2011b). The existence of the \textit{WWR} is particularly problematic because it can exacerbate losses, since the exposure to credit risk can be higher at precisely the time when the probability of default increases. Alternatively, the probability of default may increase as the amount owed by a counterparty increases.

The results suggest that by not incorporating the dependence between credit risk components, many models underestimate the potential loss due to the counterparty credit risk. The analysis also indicates that the \textit{WWR} is relevant, therefore the adverse and joint fluctuations of risk components should be evaluated for both risk management and for pricing purposes. Moreover, since the calculation of the \textit{CVA} depends on the modelling, for example, of the dependency relationships of credit risk components, the diffusion process of market risk factors, and volatility of risk factors, counterparty credit risk assessments can also be subject to risk modelling. In this sense, the study also shows, for the specific case of the \textit{CVA} of swaps, that the definition of the dynamics of the interest rates diffusion process also influences the results of counterparty credit risk analysis.

The remainder of this study is structured as follows: Section 2 presents the notation and definitions for measuring counterparty credit risk and the \textit{CVA}. In Section 3, the \textit{CVA} calculation of interest rate swaps with the three different dependency models is presented. Section 4 describes the seven different interest rate models used for measuring the \textit{CVA}. Section 5 describes the data and the empirical calibration method applied. Section 6 presents the results of the estimated volatility and the \textit{CVA}. Section 7 provides final remarks and conclusions, with suggestions for further extensions of the work.

\section{2. Modelling tail dependence in counterparty credit risk}

Let us consider two counter parties denoted by \textit{B} (the bank) and \textit{C} (the client). In this paper, we will use a risk-neutral probability measure, typically associated with pricing procedures, in contrast to the measure of real probability, usually linked to risk measurement mechanisms (Brigo et al., 2013b; Cherubini, 2013).\textsuperscript{7}

\textsuperscript{6}Recent counterparty credit risk mathematical models such as Brigo and Chourdakis (2009), Hull and White (2012), Brigo et al. (2013a), Brigo et al. (2013), and Hull and White (2014) consider effects from collateralization, netting rules, and rehypothecation additional to the \textit{WWR}; their use could provide valuable empirical insights about multivariate dependency effects on credit.

\textsuperscript{7}Further details on the suitability of a risk-neutral probability measure with respect to the real-world probability measure can be seen, for example, in Gregory (2012) and Brigo et al. (2013b). Once having established the probability space and the analysis of structure based on a risk-neutral world, let us now discuss the concept of the \textit{CVA}, emphasising the unilateral model, in which only one of the parties incurs in credit risk, since the other party has a negligible credit risk.
Notice that $B$ does not necessarily represent a bank, but this notation makes it easier to adapt the discussion to market conventions. The bank or dealer ($B$) provides a product to the customer ($C$), which has a certain financial need. Given that, in most cases, the dealers have higher risk management capabilities, we will establish that the analysis of counterparty credit risk is being conducted by counterparty $B$.

### 2.1. Credit valuation adjustment (CVA) definition

We define, following Brigo et al. (2013b) $\Pi(t, T)$ as the net cash flows from $t$ to $T$ of a portfolio of financial products that $B$ traded with $C$, discounted to the present value, and taking into consideration that $C$ is not exposed to the credit risk of $B$. In this formulation, $t$ is the time of the valuation of the portfolio and $T$ is the tenor of the further cash flow to be exchanged between the counterparties. In the case of a portfolio with derivatives, $T$ is the maturity of the contract with the greater expiration date.

The value of this portfolio in terms of price at time $t$ is the expected value of $\Pi(t, T)$, i.e., $NPV(t) = \mathbb{E}_t[\Pi(t, T)]$, where $NPV$ is the net present value. One can therefore consider that the fair value, without credit risk, of this portfolio of trades for counterparty $B$ is worth $NPV(t)$. Many derivatives’ pricing models were developed from this premise of no credit risk of both counterparties.

From a practical standpoint, however, the assumption of portfolios without credit risk is unrealistic, especially in the context of more complex operations such as the case of derivatives’ transactions. Thus, although we assume that bank $B$ has zero probability of going bankrupt, i.e., is default-free, client $C$ can go bankrupt. This assumption of the asymmetry of credit risk, whilst unreasonable, was usual in models previous to the 2007/2008 crisis, since it was considered that, for instance, AAA dealers were attributed an extremely low probability of not meeting their obligations. In general, the risk that client $C$ was exposed to the credit quality was considered negligible. The bankruptcy of several high-reputation international banks during the subprime crisis showed that even banks with AAA ratings could have difficulty in paying debts to their counterparties.

For modelling purposes, we consider that, as discussed, only bank $B$ is exposed to the credit risk of the counterparty. We will analyse the value of a portfolio $P$ from the point of view of the bank, with a possible default of the counterparty $C$ in instant $\tau_C$. Initially, two scenarios can be analysed: (1) the default occurs in $\tau_C > T$ or (2) the default occurs in $\tau_C \leq T$, as in Brigo et al. (2013b).

In the first case (1), the portfolio cash flows would not be affected by the default that would occur only after the expiration of the contract with higher maturity and therefore the evaluation follows the structure of a no counterparty credit risk loan portfolio given by $\Pi(t, T)$.

In the second case (2), when the event of default occurs before maturity or at maturity $T$, the result of $B$ is associated with the present value of the flows that occur until default $\Pi(t, \tau_C)$ and the amount to be paid or received due to the mark-to-market value of the portfolio. In this context, two situations should be considered (Brigo et al., 2013b):
1. If the portfolio value is negative, i.e., if $NPV(\tau_C) < 0$, then counterparty $B$ is in a losing position and pays cash flows to counterparty $C$, even if $C$ has defaulted and has not paid its obligations to creditors. The value to be paid by $B$ corresponds to $NPV(\tau_C) < 0$. Note that one of the assumptions of the model implies that $B$ is default-free and will entirely honour its obligation to the counterparty; and

2. If the portfolio value is positive, i.e., if $NPV(\tau_C) \geq 0$, counterparty $B$ receives $RR_C \cdot NPV(\tau_C)$, where $RR_C$ represents the recovery rate.

Combining the various scenarios of the net cash flows in relation to the default of $C$, then the value of the portfolio of $B$ exposed to credit risk of $C$ can be defined as:

$$
\tilde{\Pi}(t, T) = \mathbb{I}_{(\tau_c > T)}\Pi(t, T) + \mathbb{I}_{(t < \tau_C \leq T)}\left[\Pi(t, \tau_C) + D(t, \tau_C)\left(\min(NPV(\tau_C), 0) + RR_C \cdot \max(NPV(\tau_C), 0)\right)\right].
$$

(1)

According to the Basel Committee, for the specific case of a derivative contract, the $CVA$ is typically defined as the difference between the value of the product assuming that the counterparty does not have default risk and the value of the product subject to counterparty default risk (BCBS, 2011a).

Let us now examine the definition of the $CVA$, following Brigo et al. (2013b). Let $\Pi(t, T)$ and $\tilde{\Pi}(t, T)$ be the net present values of the cash flows, measured at time $t$, of a portfolio maturing at $T$ negotiated with a counterparty without credit risk and with credit risk, respectively. The $CVA$, i.e., the value of the portfolio value adjustment, depending on the counterparty credit risk, is given by the expected value of the difference between $\Pi(t, T)$ and $\tilde{\Pi}(t, T)$:

$$
CVA_C = \mathbb{E}_t[\Pi(t, T) - \tilde{\Pi}(t, T)].
$$

(2)

From the definition in Eq. 2, one can get the unilateral $CVA$.

**Proposition 2.1.** The credit value adjustment when only one of counterparties has credit risk is given by:

$$
CVA_C^B = \mathbb{E}_t[\mathbb{I}_{(t < \tau_C \leq T)} \cdot LGD_C \cdot EAD_C],
$$

(3)

where,

$LGD_C = 1 - RR_C$ is the loss given default (LGD); and,

$EAD_C = (NPV(\tau_C))^+$ is the exposure to counterparty credit risk in the instant of default (EAD).

**Proof.** See supplementary material detailing the discussion in Brigo et al. (2013b).
If we consider that the counterparty credit exposure, the loss given default, and the instant of default are independent and establishing $\mu_{PD_C} = \mathbb{E}[\mathbb{1}_{t < \tau_C \leq T}]$, $\mu_{LGD_C} = \mathbb{E}_t[LDG_C]$ and $\mu_{EAD_C} = \mathbb{E}_t[EAD_C]$, then the adjustment due to credit risk can be rewritten, in a simple way, as the product of the mean values of $PD$, $LGD$ and $EAD$:

$$CVA_B^C = \mu_{PD_C} \cdot \mu_{LGD_C} \cdot \mu_{EAD_C}. \quad (4)$$

In Basel II, although the primary concern is not with the pricing of counterparty credit risk, $PD$, $LGD$, and $EAD$ are called credit risk components and are used to estimate losses in traditional credit portfolios composed of loans and financing operations. Those components are fundamental for the calculation of regulatory capital requirements – for example, expected losses ($EL$) and unexpected losses ($UL$). It is important to notice that, in Basel II, a similar formula to Eq. 4 is used to estimate the expected loss of traditional credit portfolios.

