Ambiguity Aversion as Limits to Arbitrage*

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Abstract

Tests of a linear factor models can be interpreted as mean-variance efficiency tests of factor portfolios. An asset with statistically significant alpha implies efficiency gain relative to factor portfolios. We show that the efficiency gain must be sufficiently high to entice ambiguity averse investor to hold the asset and, in that sense, ambiguity aversion presents limits to arbitrage. We propose a new method to evaluate the strength of violations of linear factor models by using asset exclusion conditions from the optimal portfolio of the ambiguity averse investor. The threshold for excluding assets may present a more stringent test and may reverse standard ambiguity-neutral result: assets with statistically significant alpha may be excluded from the optimal portfolio under reasonable ambiguity aversion. We apply this criterion to several “anomaly” portfolios and find that some of them do not violate common linear factor models under reasonable ambiguity aversion.

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1 Introduction

Empirical tests of linear factor models typically use no-arbitrage restrictions and failure of a test implies existence of a portfolio with statistically significant abnormal return (alpha). However, such a portfolio may be of limited value for investors concerned with model uncertainty or precision of estimated parameters required for its implementation. Although such additional criteria based on optimal portfolio choice may suggest that empirical violations of no-arbitrage restrictions are "not too strong", they lack formal interpretation associated with standard statistical tests. We show that for an ambiguity averse investor, the gain from an abnormal return portfolio must be sufficiently high to entice him to hold such a portfolio and, in that sense, ambiguity aversion represents limits to arbitrage.\footnote{Ambiguity aversion, or aversion towards uncertainty about prior distribution, is distinct from risk aversion for a given prior distribution. First illustrated in the seminal paper by Ellsberg (1961), ambiguity aversion has been applied in finance literature to a wide range of problems.} We use this property to propose a new criterion for the strength of violation of no-arbitrage restrictions in linear factor models. We also demonstrate the application of the method on a number of portfolios with statistically significant alphas and find that some of them may be excluded from the optimal portfolios for reasonable levels of ambiguity aversion.

Our approach relies on the well-known link between tests of no-arbitrage restrictions in linear factor models and tests of portfolio mean-variance efficiency. If the no-arbitrage restrictions under a linear factor model hold for a set of test portfolios, then the test portfolios have zero alphas and the optimal portfolio contains only factor assets. A portfolio with statistically significant abnormal return (alpha) constitutes a violation of the no-arbitrage restriction and can be used to improve efficiency of the optimal portfolio constructed only from factors. This standard result applies to ambiguity-neutral investor whenever the abnormal return (alpha) is different from zero. The same reasoning holds if investor is ambiguity averse but with the caveat that portfolio alpha must be sufficiently large and statistically significant considering the uncertainty about the portfolio's expected return. If the ambiguity averse investor chooses to hold only factor assets, then no-arbitrage violation is not "strong enough". This suggests that we can use the threshold for asset exclusion from the optimal portfolio for ambiguity averse investor to formally evaluate the strength of violation of the no-arbitrage restriction. The exclusion threshold has a standard statistical interpretation and is related to a commonly used test for portfolio efficiency proposed by Gibbons, Ross and Shanken (1989, GRS henceforth).

We use the framework developed in Garlappi, Uppal and Wang (2007) for mean-variance portfolio selection when the investor is ambiguity averse (in the sense of Knight, 1921) towards the assets' expected returns. The investor forms portfolios using benchmark (factor) and non-benchmark assets. The benchmark assets are used to specify a linear factor model. The investor beliefs are characterized by confidence areas of expected returns for benchmark and non-benchmark assets.
The investor solves for optimal portfolio under the worst-case beliefs using a max-min criterion. One of the distinctive characteristics of this portfolio choice problem is that when the mean returns of non-benchmark assets are sufficiently ambiguous, they may be excluded from the optimal portfolio. We propose to use the condition for asset exclusion from the optimal portfolio of the ambiguity averse investor to evaluate the strength of violation of the no-arbitrage restrictions imposed by a factor model.

We show that there are beliefs such that a portfolio with statistically significant abnormal return is undesirable to the ambiguity averse investor. Specifically, the GRS statistic could indicate statistically significant alpha and nevertheless an ambiguity averse investor would optimally avoid such asset. Thus, the exclusion criteria is non-redundant relative to standard ambiguity neutral test. Importantly, the ambiguity aversion, expressed in terms of confidence interval for asset mean, may be similar or even lower than the confidence interval the investor is using to evaluate significance of the GRS statistic. Thus, in general, the exclusion of the anomaly portfolio is not an artifact of an unduly high ambiguity aversion.

The exclusion threshold defines a confidence area in portfolio mean and can be characterized by $F$-distribution under standard assumptions. For a given investor confidence in factor portfolio expected returns we can compute the lowest confidence threshold for test portfolio mean which is sufficient not to deviate from the benchmark assets. Thus, the threshold indicates the least ambiguity about the mean return which is necessary to exclude the anomaly portfolio. If the exclusion threshold is low, i.e. below standard statistical confidence levels, then a reasonably ambiguity averse investor would not find the anomaly portfolio attractive and vice versa.

We use our methodology to investigate a number of well known cross-sectional anomalies in the US stock market. These portfolios have large and statistically significant alphas and their average returns are hard to explain using standard factor models. For each anomaly we have two portfolios, short-side and long-side with negative and positive alpha, respectively. We add these portfolios, one at a time, to the factor portfolios of each asset pricing model to evaluate the strength of the anomaly under exclusion criterion. As benchmark models we use one-factor (market index) CAPM, Fama-French 3-factor model which adds size and value factors, and a four-factor model (Fama-French 3-factor model plus momentum factor).

For the case when investors are fully confident in factor means, moderate uncertainty about non-benchmark mean returns (defined as confidence intervals of below 90%-tile) is sufficient to exclude many of the long-side anomalies under the factor models we consider. When the confidence in the factor expected returns is eroded, long-side anomaly portfolios become more attractive as they

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2 Also, as we discuss in section 3, the exclusion threshold has an analogy with appraisal ratio in Treynor and Black (1973) active portfolio model.

3 We are grateful to Jianfeng Yu and Long Chen for proving data on anomaly portfolios returns.
help diversify ambiguity of the factors. On the short-side the anomaly portfolios are often included when investor is confident about benchmark means because factors are required to hedge the short positions. When confidence in the factor expected returns is lower, short-side anomaly portfolios are easier to exclude from the optimal allocation because factors are less desirable as a hedge. When uncertainty about the factor means is high (above 90%-tile confidence interval), all short-side anomalies can be excluded for moderate to low levels of ambiguity about their mean returns. These results suggest that no-arbitrage violations under the multi-factor models are considerably weaker, and in many cases are within limits to arbitrage when evaluated by an ambiguity averse investor.

From the theoretical perspective our paper is related to the literature analyzing the effects of investor concerns about estimated parameters and asset pricing model used to construct optimal portfolio. Pastor (2000) considers a portfolio choice problem of an investor who combines a prior belief in a factor model with asset return data which may potentially include deviations from factor pricing. Pastor and Stambaugh (2000) use such a portfolio choice problem to compare different asset pricing models from the perspective of efficiency gains from implied optimal portfolios. Both of these papers consider expected utility (mean-variance) investor with Bayesian updating of beliefs. Wang (2005) considers a portfolio problem where a mean-variance investor is using max-min criterion over the worst-case belief about the potential deviations from the factor model in the data. The set of prior beliefs is described by a precision parameter about asset alphas and the investor is using Bayesian updating to incorporate data potentially violating the asset pricing model. Wang shows that optimal portfolio is obtained by appropriately shrinking mean and variance of the distributions implied by the data towards those restricted by the model. Garlappi, Uppal and Wang (2007) also consider the max-min model of mean variance investor but they parameterize the set of investor beliefs using confidence areas about the means of factors and test assets portfolios. This structure allows for the analysis of both model and parameter uncertainty. We use it for our analysis of the tradeoff between deviations from the model and uncertainty about estimated expected returns.