The definition of the $CVA$ given by Eq. 3, although it is more complex, since it involves portfolios of products that may have different characteristics, follows the logic of expected loss. In this context, the $CVA$ is an estimate of expected losses calculated from the probability of the counterparty defaulting, the actual loss in the case of default, assuming that there is a certain recovery rate, and the value that an agent has to receive from the counterparty which defaulted.

### 2.2. Credit and equity risk tail dependence

$WWR$ in a bivariate factor setting is related to the lower-lower tail risk of two factors being greater than the initial estimates. Nevertheless, we are interested in other multivariate dependence studies, for example, the dependence between credit and equity markets. For this purpose we use a Markov two-regime switching model (Hamilton, 1989) to assess the dependence between $PD$, $EAD$, and the stock market.

Vassalou and Xing (2004) used the Merton (1974) structural model to find evidence of the relationship between equity returns and default risks; their results suggest that a separation of the dataset into a two-regime equity could be important for the calculation of the $CVA$. Ang and Chen (2002) and Longin and Solnik (2001) found that there exist asymmetries between the bull and bear stock markets when the upper-upper (bull) and lower-lower (bear) tail returns correlations of stocks and the market are tested. Furthermore, Ang and Bekaert (2002) studied the impact of diversification in a Markov two-regime (Bull vs. bear) setting of (i) all-equity portfolios and (ii) equity plus a conditional risk-free asset, finding that when higher lower-lower tail dependence is considered only portfolios with the conditional risk-free asset have a significant difference, as compared to those portfolios that dismiss the existence of a two-regime setting.

In our study we will divide the data into two regimes: (i) a regime considered to be “normal” with average mean-volatilities, (ii) a regime considered to be of “crisis” with different mean to model asymmetry and higher volatility. Let $y_t$ be the stock univariate process, $x_{i,t}$ be the explanatory variables, $x_{i,t}^{nS}$ be the subset
of $x_{i,t}$ that has non-switching variables, $x_{i,t}^{nS}$ be the subset of $x_{i,t}$ with the switching variables, $S_t \in \{1, \ldots, k\}$ be a variable that represents the state at time $t$, and $\beta_i, \phi_{j,S_t}$ be the parameters; a Markov regime switching process can be defined as:

$$y_t = \sum_{i=1}^{N_n} \beta_i x_{i,t}^{nS} + \sum_{j=1}^{N_s} \phi_{j,S_t} x_{j,t}^S + \epsilon_t,$$

(5)

$$\epsilon_t \sim P(\Phi_{S_t}),$$

(6)

where $P(\Phi_{S_t})$ is the distribution assumed for the innovations. Consider a two-regime model with differences in mean and variance between the regimes, with Gaussian error innovations, then $k = 2$ and Eq. 5 reduce to:

$$y_t = \beta_{1,S_t} x_{1,t} + \beta_{2,S_t} x_{2,t} + \epsilon_t,$$

(7)

$$\epsilon_t \sim N(0, \sigma_{S_t}^2).$$

(8)

To implement the Markov two-regime switching model we used the MATLAB MS_Regress® toolbox from Perlin (2010). Once the data has been divided into the two regimes, we tested the PD and the EAD dependence on each regime.

3. Pricing of swaps exposed to counterparty credit risk

As already discussed, the CVA in Eq. 3 includes the model from Basel III as a special case. The CVA in Eq. 3 also incorporates, as a particular case, the seminal model of Sorensen and Bollier (1994), which analysed the pricing of swaps subject to counterparty credit risk. Despite the fact that the CVA can be applied to portfolios of financial products traded with a certain counterparty, the modelling of individual transactions is relevant as it allows us to focus on the pricing of specific operations, without any influence from other products in the portfolio.

For example, derivatives’ portfolios commonly have trades that may have cash flows in opposite directions, market-to-market from netted positions. Thus, an interest rate derivative may have cash flows that match those of a commodity derivative, traded with the same counterparty in the over-the-counter (OTC) market. From the point of view of counterparty credit risk analysis, the cash flows from these trades can be grouped together. In addition, various market risk factors can be interrelated, implying a diversification effect, which in turn can mitigate some of the counterparty credit risk. However, from the perspective of analysis of the derivative itself, each product can be viewed individually. Thus, the analysis of specific products is also relevant for pricing derivatives’ products subject to the credit risk of the counterparty, as the sum of the
CVAs of the portfolio may differ from the sum of the CVAs of the derivatives that comprise the portfolio.

### 3.1. Swap analysis

We will focus our analysis of the $CVA$ on traditional swaps, taking into account the discussion in Cherubini (2013). We will initially consider bank $B$ holding a long/short position in the swap. As discussed in Section 2, we will establish, for example, that bank $B$ does not impose credit risk on its counterparty $C$.

However, client $C$, i.e., the counterparty of bank $B$, may go bankrupt.

Without loss of generality, the notional value is defined as 1 monetary unit. Following Cherubini (2013), the net present value \( NPV_B(t) \), measured in $t$, associated with bank $B$’s cash flows is given by:

\[
NPV_B(t) = \sum_{i=1}^{n} D(t, t_i) \delta_i (f(t, t_{i-1}, t_i) - r_s(t_0, t_n)),
\]

(9)

where:

- \( D(t, t_i) \) is the discount factor of cash flows in a risk-neutral world;
- \( \delta_i = t_i - t_{i-1} \) is the difference between the tenor of two subsequent cash flows;
- \( r_s(t_0, t_n) \), denoted by the swap rate, is the spot rate that will be applied to the fixed leg of the swap; and
- \( f(t, t_{i-1}, t_i) \) is the forward rate, established in $t$, related to the spot rate that will be be applied to cash flows in $t_{i-1}$ with maturity in $t_i$.

Note that the swap rates $r_s$ and the forward rates $f$, as already indicated, are given in the same unit of time horizons as $t$. At the time of the trade $t_0$, the swap should reflect a balance between the expectations of accrued interest through a fixed rate $r_s$ and forward rates $f$ until the maturity of the operation. Thus, the swap rates $r_s(t_0, t_n)$ is the fixed rate that corrects the notional value each period, remaining constant between the whole interval from the moment of the operation $t_0$ until maturity $t_n$. The forward rates $f(t_0, t_{i-1}, t_i)$ represent the interest rates that the market, at instant $t_0$, considers appropriate to be in place from $t_{i-1}$ for the period $\delta_i$, taking into account the terms of the swap.

**Proposition 3.1.** Under a no-arbitrage assumption, the $CVA$ for bank $B$, with a long position in the swap, and exposed by counterparty credit risk of $C$, is given by:

\[ CVA_B(t) = LGD_C \cdot \sum_{j=1}^{n} (S_C(t_j) - S_C(t_{j-1}))D(t, t_j)EE_j. \]

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8 The model in Eq. 10 also incorporates the Basel guidelines. Cherubini (2013) shows that if we define \( EE_j = \frac{A(t_j, t_{j-1})}{D(t_j, t_{j-1})} = Q[\max(r_s(t_j, t_{j-1})), 0] \), then, \( CVA_B(t) = LGD_C \cdot \sum_{j=1}^{n} (S_C(t_j) - S_C(t_{j-1}))D(t, t_j)EE_j. \)

Assuming a discretisation that considers the exposure as an average between two periods, \((EE_{j-1} + EE_j)/2\), one can obtain the counterparty credit risk indicated by the Basel Committee. Thus, the model from Cherubini (2013) also includes other simpler definitions of $CVA$.

9 The important contribution from Cherubini (2013) for the calculation of the $CVA$ involves representation of a joint probability using copulas, which allows an analysis of many other derivatives’ products and the study of the relationships among

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9
Proof. See supplementary material, detailing the discussion in Cherubini (2013).

\[ CVA^B_C(t) = LGDC \cdot \sum_{j=1}^n A(t, t_j, t_n) \cdot \int_{r_s(t_0, t_n)}^\infty \tilde{C}(1 - Q(u), S_C(t_{j-1}) - S_C(t_j)) \, du. \] (10)

3.2. Analysis of dependence

The model in Eq. 10, defined using copulas, can be used to analyse several relationships between the probability of default defined by \((S_C(t_{j-1}) - S_C(t_j))\) and the exposure at default defined by \(\int_{r_s(t_0, t_n)}^\infty (1 - Q(u)) \, du\). Considering initially the instant of default independent of the exposure at default, then the copula function to be used is \(\tilde{C}(u, v) = uv\).

Since \(\int_{r_s(t_0, t_n)}^\infty \tilde{Q}(u) \, du = EQ[\max(r_s(t_j, t_n) - r_s(t_0, t_n), 0)]\), with \(\tilde{Q} = 1 - Q\), the expectation represents the price of an option on a swap, since the counterparty would have the right to get into a swap in favourable market conditions. Then the adjustment related to counterparty credit risk is given by:

\[ CVA^B_C(t) = LGDC \cdot \sum_{j=1}^n (S_C(t_{j-1}) - S_C(t_j))SwOp(t, t_j, r_s(t_0, t_n)), \] (11)

where \(SwOp(t, t_j, r_s(t_0, t_n))\) is the price in \(t\) of an option on a swap, in which bank \(B\) has the right to go long in a swap at \(t_j\), paying a fixed rate \(r_s(t_0, t_n)\).