In the above models, the investor typically balances the optimal portfolio between what is prescribed by the factor model and what is suggested by the data. The strength of prior belief in the model and the precision of the estimates from the data determine how much investor is willing to deviate from the benchmark portfolio and include data-driven active component. Such

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5 Specifically, the investor’s prior on alpha is a normal distribution with zero mean and a variance $\vartheta \Sigma$ where $\vartheta$ is used to parameterize the beliefs in the factor model from dogmatic $\vartheta = 0$ to $\vartheta = +\infty$ which indicates no belief in the model validity.
deviations can be expressed through portfolio weights, through Sharpe ratio efficiency gains or through certainty equivalent gains. Stronger deviations from the asset pricing model in the data typically result in larger deviations of portfolio from benchmark and larger expected certainty equivalent gains. Thus, investors always take advantage of the abnormal return detected in the data, but to a different degree which depends on the strength of the prior belief in the factor model and the extent of its violations in the data. Using this method we can apply the scale of expected certainty equivalent gains to rank the strength of abnormal returns in the data as is done in Pastor and Stambaugh (2000) and Wang (2005). However, such ranking still leaves open the question of what constitutes economically large gain and, correspondingly, sufficiently large violation of no-arbitrage restriction under a factor model. The methodology we propose helps to address this issue by explicitly computing the least amount of ambiguity about the mean return necessary to exclude the anomaly portfolio. If the required minimum ambiguity is high, in a conventional statistical sense, then the anomaly portfolio presents a significant violation of the no-arbitrage restrictions and vice-versa.

Our paper also contributes to empirical tests of linear factor models by providing an additional diagnostic tool. Portfolio anomalies detected in the data are often used to motivate research of new risk factors or incorporate behavioral biases in asset pricing. The rationale for this is that if portfolio strategies with significant alphas are present in the data and their abnormal returns persist over a long time, there must be explanations for such abnormal returns beyond standard frictions-based limits to arbitrage. The metric we propose in this paper helps to look at anomaly portfolios from a different angle than a standard test for significance of alpha. The criterion we propose is non-redundant and some anomalies with statistically significant alpha may appear weak under the asset exclusion criterion. Our methodology thus can help identify portfolios with stronger violations of the factor models and focus research attention on these worthwhile cases.

2 Mean-variance portfolio choice with ambiguity aversion

We begin with an overview of the framework developed in Garlappi, Uppal and Wang (2007, GUW henceforth). They analyze mean-variance portfolio selection problem of the ambiguity averse investor with multiple priors about assets’ expected returns. Multiple priors beliefs arise from estimation error and the investor is assumed to be ambiguity averse in the sense of Knight (1921), that is to maximize the objective function under the worst-case beliefs.

The investor has access to \( N \) risky assets, \( K \) risky benchmark portfolios, and a risk free asset. Denote \( r_{at} \) as the \( N \times 1 \) vector of assets excess returns at time \( t \) and \( r_{bt} \) as the \( K \times 1 \) vector of

\[^{6}\text{We only present the elements relevant for our analysis which rely on Proposition 3 and section 1.2.4 in Garlappi, Uppal and Wang (2007). For discussion, proofs, and derivations we refer the readers to the original source.}\]
benchmarks excess returns. The relationship between $r_{at}$ and $r_{bt}$ is captured by the return generating process:

$$r_{at} = \alpha + \beta r_{bt} + u_t, \quad var(u_t) = \Omega,$$

where $\alpha$, $\beta$, and $u_t$ are, respectively, the $N \times 1$ vector of intercepts, $N \times K$ matrix of regression coefficients, and $N \times 1$ vector of residuals with variance-covariance matrix $\Omega$. The mean and variance of the asset and benchmark returns are

$$\mu = \begin{bmatrix} \mu_a \\ \mu_b \end{bmatrix} = \begin{bmatrix} \alpha + \beta \mu_b \\ \mu_b \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ba} \\ \Sigma_{ab} & \Sigma_{bb} \end{bmatrix} = \begin{bmatrix} \beta \Sigma_{bb} \beta^T + \Omega & \beta \Sigma_{bb} \\ \Sigma_{bb} \beta^T & \Sigma_{bb} \end{bmatrix}.$$  

An ambiguity averse investor has multiple-prior beliefs about the expected returns of the assets and benchmarks. The underlying assumption is that investor does not know true expected returns and must estimate them from the data. Bewley (1988) has argued that in this case standard statistical confidence intervals could be used to represent uncertainty of estimated parameters.\(^7\) Formally, the beliefs are represented by the confidence areas centered around sample estimates of $(\hat{\mu}_a, \hat{\mu}_b)$ as follows:

$$\xi_a \geq (\hat{\mu}_a - \mu_a)\Sigma_{aa}^{-1}(\hat{\mu}_a - \mu_a),$$

$$\xi_b \geq (\hat{\mu}_b - \mu_b)\Sigma_{bb}^{-1}(\hat{\mu}_b - \mu_b).$$

When $\xi_a = \xi_b = 0$, the constraints collapse to a single prior used in the standard ambiguity neutral model. If the returns are normally distributed and the variance-covariance matrix is estimated then the right-hand sides of the constraints (4) and (5), after scaling adjustment, are distributed as $F(N, T - N)$ and $F(K, T - K)$, respectively, where $T$ is the sample size.\(^8\) If we denote critical values associated with these $F$-distributions as $v_a$ and $v_b$, respectively, we can rewrite the constraints as follows:

$$\xi_a \equiv v_a \frac{N(T - 1)}{T(T - N)} \geq (\hat{\mu}_a - \mu_a)^\top \Sigma_{aa}^{-1}(\hat{\mu}_a - \mu_a),$$

$$\xi_b \equiv v_b \frac{K(T - 1)}{T(T - K)} \geq (\hat{\mu}_b - \mu_b)^\top \Sigma_{bb}^{-1}(\hat{\mu}_b - \mu_b).$$

\(^7\)The literature on parameter and model uncertainty in portfolio choice has focused on the uncertainty about mean returns rather than variance-covariance matrix primarily for two reasons. First, means are much harder to estimate in a sense that they are less precise than the second order moments for the same amount of data (Merton, 1980). Second, errors in estimated means have the first order effect on portfolio efficiency. For example, Chopra and Ziemba (1993) express cost of parameter errors in cash certainty equivalent and estimate that estimation errors in means are as much as ten times costlier than errors in variance and twenty times costlier than errors in covariances.

\(^8\)For simplicity of notation we do not distinguish sample size for benchmark and non-benchmark assets but they are generally considered to be different and in the empirical analysis of anomalies we allow for this difference.
Such formulation of the problem allows for different degrees of confidence in the mean returns of assets and factors. This is a useful distinction which, as pointed out by GUW, can be understood from both statistical and institutional perspective. There could be different sample lengths available for factors and assets, and portfolios of factors and assets could have different degrees of diversification resulting in varying degree of uncertainty about mean estimates. Factor portfolios could also be used as benchmarks for compensation and retention of portfolio managers and in this case uncertainty about factor expected returns may be relatively less important to portfolio managers than uncertainty about expected returns of non-benchmark assets.