As noted by Cherubini (2013), this result is analogous to the work of Sorensen and Bollier (1994), which is one of the first and most influential papers on the counterparty credit risk of derivatives. According to Sorensen and Bollier (1994), the exposure of counterparty \(B\) to the credit risk of counterparty \(C\) in a swap contract is equivalent to a series of European style options on swaps that are exercised by \(B\) to get reimbursed from losses due to the default of \(C\). It is important to emphasise that the model from Sorensen and Bollier (1994) works under conditions of independence among the credit risk components and aggregates \(LGD\) and \(EAD\) in a single element.

Assuming now that there is dependence among the components of counterparty credit risk, more particularly, between \(PD\) and \(DL\), considering that exposure may depend on the time of default that, in turn, derives from the survival probability of a counterparty in a given time interval. Analysing the \(WWR\) from the perspective of counterparty \(B\) holding a long position in the swap, the exposure to credit risk increases when the value of the swap increases, i.e., the greater \(r_s(t_n)\) compared to \(r_s(t_0, t_n)\), the higher the value of the position for the bank, but the higher the credit risk to which the bank is exposed.

Following Cherubini (2013), \(Q(u) = P(r_s(t_0, t_n) \leq u)\) is the risk-neutral probability of the swap rate being less than a given value \(u\) and considering an average default intensity for each term \(t_j\) given by \(\lambda_t = s_i/LGD\), with survival probability \(t_i\) defined by \(S(t_i) = \exp(-\lambda t_i)\), then: \(P(r_s(t_0, t_n) > u) \leq t_j \leq r_{C} = \tilde{C}(1 - Q(u), S_C(t_{j-1}) - S_C(t_j))\), where:

\(\tilde{C}(1 - z, v)\) is the copula function associated with the joint event of the swap rate in a given instant \(t\) being higher than the swap rate in the trade date \(t_0\) and the counterparty \(C\) getting into default in an instant between \(t_{j-1}\) and \(t_j\) (Cherubini, 2013).
A situation of WWR involves, for the buyer of the swap, a scenario in which the interest rate increases, implying greater exposure to credit risk and, at the same time, a higher likelihood of default, given by the difference between the survival probability of $C$ between the two periods. Considering an extreme case, defined by the copula $\tilde{C}(u, v) = \min(u, v)$, following Cherubini (2013):

$$\tilde{C}(1 - Q(u), S_C(t_{j-1}) - S_C(t_j)) = \min(1 - Q(u), S_C(t_{j-1}) - S_C(t_j))$$  \hspace{1cm} (12)

**Proposition 3.2.** Consider a perfect dependence between the exposure and the default of the counterparty. In the worst case scenario (WWR) the CVA of a long position in a interest rate swap is (Cherubini, 2013):

$$CVA^B_C = LGD_C \cdot \sum_{j=1}^{n} \left( \max(k(t_j) - r_s(t_0, t_n), 0) (S_C(t_{j-1}) - S_C(t_j)) 
+ SwOp_P(t, t_j, \max(r_s(t_0, t_n), k(t_j))) \right), \hspace{1cm} (13)$$

where:

$SwOp_P(\cdot)$ represents the value of a payer swaption, in which one has the right, but not the obligation, to get into a swap paying a fixed rate and receiving a floating rate,

and the worst case scenario CVA of a a short position in a interest rate swap is:

$$CVA^S_C = LGD_C \cdot \sum_{j=1}^{n} \left( \min(r_s(t_0, t_n) - k^*(t_j), 0) (S_C(t_{j-1}) - S_C(t_j)) 
+ SwOp_R(t, t_j, \min(r_s(t_0, t_n), k^*(t_j))) \right), \hspace{1cm} (14)$$

where:

$SwOp_R(\cdot)$ represents the price of a receiver swaption, in which the investor has the option to get into a swap paying a floating rate and receiving a fixed rate.

**Proof.** See supplementary material, detailing the discussion in Cherubini (2013).

Thus, through the use of extreme copulas, the model of Cherubini (2013) enables the study of a swap price considering the credit risk of the counterparty, including situations of dependence or independence between the probability of default and exposure at default.
An additional dependency model is tested. We replaced the copula in Eq. 12 by a Gaussian copula with the purpose of testing different dependency models:

$$\tilde{C}_G(1 - \mathbb{Q}(u), S_C(t_{j-1}) - S_C(t_j)) = \Phi \left( \Phi^{-1}(1 - \mathbb{Q}(u))^{-1}, \Phi^{-1}(S_C(t_{j-1}) - S_C(t_j)) \right)$$

(15)

Using the Proposition 3.2, it is straightforward to approximate the CVA of a long position interest rate swap position with a Gaussian copula for the dependence between the EAD and the PD by (Cherubini, 2013):

$$CV^A_C = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \cdot \Phi \left( \Phi^{-1}(S_C(t_{j-1}) - S_C(t_j)), \Phi^{-1}(SwOp_R(t, t_j, r_s(t_0, t_n))) \right),$$

(16)

and CVA for a short position in the interest rate swap by:

$$CV^G_B = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \cdot \Phi \left( \Phi^{-1}(S_C(t_{j-1}) - S_C(t_j)), \Phi^{-1}(SwOp_P(t, t_j, r_s(t_0, t_n))) \right),$$

(17)

3.3. Pricing of swaptions

In order to apply the model from Cherubini (2013) for swaption pricing, we will briefly describe valuation methods for bonds, swaps, and options on swaps, using interest rates diffusion processes. Let $r(t)$ be an interest rate diffusion process. We consider a general formula for the pricing of a zero coupon bond, with unity face value at maturity. Then the price of the zero coupon bond is:

$$P(t, T) = \mathbb{E}_t \left[ \exp \left( - \int_t^T r(s) ds \right) \right].$$

(18)

Following Brigo and Mercurio (2006), Eq. 18 can be written as:

$$P(t, T) = A_1(t, T) \exp(-B_1(t, T)r(t)),$$

(19)

where:

$$A_1(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp\left[ B_1(t, T) f^M(0, t) - \frac{\sigma^2}{2\alpha}(1 - \exp(-2\alpha t))B_1(t, T)^2 \right];$$

$$B_1(t, T) = \frac{1}{a}[1 - \exp(-a(T - t))];$$

$P^M(0, x)$ is the discount factor for a cashflow in $x$, obtained from market data; and

$f^M(0, x)$ is the instantaneous interest rate observed in the market in 0 with expiration in $T$. 

12
Considering Eq. 13 and Eq. 14, the CVA depends on the price of swaptions. Then the payer swaption and the receiver swaption prices are given, respectively, by (Brigo and Mercurio, 2006): 

\[
\text{SwOp}_P(t, T, X) = \sum_{i=1}^{n} c_i \text{Put}(t, T, t_i, X_i),
\]

\[
\text{SwOp}_R(t, T, X) = \sum_{i=1}^{n} c_i \text{Call}(t, T, t_i, X_i),
\]

where:

\[c_i = X \delta_i, \text{ for } i = 1, \ldots, n \text{ with } \delta_i = t_i - t_{i-1}.
\]

The values in \( t \) of European Call and Put options with strike price \( X \), and maturity \( T \), associated with a zero-coupon bond as the underlying asset which expires in \( S \), are given, according to Brigo and Mercurio (2006), respectively by:

\[
\text{Call}(t, T, S, X) = P(t, S) \Phi(h) - XP(t, T) \Phi(h - \sigma_p),
\]

\[
\text{Put}(t, T, S, X) = XP(t, T) \Phi(h + \sigma_p) - P(t, S) \Phi(-h),
\]

where:

\[
\sigma_p = \sqrt{\frac{1-\exp(-2a(T-t))}{2a}} B(T, S)
\]

\[
h = \frac{1}{\sigma_p} \ln\left(\frac{P(t, S)}{P(t, T) X}\right) + \frac{\sigma_p}{2}
\]

To calibrate parameters and price these options, we also need data of derivatives on swaps. Regarding the swap pricing procedure, we use a local volatility model, in which parameters \( k(t_j) \) and \( k^*(t_j) \) are calculated as follows (Cherubini, 2013):

\[
k(t_j) = \text{fsr}(t, t_j, t_n) \cdot \exp\left[-\frac{\sigma_{j,n}}{2} - \Phi^{-1}(S_C(t_{j-1}) - S_C(t_j)) \sigma_{j,n} \sqrt{t_j - t}\right],
\]

\[
k^*(t_j) = \text{fsr}(t, t_j, t_n) \cdot \exp\left[-\frac{\sigma_{j,n}}{2} + \Phi^{-1}(S_C(t_{j-1}) - S_C(t_j)) \sigma_{j,n} \sqrt{t_j - t}\right],
\]

where:

\[
\text{fsr}(t, t_j, t_n) \text{ is the forward swap rate, i.e., the rate defined in instant } t \text{ for a swap beginning in } t_j \text{ and expiration in } t_n;
\]
4. Description of interest rate models

The counterparty credit risk in interest rate swap transactions depends on the diffusion process of the relevant risk factors. Thus, the assessment of the CVA is influenced not only by choosing the form of dependence between the risk components \(PD, LGD,\) and \(EAD\) but also by the model used to express the dynamics of interest rates. According to Buetow et al. (2001), there are two distinct approaches to the structuring of stochastic differential equations to model the behaviour of interest rates: (a) equilibrium models and (b) no-arbitrage models.