Let \( w \equiv (w_a, w_b) \) be the \((N + K) \times 1\) vector of portfolio weights. The investor objective is to maximize mean-variance utility under the worst-case beliefs subject to the constraints (4) and (5):

\[
\max_w \min_{\mu_a, \mu_b} \mu^\top \mu - \frac{\gamma}{2} w^\top \Sigma w \\
\mu_a, \mu_b \quad \text{s.t.} \quad (4) \text{ and } (5)
\]

where \( \gamma \) is the coefficient of risk aversion. As shown by GUW, the solution to this problem is given by:

\[
w_a = \max \left[ 1 - \sqrt{\frac{\xi_a}{g(w_b)^\top \Sigma_{aa}^{-1} g(w_b)}}, 0 \right] \frac{1}{\gamma} \Sigma_{aa}^{-1} g(w_b),
\]

\[
w_b = \max \left[ 1 - \sqrt{\frac{\xi_b}{h(w_a)^\top \Sigma_{bb}^{-1} h(w_a)}}, 0 \right] \frac{1}{\gamma} \Sigma_{bb}^{-1} h(w_a).
\]

Where

\[
g(w_b) = \hat{\mu}_a - \gamma \Sigma_{ab} w_b
\]

\[
h(w_a) = \hat{\mu}_b - \gamma \Sigma_{ba} w_a.
\]

As discussed by GUW, assets will be excluded from the optimal portfolio if investor is using constrained estimate of the mean \( \hat{\mu}_a = \hat{\beta} \hat{\mu}_b \) and the ambiguity of factors’ expected returns is zero \( \xi_b = 0 \) while that of the assets is positive \( \xi_a > 0 \). That is, an investor who believes in the model and is fully confident about factors’ means but uncertain about assets’ means would not hold the assets. As we show below, asset exclusion may also occur when investor does not believe in the model and, in this case, the asset exclusion can be used to test the strength of deviations from the model.

### 3 Ambiguity aversion and asset exclusion criteria

There is a well known connection between the tests of linear factor models and tests of portfolio mean-variance efficiency. If a factor model generates zero (statistically insignificant) Jensen’s alphas
for a set of assets, then the optimal mean-variance portfolio can be constructed only from factors, that is, the additional assets do not improve portfolio efficiency. If some assets (active portfolios) have statistically significant alphas, the assets can be used to improve efficiency of the factor-based portfolio. We now show that the same argument applies to the ambiguity averse investor but with the caveat that alphas must be sufficiently large and significant in view of uncertainty about the active portfolio’s expected return. If the ambiguity averse investor chooses to hold only factor portfolios, then the violations of the no-arbitrage restrictions are not “strong enough” and we could use the lowest beliefs threshold $\xi_a$ required for asset exclusion to characterize the strength of the deviation. To see if the exclusion threshold is reasonable, we can convert $\xi_a$ to a corresponding critical value $\varepsilon_a$ in (6) and check whether such critical value is within an acceptable confidence area of the corresponding $F$-distribution.

Our goal is to understand how investor confidence about the mean return of active portfolio affects his decision to hold it, therefore we do not impose a priori model constraints on the estimated mean. By imposing constrained estimate of the mean, the characterization of uncertainty as shown by inequality (4) does not allow us to analyze the tradeoff between the deviation from the model (alpha) and the ambiguity of asset means because they are essentially the same. Specifically, with $\hat{\mu}_a = \hat{\beta}\hat{\mu}_b$, $\xi_a$ in inequality (4) defines how far $\mu_a$ can deviate from the model and thus mechanically determines the range of admissible alphas. In this case, $\xi_a = 0$ indicates dogmatic belief while $\xi_a = +\infty$ indicates no belief in the model.

To serve our purpose, unconstrained estimate of $\hat{\mu}_a$ is introduced to capture deviations from the factor model in the portfolio choice problem. In this case, $\xi_a = 0$ corresponds to using point estimate of the mean as in the standard mean variance problem, and $\xi_a = +\infty$ corresponds to not knowing the mean. Thus, the investor does not rule out that some assets may have non-zero alpha, but he is also aware that mean returns may be estimated with error. This tradeoff determines whether to hold the assets.

Consider an investor who has estimated the return generating process in (1) and potentially found a deviation from a factor model so that $\hat{\mu}_a = \hat{\alpha} + \hat{\beta}\hat{\mu}_b$ and $\hat{\alpha} \neq \mathbf{0}_{N \times 1}$. Using the solutions from (9) and (10), the conditions for assets exclusion from the optimal portfolio $w_a = \mathbf{0}_{N \times 1}$ are as follows:

\begin{align*}
\xi_a & \geq g(w_b)^\top \Sigma_{aa}^{-1} g(w_b) \quad (13) \\
\xi_b & < h(w_a)^\top \Sigma_{bb}^{-1} h(w_a) = \hat{\mu}_b^\top \Sigma_{bb}^{-1} \hat{\mu}_b \quad (14) \\
g(w_b) & = \hat{\mu}_a - \gamma \Sigma_{ab} w_b \quad (15) \\
h(w_a) & = \hat{\mu}_b \quad (16)
\end{align*}
Using (10) and (16) we obtain the optimal factor portfolio weight $w_b$:

$$
w_b = \left(1 - \sqrt{\frac{\xi_b}{\mu_b^\top \Sigma_{bb}^{-1} \mu_b}}\right) \frac{1}{\gamma} \Sigma_{bb}^{-1} \hat{\mu}_b
$$

(17)

Note that this weight is a classical mean-variance efficient portfolio of the factors $\frac{1}{\gamma} \Sigma_{bb}^{-1} \hat{\mu}_b$ scaled down by a multiple in parenthesis which is less than one (since (14) holds). This is intuitive because factors are the only risky assets held in this case and the scaling factor reflects ambiguity concerns about factor means (recall that there is also a risk free asset available). Substituting (17) into (15) and using $\hat{\mu}_a = \hat{\alpha} + \beta \hat{\mu}_b$ and the definition of $\beta \equiv \Sigma_{ab} \Sigma_{bb}^{-1}$ gives:

$$
g(\xi_b) = \hat{\alpha} + \beta \hat{\mu}_b - \left(1 - \sqrt{\frac{\xi_b}{\mu_b^\top \Sigma_{bb}^{-1} \mu_b}}\right) \Sigma_{ab} \Sigma_{bb}^{-1} \hat{\mu}_b
$$

$$
= \hat{\alpha} + \sqrt{\frac{\xi_b}{\mu_b^\top \Sigma_{bb}^{-1} \mu_b}} \beta \hat{\mu}_b
$$

(18)

Inequalities (13) and (14) and equation (18) characterize the beliefs for which ambiguity averse investor will optimally hold only factor benchmarks without the assets. Before we proceed to analyze this case further we present the remaining solution possibilities.

To illustrate the structure of the solution we define four areas for the optimal portfolio weights: (i) both factors and assets are held; (ii) only factors are held; (iii) only assets are held; (iv) no risky assets are held. For low levels of confidence (wide confidence areas), when $\xi_b$ is such that inequality (14) is reversed and a similar inequality for assets also holds:

$$
\xi_a > \hat{\mu}_a^\top \Sigma_{aa}^{-1} \hat{\mu}_a
$$

$$
\xi_b > \hat{\mu}_b^\top \Sigma_{bb}^{-1} \hat{\mu}_b
$$

both factors and assets are excluded and investor only holds the risk free asset. The conditions when factors, rather than assets, are excluded from the optimal portfolio are symmetric to those in (13)-(17) and are shown in Appendix A. The remaining values of $\xi_a$ and $\xi_b$ define the area where investor holds both factors and assets together in the optimal portfolio.