The equilibrium models seek to establish mechanisms of pricing of debt securities under an analytical framework based on market equilibrium, which specifies the market price of risk (Buetow et al., 2001; Vasicek, 2007). Some equilibrium models are traditionally studied, such as Vasicek (1977), Brennan and Schwartz (1979), Cox et al. (1985), Longstaff (1989), Longstaff (1992), and Dai and Singleton (2003).

In contrast, no-arbitrage models use market prices to generate a grid of possible values of interest rates, by which the theoretical price of a debt security is equal to the price observed in the market. Traditional models of non-arbitration are Ho and Lee (1986), Black et al. (1990), Hull and White (1990), Black and Karasinski (1991), Heath et al. (1992), and Hull and White (1993). Importantly, no-arbitrage models allow parameters to be calibrated according to the prices traded in the market, and therefore offer greater comparability to values traded by practitioners.

The CVA calculated in Eq. 10 establishes a model in which the instant of default, associated with a survival function and therefore with the probability of default \(PD\), can be studied jointly with the exposure at default \(EAD\).

We now calculate the CVA of an interest rate swap, using interest rates and volatilities’ market data. We focus on the analysis of no-arbitrage interest rate models. To model the diffusion process of interest rate \(r\), we will use three one-factor short-term interest rate models: Black et al. (1990), Hull and White (1990), and Black and Karasinski (1991), the multifactor interest rate model of Heath et al. (1992), the two-additive-factor interest rate model of Brigo and Mercurio (2003) known as \(G2++\), the stochastic volatility model of Hagan et al. (2002) better known as \(SABR\), and the \textsc{LIBOR} market model of Brace et al. (1997).

4.1. One-factor short-term interest rate models

The model from Black et al. (1990), although initially defined for a discrete analysis, can be adjusted to incorporate a continuous stochastic differential equation:

\[
d\ln(r(t)) = \Theta(t)\, dt + \sigma(t)\, dW(t),
\]
where:

- \( r(t) \) is the instantaneous interest rate;
- \( \Theta(t) \) is the mean reverting equilibrium;
- \( \sigma(t) \) is the instantaneous short rate volatility; and
- \( W(t) \) is a Brownian motion.

Hull and White (1990) and Hull and White (1994b) present an extension of the Vasicek (1977) model, in which interest rate dynamics are given by:

\[
 dr(t) = \left[ \vartheta(t) - \alpha(t)r(t) \right] dt + \sigma(t) dW(t),
\]  

(25)

where:

- \( \vartheta(t) \) is the mean reverting equilibrium; and
- \( \alpha(t) \) is the rate on which \( r(t) \) reverts to \( \vartheta(t) \).

Black and Karasinski (1991) established that the dynamics of the spot rates follow a generalised model from Black et al. (1990):

\[
 d \ln(r(t)) = \left[ \theta(t) - a(t) \ln(r(t)) \right] dt + \sigma(t) dW(t),
\]  

(26)

where:

- \( \theta(t) \) is the mean reverting equilibrium; and
- \( a(t) \) is the rate on which \( \ln(r(t)) \) reverts to \( \vartheta(t) \).

Another expression for this model is:

\[
 d \ln(r) = a(t) \left[ \ln(\mu(t)) - \ln(r(t)) \right] dt + \sigma(t) dW(t),
\]  

(27)

with \( \ln(\mu(t)) \) is the mean reverting level; if \( \log(r) > \log(\mu(t)) \) is superior then the interest rate will decrease, and if \( \log(r) < \log(\mu(t)) \) then the interest rate will increase. The resulting interest rate will have a lognormal distribution, that is the continuous time limit of the Black et al. (1990) model. Binomial and trinomial trees will be used to price the short rate models.
The model from Hull and White (1990) can be calibrated to the term structure of interest rates and to the spot or forward term structure of volatility. However, a perfect adjustment to the term structure of interest rates may imply problems in the adjustment to the volatility term structure, since not all volatilities extracted from market prices are relevant, due to the lack of liquidity of financial products and to the fact that the future volatilities’ term structure in Eq. 25 may not follow a realistic traditional shape (Carverhill, 1995; Hull and White, 1995; Brigo and Mercurio, 2006).

4.2. Heath, Jarrow, and Morton (1992) model

Taking into account the general model, given by Eq. 25, Heath et al. (1992) developed an analysis that follows Hull and White (1990) and Hull and White (1994a), using an extension of the Vasicek (1977) model in which $\alpha$ and $\sigma$ are positive constants and $\vartheta$ can be calibrated to adjust to the implicit market interest rates.

In contrast with the previous models, the interest rate dynamics in Heath et al. (1992) follow the diffusion process, given a maturity $T$:

$$
\frac{df(t, T)}{dt} = \alpha(t, T) dt + \sigma(t, T) dW(t),
$$

with $f(0, T) = f_M(0, T)$ representing the term structure of interest rates given by the market price in $t = 0$, where:

$W = (W_1, \cdots, W_N)$ follows a Brownian motion of $N$ dimensions;

$\sigma(t, T) = (\sigma_1(t, T), \cdots, \sigma_N(t, T))$ is a vector of adaptive processes related to volatilities; and

$\alpha(t, T)$ is an adaptive process, chosen from the vector of volatilities $\sigma$ and the rate of evolution of the dynamic of the $N$ zero coupon bonds selected to the pricing model.

Instead of modelling the short-term interest rate, Heath et al. (1992) use the whole interest rate term structure information by modelling the instantaneous forward rate curve. The forward rate curve includes information of the future rates that eventually will become the short rate at some point in time.

4.3. Linear Gaussian two-additive-factor (G2++) model

Given the widespread use of the Hull and White (1993) model, Brigo and Mercurio (2003) developed an extension with two linear Gaussian factors to overcome certain smile modelling limitations without losing the simplicity of the model. Let $x_g(t), y_g(t)$ be two Gaussian factors, the interest rate evolution is denoted by:

$$
r(t) = x_g(t) + y_g(t) + \phi(t),
$$

$$
\frac{dx_g(t)}{dt} = -a(t)x_g(t) dt + \sigma(t) dW_1(t),
$$

$$
\frac{dy_g(t)}{dt} = -b(t)y_g(t) dt + \eta(t) dW_2(t),
$$

16
with \( x_g(0) = y_g(0) = 0 \), where \( \sigma(t) \) and \( \eta(t) \) are the corresponding volatilities associated with the factors \( x_g(t), y_g(t) \), respectively. The G2++ model is highly tractable and we will provide some of the swaption’s calculations for measuring counterparty credit risk CVA using a G2++ analytical formula. We also provide swaptions calculations using a numerical simulation method.

4.4. LIBOR market model

Interest rate market models were created to match the prices observed in some interest rate derivatives such as caps and swaptions. These derivatives are priced by practitioners with the Black and Scholes (1973); Black (1976) formulae, as in the Black and Karasinski (1991) model. The forward-LIBOR (London Interbank Offered Rate) market model, or just the LIBOR model, was developed by Brace et al. (1997) as an extension of Heath et al. (1992) when modelling simple but observable \( M \) forward rates \( f_i(t), i \in \{1, \ldots, M\} \). Let \( f_i(t) \) represent the multivariate Gaussian process of the forward rates between \( t_i - 1 \) and \( t_i \), \( \sigma_i(t) \) the instantaneous volatility of \( f_i(t) \), \( \tau_i = t_i - t_{i-1} \), and \( dW_i(t) dW_j(t) = \rho_{i,j} \), then the Gaussian LIBOR market model is defined by:

\[
\frac{d f_i(t)}{f_i(t)} = \xi \sigma_i(t) d \sum_{k=I_B}^{I_E} \frac{\rho_{i,k} \tau_k \sigma_k(t)f_k(t)}{1 + \tau_k f_k(t)} \, dt + \sigma_i(t) dW_i(t),
\]

(30)

where

\[
I_B = i + 1, I_E = j, \xi = 1, i < j,
\]

\[
\xi = 0, i = j,
\]

\[
I_B = j + 1, I_E = i, \xi = -1, i > j.
\]

LIBOR market models are priced using a numerical simulation.

4.5. Stochastic alpha, beta, rho (SABR) market model

The SABR model developed by Hagan et al. (2002) is an extension of the market models where the forward rates are modelled using a modified constant elasticity of variance (CEV) diffusion with stochastic volatility. Three parameters define this model: \( \alpha \) represents the vol-vol or volatility of the volatility, \( \beta \) represents a leverage effect of increasing volatility with falling prices (and vice versa), and \( \rho \) the correlation between the forward rate diffusion and the volatility diffusion; the SABR model can then be defined by:
Numerical simulations will be used to price swaptions under the LIBOR model.

5. Data description and estimation procedure

For the research we used different datasets. The first dataset is used for calculating the zero-coupon interest rate term structure.

5.1. Zero-coupon interest rate term structure estimation

To determine the implied zero coupon rates we use daily closing prices data from the US LIBOR from overnight up to 6 months for the short-term part of the curve, and US interest rate swap markets for maturities that range from 1 year up to 40 years for the medium- to the long-term part of the term structure. LIBOR represents the average interest rate at which the banks in London will lend between them in American dollars. As the fixed-income markets are in general Over-The-Counter (OTC), then we use the data provided by the Bloomberg® platform. A detailed description of the instruments is in Tab. 1.