We use a simple case with one factor and one asset to understand solution structure. Figure 1 schematically illustrates four different solution regions with borders marked by solid lines. The case with $\alpha > 0$ is shown on the left panel and $\alpha < 0$ is shown on the right panel. When investor is reasonably confident in both factors and assets he will be including both in the optimal portfolio (region i). When the ambiguity about both benchmark and non-benchmark assets is high, it is

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9This follows immediately from the optimal weight expressions in (9) and (10) and the definitions of $g(w_b)$ and $h(w_a)$ when we set $w_a = 0_{N \times 1}$ and $w_b = 0_{K \times 1}$. 

optimal to withdraw from the risky assets altogether (region iv). And assets (region ii) or factors (region iii) could be excluded when the relative ambiguity with respect to the other group is high. The case of positive alpha is relatively intuitive: as long as confidence intervals in factors and assets $\xi_b$ and $\xi_a$ are not “too wide” and they are close to each other, the optimal portfolio includes both factors and non-factor assets.

The negative alpha case is more nuanced and the region i where both factors and assets are held consists of two distinct parts. For negative $\alpha$ asset there exists a confidence level $\xi_b$ for factor portfolios such that $g(\xi_b) = 0$ in (18). This implies from (13) that assets are excluded for any confidence level $\xi_a \geq 0$. Intuitively this happens because negative alpha assets would have to be short in the optimal portfolio and their factor exposure is hedged by increasing position in factor assets. As factor means become more uncertain, non-benchmark assets which require shorting become less desirable. Thus, the first part of region i is in the bottom left corner of the figure as shown by the arrow. As factor uncertainty $\xi_b$ rises further, there again emerges the area where assets with negative alpha would be held for some beliefs. This is because assets and factors reverse their roles: shorted assets become beneficial for hedging highly uncertain factor portfolios. This property will play a role in understanding empirical results for portfolios with negative $\alpha$ presented later.\footnote{Mathematically, the reason why sign of $\alpha$ matters for the shape of the region can be traced to the inequality for quadratic form in (13). Note that $g(\xi_b)$ in (18) is negative when $\xi_b = 0$ and $\alpha < 0$. As $\xi_b$ is increasing, there is a non-monotonicity in the quadratic form. When $\alpha$ is positive, then the quadratic form is always monotonically increasing for $\xi_b > 0$.}

The boundary between the areas (i) and (ii) in Figure 1 characterizes assets exclusion condition in (13) and presents a particular interest. We now relate it to the tests of mean-variance efficiency studied by Gibbons, Ross and Shanken (1989). In the mean-variance portfolio theory, the expressions $\hat{\mu}^\top \Sigma^{-1} \hat{\mu}$ and $\hat{\mu}_b^\top \Sigma_{bb}^{-1} \hat{\mu}_b$ correspond to the square of in-sample Sharpe ratio of the efficient portfolios, respectively, for assets and factors jointly, and for factors only. As shown in GRS, these quantities are related as follows\footnote{GRS derived this decomposition for a one-factor model. In Appendix A we show that it holds in a multi-factor model.}:

$$\hat{\mu}^\top \Sigma^{-1} \hat{\mu} = \hat{\mu}_b^\top \Sigma_{bb}^{-1} \hat{\mu}_b + \hat{\alpha}^\top \Omega^{-1} \hat{\alpha}$$

The squared maximum Sharpe ratio achieved by combining assets and factors in the optimal portfolio can be decomposed to the squared maximum Sharpe ratio of the factors plus the additional term, $\hat{\alpha}^\top \Omega^{-1} \hat{\alpha}$, which is the GRS statistic. Specifically, when returns are normally distributed, the GRS statistics has an F-distribution under the null hypothesis $H_0$: $\alpha = 0_{N \times 1}$:

$$\frac{(T - N - K) \ \hat{\alpha}^\top \Omega^{-1} \hat{\alpha}}{N \ \underset{1 + \hat{\mu}_b^\top \Sigma_{bb}^{-1} \hat{\mu}_b}{\sim} F(N, T - N - K)}$$

$$\text{(20)}$$
where $T$ is the number of time series observations of the assets’ returns, $N$ is the number of test assets, and $K$ is the number of benchmark portfolios.\(^{12}\) This statistic can be used to evaluate factor models based on a set of chosen benchmarks. Let $\varepsilon_A$ denote some critical level associated with the GRS statistic and consider it significant if:

$$\frac{(T - N - K)}{N} \hat{\alpha}^\top \Omega^{-1} \hat{\alpha} \geq \varepsilon_A$$