The data span from 10 May 2005 to 29 January 2015 (Fig. 1). Maturities for the selected LIBOR instruments range from spot (overnight rate) to 6 months. Swap rates are provided as the implicit rates from the interest rate swap between two counterparties on which the payer will have to pay a fixed rate and will receive the spot LIBOR rate. Bloomberg® swap rates are calculated from the Treasury bonds’ mid-prices and the quoted swap spreads.

Discount factors and forward rates are estimated from the zero-coupon rates (swap rates). With discount factors and forward rates, the fixed rate of the interest rate swap (the swaption strike price) can be calculated. Similarly, forward swap rates (FSR) with an specific tenor (2, 5, and 10) are calculated using discount factors and forward rates. Finally, the interest rate term structure is estimated using a Nelson and Siegel (1987) curve fitting, that will be useful for extrapolating the rates of intermediate maturities (Fig. 2).

\[
\frac{d f_i(t)}{f_i(t)^\beta} = \sigma(t) d W_i(t), \quad (31)
\]

\[
d \sigma(t) = \alpha \sigma(t) d Z_i(t),
\]

\[
d W_i(t) d Z(t) = \rho d t.
\]
5.2. *Implied risk-neutral volatilities estimation*

The second dataset used in this research is the risk-neutral interest rate volatility, extracted from the caps/floors volatility (Hagan and Konikov, 2004). This risk-neutral market volatility defined by the Black (1976) volatility surface is used for the calibration of the different interest rate models utilized in pricing the CVA. The volatility is extracted for the caps/floors with strikes of 1%, 2%, 3%, 4%, 5%, 6%, 7%, 8%, 9%, 11%, 12%, 13%, and 14%, with maturities that range from 2 up to 10 years. Fig. 4 shows the market caps/floors volatility for the 1% and 5% strikes. The data were collected for the period from 10 May 2005 to 1 August 2013. In Fig. 3 we observe the volatility surface for 10 May 2005.\(^\text{10}\)

5.3. *Interest rate models calibration*

The interest rate models are priced using binomial and trinomial tree numerical methods. The Black caps/floors volatility and Black cap prices are used for estimating the parameters of the interest rate models. In the case of the Black et al. (1990) model, \(\Theta(t), \sigma(t)\) are the parameters to be estimated. \(\Theta(t)\) is calibrated with the zero-coupon interest rate term structure and \(\sigma(t)\) is calibrated with the Black cap prices using a least squares method, implemented in the function `capbybdt()` of MATLAB. In the case of the Hull and White (1990) model, the parameter \(\vartheta(t)\) is calibrated with the zero-coupon interest rate term structure and \(\alpha(t), \sigma(t)\) parameters are estimated from the Black prices by a least squares method implemented by the function `hwcalbycap()` of MATLAB. The Heath et al. (1992) model has parameters \(\sigma(t, T), \alpha(t, T)\) that are calibrated with the zero-coupon interest rate term structure by the least squares method using the function `capbyhjm()` of MATLAB.

\(^{10}\)Duyvesteyn and de Zwart (2015) calculate the volatility risk premium of the interest rate term structures by constructing synthetic derivatives (straddles) of the interest rate swaptions in the risk-neutral measure. Their conclusions will be important when considering our results for hedging in the physical measure.
6. Results

Let us now analyse the empirical calibrated CVA of swaps, using the models of Black–Derman–Toy (BDT; Black et al., 1990), Black–Karasinski (BK; Black and Karasinski, 1991), Hull–White (HW; Hull and White, 1993), Heath–Jarrow–Morton (HJM; Heath et al., 1992), G2++ (Brigo and Mercurio, 2003), LIBOR (Brace et al., 1997), and SABR (Hagan et al., 2002).

To obtain the term structure of interest rates we use daily data from LIBOR rates of up to 6 months to short term and the swap rates in the US market, with maturities between 1 and 50 years for the medium and long term. The interest rates for given maturities are interpolated. References on theoretical and empirical aspects concerning the implementation and calibration of interest rate models can be found in Bjerksund and Stensland (1996), Boyle et al. (2001), Hagan and Konikov (2004), Leippold and Wiener (2004), Galluccio et al. (2007), La Chioma and Piccoli (2007), and Keller-Ressel et al. (2012).

6.1. Empirical risk-neutral calibrated volatility

Fig. 5 shows the calibrated volatility obtained by using each of the BDT, HW, BK, and HJM interest rate models. One can observe a large difference in volatility estimates obtained from distinct models of interest rates. Thus, the calibration of parameters from market data is very sensitive to the model used and can have a considerable impact on the calculation of the CVA.\textsuperscript{11} HW and HJM calibrated volatilities seem to peak during the crisis near September 2008, and steadily decrease after then to the levels of 2005. BDT and BK calibrated volatilities seem to have increased since mid-2007 until recently. BDT and BK volatility behaviour is the result of the log prices modelling of interest rates, where recent lower short-term interest rates will have a higher impact than higher interest rates.

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Please insert Fig. 5 here.

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Fig. 6 shows the results of the calibrated volatility obtained with the G2++, LIBOR, and SABR interest rate models. The calibrated volatility of G2++ is composed of two terms: \( \sigma(t) \) the volatility of the first factor that reproduces the volatility of the events of the 2007/2008 financial crisis and seem to stabilise after mid-2009, and \( \eta(t) \) that behave as a counterpart of \( \sigma(t) \) for the second linear factor, showing that the volatility of the second factor decreases during the crisis periods as evidence of a “contagion effect” and high tail dependence, resulting in “only one-factor” behaviour. LIBOR and SABR volatility surfaces are richer in structure and both seem to have increased since 2007/2008. The calibrated volatility surface of the LIBOR model is decreasing for higher maturities (higher for lower maturities); SABR seems to have a

\textsuperscript{11}In the models used, the volatility estimates are derived from at-the-money (ATM) options. However, it is important to note that different strike prices for the interest rate options involve different estimates of implied volatility. Thus, an important element in the calculation of the CVA also involves choosing the correct volatility model.
hump-shaped surface volatility relatively stable during the entire period, but highly volatile in the short end for the 2012/2013 period due to the decisions of the Federal Reserve over the short rate.

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Please insert Fig. 6 here.
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In general, different interest rate models adopt certain characteristics of the volatility cube (Fig. 4) extracted as in Hagan and Konikov (2004). BDT, BK, and LIBOR are more sensitive to the short-term rate and adjust more to the low strike (1%) caps/floors volatility, while HW, HJM, and SABR are balanced sensitive in all maturities and adjust more to the medium strike (5%) caps/floors volatility (Fig. 4).

6.2. WWR CVA results

Fig. 7, 8, and 9 present the 10-year interest rate swap counterparty credit risk CVA in the WWR scenario for the seven different interest rate models using the maximum positive dependence Fréchet upper bound copula. Fig. 7a plots the short-term rate models’ WWR CVA of the interest rate swap long positions that yield around an average of 12% during the May 2005 to August 2013 period and start to increase by the end of 2007, reaching a maximum of 25% during the 2008 crisis and decreasing to an average level of 5% after its peak. BDT is the model that produces the higher WWR CVA estimation and BK is the one that produces the lowest WWR CVA estimation of the three models, all three models having similar behaviour before and after the crisis. In light grey we highlight the “high volatile–larger mean” regime found by applying the Markov two-regime switching model as in Eq. 7 of Section 2.2. We can observe that the two-regime switching accurately explains the 2007/2008 credit crisis and the 2011 Euro zone Greek bailout crisis. Fig. 7a plots the short-term rate models WWR CVA of interest rate swap short positions. In comparison to the WWR CVA of interest rate swap long positions, all three short-term rate models have similar behaviour before, during, and after the crisis starting from 25% before the 2007/2008 crisis and reaching the maximum at 50% shortly afterwards. An explanation is that before the crisis there was a relatively flat interest rate term structure that later got transformed into a stepped term structure after 2008 with the consequence that the interest rate swap short positions paid an extreme low short-term rate and received a higher medium- and long-term interest rate for a long time. The market perceives this anomaly in the behaviour of the interest rates and penalises it with higher counterparty credit risk, with important implications for the Federal Reserve decisions on increasing the systemic risk by holding a low level of the short-term rate.

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Please insert Fig. 7 here.
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Fig. 8 plots the 10-year interest rate swap WWR CVA results for the HJM, LIBOR (LMM), and SABR interest rate models. The WWR CVA of interest rate swap long positions (Fig. 8a) averages 30–35% for the
three multifactor models, from a low level of 10–20% before the 2007/2008 crisis and reaching a maximum
of 45–55% shortly afterwards for HJM and LMM models, and 55% during the crisis for the SABR model.
Similar to the short-term rate models, the behaviour of the WWR CVA interest rate swap long positions
differs between models after the crisis,\textsuperscript{12} since it is the LMM model that produces the higher WWR CVA
estimates. The WWR CVA interest rate swap short position (Fig. 8b) is similar for the three multifactor
models, starting from 12–20% before the financial crisis to reach a maximum of 50% shortly afterwards,
with LMM having the largest WWR CVA estimate. The steady high levels of WWR CVA after the crisis
demonstrate a market premium for the Federal Reserve policy of lower levels for the short-term interest rate,
increasing the interest rate swap short positions’ counterparty credit risk.

Please insert Fig. 8 here.

Fig. 9 shows the 10-year interest rate swap WWR CVA results for the G2++ interest rate model. The
WWR CVA of interest rate swap long positions (Fig. 9a) implied by this model seems to have a two-regime
behaviour: (i) one regime with an average WWR CVA of 10% steady during the during the sample data
period from 10 May 2005 to 1 August 2013, and (ii) one regime with an average WWR CVA of 60–100%
during mid-2005 and during the financial crisis. This is the only model that detects some counterparty risk
distress during the second semester of 2005, and this could be due to the flattening of the interest rate term
structure and the stopping of short-term rates increase by the Federal Reserve, signalling an initial credit
risk distress. The WWR CVA of interest rate swap short positions (Fig. 9b) reveals a similar behaviour to
the WWR CVA of long positions with a two-regime scenario and similar values for each regime. The regime
appears as a result of the exploding G2++ first factor volatility $\sigma(t)$, as shown in Fig. 6a, as the result of a
contagion, and in consequence we observe that only one factor having influence on the model.

Please insert Fig. 9 here.

6.3. CVA tail dependence

The main purpose of our empirical study is to empirically calculate the WWR CVA counterparty credit
risk (Cherubini, 2013) of interest rate swaps and determine differences from earlier independent PD and EAD
modelling (Sorensen and Bollier, 1994), and Gaussian copula dependence modelling for CVA calculations (Li,
2000). The dependency structure of PD and EAD is considered a tail dependency as the event of default

\textsuperscript{12}Feldhütter and Lando (2008) found that the convenience yield is responsible for a great part of the interest rate swap
spreads. It will be interesting to re-apply the study following the 2007/2008 financial crisis to measure the impact of credit risk
and specific factors in comparison with the convenience yield weight in determining the interest rate swap spread.
occurs only in the lower extreme of the life distribution of the institution. Independent CVA, dependent WWR CVA, and Gaussian copula dependent CVA are calculated with Eq. 4, 13, and 16 respectively.

A calculation of the CVA Gaussian copula PD and EAD dependence model requires the estimation of an additional parameter, the time varying correlation. If WWR by definition is the co-monotone variation of the dependent risk factors, then we assume as the worst-case scenario a Gaussian copula correlation of $\rho = 0.9$.\textsuperscript{13}

In Fig. 10 we find the estimated Gaussian copula correlation for the different windows’ size ($N = 7, 8, 9, 10$) used to estimate the empirical marginal densities.\textsuperscript{14} The $\rho(t)$ parameter of the Gaussian copula in Eq. 15 is estimated with the \texttt{copulafit(·)} function of MATLAB that calculates the sample correlation of the inverse \textit{cdf} of the PD and EAD empirical marginal densities. The EAD marginal density is proxied by the average LGD in Tab. 3 of Jacobs (2012) from Moody’s Ultimate LGD Database 1987–2007; the PD marginal density is proxied by the proportion of delisted companies in the CRSP database. Fig. 10 shows an increasing correlation; nevertheless the estimated value never reaches the assumed worst-case scenario of $\rho(t) = 0.9$.\textsuperscript{13}

In a robustness check, we tested with a correlation of $\rho = 0.99$ for the Gaussian copula dependence and the interest rate swaps WWR CVA was still underestimated in all interest rate models.\textsuperscript{14}

Interest rate swap CVA valuations can differ not only because of the dependency structure and the interest rate models used for its calculation, but because of the tenor of the instrument, the direction of the position (long/short), and the conditions of the market (crisis/non-crisis). Tab. 2 presents a panel with the results of the estimated CVA from 10 May 2005 to 1 August 2013 by (i) dependence structure: independent PD and EAD of long interest rate swaps (CVA$^{BL}$), Gaussian copula dependent PD and EAD (CVA$^{BL}_G$), and worst-case scenario WWR (CVA$^{BL}_{WWR}$); (ii) interest rate swap positions, long (CVA$^{BL}$) and short (CVA$^{BS}$); (iii) interest rate model: BDT, HW, BK, HJM, G2++, LMM, and SABR; (iv) interest rate swap tenor, short term (2 years), medium term (5 years), and long term (10 years). In Tab. 3 and 4 we have the same CVA calculations for subsets of (i) non-crisis periods and (ii) crisis periods, respectively, applying the Markov two-regime switching model as in Eq. 7 for dividing the sets. We observe that in Tab. 2, 3, and 4 there is a significant difference when using Gaussian copula and WWR copula dependence vs. the independent modelling of PD and EAD for all the different categories measured.

\textsuperscript{13}In a robustness check, we tested with a correlation of $\rho = 0.99$ for the Gaussian copula dependence and the interest rate swaps WWR CVA was still underestimated in all interest rate models.

\textsuperscript{14}A window size of $N$ will use the previous $N$ year estimates’ frequency as an approximation of the marginal density.
Tab. 5, 6, and 7 present the relative difference from Tab. 2, 3, and 4 of the three tail dependency models for the CVA of interest rate swaps calculations. For the period of 10 May 2005 to 1 August 2013 the Gaussian copula dependence CVA is between 527 and 1609% higher than the independent CVA for any interest rate model, interest rate swap tenor, and instrument position, and the WWR is about 51–362% larger than the Gaussian copula. For the crisis period, although the CVA estimates are higher, the relative difference is lower, and during the non-crisis period the relative difference is higher; this inverse effect could be due to the fact that the distributions of the marginals of the PD and EAD during the crisis have extreme values that co-vary more closely to the Gaussian copula dependence structure; however, the difference remains significant.

The most important factor when measuring the CVA of interest rate swaps was the dependency structure, followed by the tenor and the interest rate model. The results are significant when measuring counterparty credit risk for the actual regulatory capital framework, financial institutions, and risk managers.

### 6.4. Robustness checks

Due to the complexity implied in the valuation of the CVA of interest rate swaps, we conducted several robustness checks; given our space constraints, we summarise them below without full detail:

(i) We find that the estimated LGD varies with time and has a strong correlation with the EAD. We produced a new study on which the default intensity \( \lambda(t_i) \) at time \( t_i \) is modelled by:

\[
\lambda(t_i) = \frac{s(t_i)}{LGD} = 0.4,
\] (32)
where \( s(t_i) \) denotes the credit spread at time \( t_i \). In this new study the ratio of the spread and the \( LGD \) will be constant, introducing a independent factor of a constant default intensity into the CVA calculations to analyse its effect in the tail dependency behaviour. The results demonstrate that the relative difference when modelling tail dependence between \( PD \) and \( EAD \) for the \( CVA \) of interest rate swaps is reduced for the three different dependency models analysed: the estimated \( CVA \) with Gaussian copula dependence is 170–307\% higher than the \( CVA \) with independent \( PD \) and \( EAD \), compared to the original 527–1609\% estimates with time varying default intensity, for the period from 10 May 2005 to 1 August 2013. The relative difference between the \( WWR \) \( CVA \) and the \( CVA \) with Gaussian copula reduces to 0.74–160\% from 51–362\% in the original estimation with time varying default intensity. Notwithstanding, there exist significant differences in all categories between each of the three dependency models, emphasising the importance of tail dependency modelling.

(ii) The Gaussian copula dependence model required to provide a correlation parameter \( \rho(t) \). We initially defined \( \rho(t) = 0.9 \) as this is the largest empirical correlation value found when we estimated the copula correlation (Fig. 10, Section 6.3). In this robustness check, we defined the Gaussian copula constant correlation of \( \rho(t) = 0.99 \). The results demonstrate a significant difference between the \( WWR \) \( CVA \) and the \( CVA \) with Gaussian copula modelling of more than 100\% in many cases.

(iii) We analysed the calculated the independent \( CVA \), Gaussian copula dependent \( CVA \), and the \( WWR \) \( CVA \) surfaces (interest rate swap tenor vs. date) for the seven interest rate models studied, for long and short interest rate swap positions. We further analysed the differences between the \( WWR \) \( CVA \) long minus the \( WWR \) \( CVA \) of short interest rate swap positions. We found no observable difference between the seven interest rate models, with two exceptions: the two-regime behaviour of the \( G2++ \) model, and the difference between the \( WWR \) \( CVA \) of long minus short interest rate swap positions of the \( LIBOR \) model. In the case of the \( G2++ \) model, this behaviour was previously observed and is the result of one-factor contagion dominating the behaviour of the model. In the case of the inverted \( WWR \) \( CVA \) long minus short \( LIBOR \) model behaviour, for all models except \( LIBOR \), extreme low short-term interest rates favour a high premium for the counterparty credit risk of short positions in interest rate swaps holders, as they will receive a positive difference fuelled by the low-rate policy of the Federal Reserve, generating a negative surface for this difference (the \( WWR \) \( CVA \) of short positions is higher for non-market models, as they neglect the market expectations of the forward curve); however, the \( LIBOR \) model accounts for the forward expectations through market prices, and holders of long positions of interest rate swaps have been faced in this model with a steep yield curve since the 2007/2008 crisis.

15 We calculated the mean default rates of Moody’s Caa-C rated companies (Moody’s Investor Service, 2011) yielding 27.68\% with a maximum of 57.89 for 1990; we then defined a default intensity of 0.4 considered an extreme event, with 40\% of the companies defaulting in one year.
This steep yield curve is priced in the premium of counterparty credit risk long positions (the WWR CVA of long positions is higher for market models).

(iv) We tested additional sensitivity analysis for the different parameters of the CVA calculations of interest rate swaps, valued with the HW model. The effects of the local volatility $\sigma(t)$ of the swaption model, the linear combination $\rho$ between the independent and the WWR CVA, the interest rate swap tenor, and shifts in the interest rate yield curve were analysed for different specific values. We found that the volatility increases the magnitude of the CVA; a linear combination that favours the independent model will steep the medium-term CVA curve and reduce the long-term CVA curve; an increase in the tenor will produce a humped CVA curve with higher values in the mid-term portion of the curve; and an interest rate shift will negligible increase the CVA value.

7. Final remarks

In this research we tested the importance of dependency in credit risk factors, such as the probability of default PD and exposure at default EAD. Our approach involves the use of copulas linking two components of credit risk. Considering a dependency between the instant of default and the exposure value and using a copula that establishes a perfect dependence, Cherubini (2013) develops an analytical formula for the calculation of the CVA of interest rate swaps.

The use of copulas allows the study of WWR enabling the incorporation of dependency that was often overlooked, as shown in the simplified formulas described, for example, in Canabarro (2009) and Gregory (2012). In this context, this paper sought to extend the analysis of Cherubini (2013), taking into account different models of interest rate diffusion. Considering that the pricing of swaps depends on the modelling of the term structure of interest rates and their behaviour over time until the maturity of a derivative, the analysis is justified, as different assumptions can have significant impact on the estimates of CVA.

Applying interest rate models from Black et al. (1990), Hull and White (1990), Black and Karasinski (1991), Heath et al. (1992), the G2++ of Brigo and Mercurio (2003), the LIBOR of Brace et al. (1997), and the SABR of Hagan et al. (2002), we calculate the CVA of interest rate swaps using the interest rate swaption implied market volatility for the period of 10 May 2005 to 1 August 2013, and comparatively analyse the estimates of the three different models of credit risk factors dependency for the CVA calculations (independent PD and EAD, Gaussian copula dependence, and WWR copula dependence). The differences between the three different dependency models range from 51% to 1609% of the estimated CVA, revealing a clear underestimation when using regulatory independent models. The results indicate the existence of differences that can impact the price adjustments in portfolios of derivatives, although no interest rate model consistently generates higher or lower values of the CVA, suggesting a difficulty in reducing the capital requirement solely from the choice of the diffusion model of interest rates.
These results have an important consequence for the regulatory institutions and risk managers. From the point of view of capital adequacy, whether there are models that underestimate the CVA can be beneficial for the purpose of capital requirements to meet regulatory agency demands, although it increases the likelihood of non-conformities, which can later result in an increase in capital requirements. From the point of view of risk management, in the case of swaps, the dependence of the CVA on the choice of interest rate, swaptions and volatility models imply the existence of risk modelling. Thus, the analysis of various dependency models is essential for a proper assessment of the CVA, and we suggest as for future research multivariate dependency models that include all credit risk factors.

References


(a) US LIBOR and swap rates surface evolution for different maturities.

(b) US LIBOR and swap rates evolution in time.

Figure 1: This figure shows the LIBOR and swap rates evolution from May 2005 to August 2013. The historical data of LIBOR rates go from spot to 6-month maturities. The swap rates are from 1 year to 40-year maturities. The rates are extracted from Bloomberg®.
Figure 2: This figure shows the interest rate forward term structure calibrated using the Nelson–Siegel–Svensson model for 10 May 2005. The actual forward rates are in circles. The rates are extracted from Bloomberg®.

Figure 3: Volatility surface of caps/floors for the period of 10 May 2005.
(a) Volatility surface of 1% strike caps/floors from 10 May 2005 to 1 August 2013.

(b) Volatility surface of 5% strike caps/floors from 10 May 2005 to 1 August 2013.

Figure 4: This figure shows the caps/floors volatility surface for the period of 10 May 2005 to 1 August 2013.
(a) BDT calibrated volatility.

(b) HW calibrated volatility.

(c) BK calibrated volatility.

(d) HJM calibrated volatility.

Figure 5: This figure shows the calibrated volatility for four different interest rate models (BDT, HW, BK, HJM).
(a) $G2^{++}$ calibrated volatility.

(b) $LMM$ calibrated volatility.

(c) $SABR$ calibrated volatility.

**Figure 6:** This figure shows the calibrated volatility function and surface for the two-factor $G2^{++}$, $LIBOR$, and $SABR$ interest rate models.
(a) The Credit Valuation Adjustment (CVA) for a long position in an Interest Rate Swap (IRS).

(b) The Credit Valuation Adjustment (CVA) for a short position in an Interest Rate Swap (IRS).

Figure 7: This figure shows the Credit Valuation Adjustment (CVA) in an Interest Rate Swap (IRS) with one-factor short-rate models: Black, Derman, and Toy (BDT), Hull and White (HW), and Black and Karasinski (BK) in a two-regime volatility framework (Crisis period – high volatility in grey).
(a) The Credit Valuation Adjustment (CVA) for a long position in an Interest Rate Swap (IRS).

(b) The Credit Valuation Adjustment (CVA) for a short position in an Interest Rate Swap (IRS).

Figure 8: This figure shows the Credit Valuation Adjustment (CVA) in an interest rate swap with multifactor-factor interest rate models: Heath, Jarrow, and Morton (HJM), LIBOR market model (LMM), and Stochastic Alpha, Rho, and Beta (SARB) in a two-regime volatility framework (Crisis period – high volatility in grey).
The Credit Valuation Adjustment (CVA) for a long position in an Interest Rate Swap (IRS).

The Credit Valuation Adjustment (CVA) for a short position in an Interest Rate Swap (IRS).

Figure 9: This figure shows the Credit Valuation Adjustment (CVA) in an Interest Rate Swap (IRS) for the two-factor $G2++$ interest rate model in a two-regime volatility framework (Crisis period - high volatility in grey).
Figure 10: This figure shows the estimated Gaussian copula dependence correlation ($\rho(t)$) parameter for different window sizes ($N = 7, 8, 9, 10$) of the frequencies used to proxy the marginal empirical distribution of $PD$ and $EAD$. 
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<tr>
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Table 1: Interest Rate Market Data.

Note: This table displays the different fixed-income instruments used for calculating the interest rate term structure curve used in the valuation of interest rate swaps (IRS) and swaptions.
### Counterparty Credit Risk

Table 2: The Credit Valuation Adjustment of Interest Rate Swaps.

**Note:** This table displays the counterparty credit risk (CCR) measured by the credit valuation adjustment (CVA) of long-short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated for the period of 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg®.

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<th>CVA&lt;sub&gt;G&lt;/sub&gt;</th>
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### Counterparty Credit Risk

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</tr>
<tr>
<td>BK</td>
<td>0.60</td>
<td>4.27</td>
</tr>
<tr>
<td>HJM</td>
<td>0.83</td>
<td>5.91</td>
</tr>
<tr>
<td>G2++</td>
<td>0.84</td>
<td>6.20</td>
</tr>
<tr>
<td>LMM</td>
<td>1.29</td>
<td>9.61</td>
</tr>
<tr>
<td>SABR</td>
<td>0.98</td>
<td>6.87</td>
</tr>
</tbody>
</table>

*Table 3: The Credit Valuation Adjustment of Interest Rate Swaps (Crisis Period).*

**Note:** This table displays the counterparty credit risk (CCR) measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated only for the crisis period as the grey area in Fig. 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg®.
### Counterparty Credit Risk

<table>
<thead>
<tr>
<th></th>
<th>Long (IRS Fixed-rate Payer)</th>
<th>Short (IRS Float-rate Payer)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$CVA^{B_L}$</td>
<td>$CVA^{G_L}$</td>
</tr>
<tr>
<td><strong>Short-term IRS (2-years).</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BDT</td>
<td>0.06</td>
<td>0.90</td>
</tr>
<tr>
<td>HW</td>
<td>0.07</td>
<td>1.17</td>
</tr>
<tr>
<td>BK</td>
<td>0.03</td>
<td>0.62</td>
</tr>
<tr>
<td>HJM</td>
<td>0.06</td>
<td>0.97</td>
</tr>
<tr>
<td>G2++</td>
<td>0.19</td>
<td>2.43</td>
</tr>
<tr>
<td>LMM</td>
<td>0.38</td>
<td>3.81</td>
</tr>
<tr>
<td>SABR</td>
<td>0.07</td>
<td>1.00</td>
</tr>
<tr>
<td><strong>Mid-term IRS (5-year).</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BDT</td>
<td>0.33</td>
<td>3.13</td>
</tr>
<tr>
<td>HW</td>
<td>0.35</td>
<td>3.48</td>
</tr>
<tr>
<td>BK</td>
<td>0.23</td>
<td>2.37</td>
</tr>
<tr>
<td>HJM</td>
<td>0.31</td>
<td>3.06</td>
</tr>
<tr>
<td>G2++</td>
<td>0.37</td>
<td>4.02</td>
</tr>
<tr>
<td>LMM</td>
<td>0.81</td>
<td>7.13</td>
</tr>
<tr>
<td>SABR</td>
<td>0.37</td>
<td>3.45</td>
</tr>
<tr>
<td><strong>Long-term IRS (10-year).</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BDT</td>
<td>0.58</td>
<td>4.25</td>
</tr>
<tr>
<td>HW</td>
<td>0.58</td>
<td>4.53</td>
</tr>
<tr>
<td>BK</td>
<td>0.44</td>
<td>3.35</td>
</tr>
<tr>
<td>HJM</td>
<td>0.52</td>
<td>4.05</td>
</tr>
<tr>
<td>G2++</td>
<td>0.45</td>
<td>4.42</td>
</tr>
<tr>
<td>LMM</td>
<td>0.99</td>
<td>8.00</td>
</tr>
<tr>
<td>SABR</td>
<td>0.62</td>
<td>4.58</td>
</tr>
</tbody>
</table>

**Table 4:** The Credit Valuation Adjustment of Interest Rate Swaps (Non-crisis Period).

**Note:** This table displays the counterparty credit risk \((CCR)\) measured by the credit valuation adjustment \((CVA)\) of long-short positions in interest rate swaps \((IRS)\) of different maturities (2, 5, and 10 years). The \(CVA\) is calculated only for the crisis period as the non-grey area in Fig. 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy \((BDT)\), Hull and White \((HW)\), Black and Karasinski \((BK)\), Heath, Jarrow, and Morton \((HJM)\), Two-Factor Gaussian \((G2++)\), and SABR models. Rates are in percentages \(\%\). The interest rates for calculations were extracted from Bloomberg®.
## Table 5: Difference in Credit Valuation Adjustment Calculations Between Dependency Models

The table displays the difference in counterparty credit risk (CCR) calculations for different dependency models (independent, Gaussian, maximum) when measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated for the period of 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.

<table>
<thead>
<tr>
<th>Model</th>
<th>Long (IRS Float-rate Payer)</th>
<th>Short (IRS Float-rate Payer)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CVA_B^diff vs. CVA_B^fixed</td>
<td>CVA_B^fixed vs. CVA_B^fixed</td>
</tr>
<tr>
<td></td>
<td>CVA_B^G vs. CVA_B^diff</td>
<td>CVA_B^diff vs. CVA_B^diff</td>
</tr>
<tr>
<td>Short-term IRS (2-years)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BDT</td>
<td>109.78%</td>
<td>144.39%</td>
</tr>
<tr>
<td>HW</td>
<td>104.67%</td>
<td>125.41%</td>
</tr>
<tr>
<td>BK</td>
<td>104.67%</td>
<td>145.83%</td>
</tr>
<tr>
<td>HJM</td>
<td>104.67%</td>
<td>145.83%</td>
</tr>
<tr>
<td>G2++</td>
<td>104.67%</td>
<td>145.83%</td>
</tr>
<tr>
<td>LMM</td>
<td>104.67%</td>
<td>145.83%</td>
</tr>
<tr>
<td>SABR</td>
<td>104.67%</td>
<td>145.83%</td>
</tr>
</tbody>
</table>

| Mid-term IRS (5-years) | | |
| BDT     | 107.30%                     | 168.00%                      |
| HW      | 103.70%                     | 165.70%                      |
| BK      | 103.70%                     | 165.70%                      |
| HJM     | 103.70%                     | 165.70%                      |
| G2++    | 103.70%                     | 165.70%                      |
| LMM     | 103.70%                     | 165.70%                      |
| SABR    | 103.70%                     | 165.70%                      |

| Long-term IRS (10-years) | | |
| BDT     | 112.78%                     | 200.81%                      |
| HW      | 108.12%                     | 201.70%                      |
| BK      | 108.12%                     | 201.70%                      |
| HJM     | 108.12%                     | 201.70%                      |
| G2++    | 108.12%                     | 201.70%                      |
| LMM     | 108.12%                     | 201.70%                      |
| SABR    | 108.12%                     | 201.70%                      |

Note: This table displays the difference in counterparty credit risk (CCR) calculations for different dependency models (independent, Gaussian, maximum), when measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.
<table>
<thead>
<tr>
<th>Model</th>
<th>Long (IRS Fixed-rate Payer)</th>
<th>Short (IRS Float-rate Payer)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$CVA_{BL}^{B}$ vs. $CVA_{GL}^{B}$</td>
<td>$CVA_{BL}^{B}$ vs. $CVA_{WWR}^{B}$</td>
</tr>
<tr>
<td><strong>BDT</strong></td>
<td>1164.47%</td>
<td>45.94%</td>
</tr>
<tr>
<td><strong>HW</strong></td>
<td>1223.37%</td>
<td>36.81%</td>
</tr>
<tr>
<td><strong>BK</strong></td>
<td>1381.87%</td>
<td>25.28%</td>
</tr>
<tr>
<td><strong>HJM</strong></td>
<td>1184.20%</td>
<td>48.07%</td>
</tr>
<tr>
<td><strong>G2++</strong></td>
<td>873.78%</td>
<td>43.13%</td>
</tr>
<tr>
<td><strong>LMM</strong></td>
<td>832.10%</td>
<td>77.81%</td>
</tr>
<tr>
<td><strong>SABR</strong></td>
<td>1120.03%</td>
<td>63.46%</td>
</tr>
</tbody>
</table>

**Table 6**: Difference in Credit Valuation Adjustment Calculations Between Dependency Models (Crisis Period).

**Note**: This table displays the difference in counterparty credit risk ($CCR$) calculations for different dependency models (independent, Gaussian, maximum), when measured by the credit valuation adjustment ($CVA$) of long–short positions in interest rate swaps ($IRS$) of different maturities (2, 5, and 10 years). The $CVA$ is calculated only for the crisis period as the grey area in Fig. 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy ($BDT$), Hull and White ($HW$), Black and Karasinski ($BK$), Heath, Jarrow, and Morton ($HJM$), Two-Factor Gaussian ($G2++$), and $SABR$ models. Rates are in percentages (%). The interest rates for calculations were extracted from $Bloomberg$®.
## Counterparty Credit Risk

<table>
<thead>
<tr>
<th>CVA_{BL} vs. CVA_{BL}^G</th>
<th>CVA_{BL}^G vs. CVA_{WWR}^B</th>
<th>CVA_{BS}^G vs. CVA_{BS}^WWR</th>
<th>CVA_{BS}^WWR vs. CVA_{BS}^WWR</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Short-term IRS (2-years).</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BDT</td>
<td>1439.14%</td>
<td>136.85%</td>
<td>745.73%</td>
</tr>
<tr>
<td>HW</td>
<td>1481.46%</td>
<td>79.66%</td>
<td>727.57%</td>
</tr>
<tr>
<td>BK</td>
<td>1772.48%</td>
<td>161.26%</td>
<td>767.05%</td>
</tr>
<tr>
<td>HJM</td>
<td>1438.69%</td>
<td>128.49%</td>
<td>741.67%</td>
</tr>
<tr>
<td>G2++</td>
<td>1165.57%</td>
<td>73.04%</td>
<td>1622.22%</td>
</tr>
<tr>
<td>LMM</td>
<td>900.03%</td>
<td>107.78%</td>
<td>1044.23%</td>
</tr>
<tr>
<td>SABR</td>
<td>1395.00%</td>
<td>103.63%</td>
<td>745.59%</td>
</tr>
<tr>
<td><strong>Mid-term IRS (5-year).</strong></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>BDT</td>
<td>861.17%</td>
<td>126.90%</td>
<td>645.66%</td>
</tr>
<tr>
<td>HW</td>
<td>904.53%</td>
<td>81.13%</td>
<td>637.28%</td>
</tr>
<tr>
<td>BK</td>
<td>934.56%</td>
<td>128.66%</td>
<td>676.65%</td>
</tr>
<tr>
<td>HJM</td>
<td>890.40%</td>
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<td>G2++</td>
<td>985.14%</td>
<td>66.82%</td>
<td>991.05%</td>
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<tr>
<td>LMM</td>
<td>778.05%</td>
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<td>729.85%</td>
</tr>
<tr>
<td>SABR</td>
<td>837.24%</td>
<td>158.95%</td>
<td>637.48%</td>
</tr>
<tr>
<td><strong>Long-term IRS (10-year).</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BDT</td>
<td>634.36%</td>
<td>178.19%</td>
<td>573.81%</td>
</tr>
<tr>
<td>HW</td>
<td>685.75%</td>
<td>116.25%</td>
<td>574.40%</td>
</tr>
<tr>
<td>BK</td>
<td>656.56%</td>
<td>173.24%</td>
<td>603.75%</td>
</tr>
<tr>
<td>HJM</td>
<td>672.72%</td>
<td>142.54%</td>
<td>587.31%</td>
</tr>
<tr>
<td>G2++</td>
<td>871.64%</td>
<td>95.31%</td>
<td>607.54%</td>
</tr>
<tr>
<td>LMM</td>
<td>706.16%</td>
<td>108.54%</td>
<td>527.31%</td>
</tr>
<tr>
<td>SABR</td>
<td>649.97%</td>
<td>232.94%</td>
<td>569.88%</td>
</tr>
</tbody>
</table>

### Table 7: Difference in Credit Valuation Adjustment Calculations Between Dependency Models (Non-Crisis Period).

**Note:** This table displays the difference in counterparty credit risk (CCR) calculations for different dependency models (independent, Gaussian, maximum), when measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated only for the crisis period as the non-grey area in Fig. 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg®.