If the statistics is sufficiently high so that the $p$-value associated with $\varepsilon_A$ is smaller than a usual significance level, then $H_0$ is rejected because the assets significantly improve portfolio efficiency relative to the factors alone as seen from (19). If the GRS statistic is insignificant, then $H_0$ is not rejected and the efficient portfolio of factors is located within a statistically acceptable range of the minimum variance frontier spanned by both assets and factors.

The interpretation of the GRS statistic as portfolio efficiency gain is applicable for ambiguity neutral investor. For ambiguity averse investor, the efficiency gain from the assets with significant GRS statistic may not be as large. In fact, we show now that there are beliefs such that portfolios with statistically significant alphas may be excluded from the optimal portfolio of the ambiguity averse investor. Importantly, the level of ambiguity aversion towards the assets’ means could be very similar or even lower than that used to evaluate significance of alphas. The following proposition, proved in Appendix A, characterizes the conditions on beliefs under which ambiguity averse investors prefer to hold only benchmark assets (factors) even when the GRS statistics may be significant at level $\varepsilon_A$.

**Proposition 1** Given the value of the GRS statistic $\hat{\alpha}^\top \Omega^{-1} \hat{\alpha} > 0$, there exist a threshold $\bar{\xi}_b > 0$ such that for $\xi_b \in [0, \bar{\xi}_b]$ the following inequality holds:

$$\hat{\alpha}^\top \Omega^{-1} \hat{\alpha} \geq g(\xi_b)^\top \Sigma^{-1}_{aa} g(\xi_b)$$

where $g(\xi_b)$ is defined in (18). Inequality (22) is strict for $0 \leq \xi_b < \bar{\xi}_b$ and turns to equality when $\xi_b = \bar{\xi}_b$. Furthermore there exist critical levels $\varepsilon_A$ and $\varepsilon_a$ such that the GRS statistics is significant at the level $\varepsilon_A$ while condition (13) is satisfied at critical level $\varepsilon_a$:

$$\hat{\alpha}^\top \Omega^{-1} \hat{\alpha} \geq \varepsilon_a \frac{(1 + \hat{\mu}_b^\top \Sigma^{-1}_{bb} \hat{\mu}_b)N}{(T - N - K)} \geq \varepsilon_a \frac{N(T - 1)}{T(T - N)} \equiv \xi_a \geq g(\xi_b)^\top \Sigma^{-1}_{aa} g(\xi_b)$$

and the optimal portfolio includes only factor assets, i.e. $w_a = 0_{N \times 1}$.

Proposition 1 establishes that we can find critical level for the assets’ expected returns $\varepsilon_a$ such that the ambiguity averse investor will exclude the assets even though for the ambiguity neutral investor

the assets would improve the optimal portfolio Sharpe ratio with statistical significance $\varepsilon_a$. The degree of freedom scalers which multiply these critical levels in (23) are very similar when $T$ is relatively large for factors and assets and the squared maximum Sharpe ratio of the factors is small, both conditions hold in the data. Thus, depending on the confidence in the factor portfolios, the critical level for ambiguity aversion required to reject the asset may be no more stringent (or even lower) than the critical level used to test significance of the GRS statistic.

The upper bound for $\bar{\xi}_b$ implicitly defined through inequality (22) determines the range of ambiguity in the factors such that it is possible to reject the assets while the exclusion threshold is still below the GRS statistic. Exclusion still occurs for higher values of $\xi_b > \bar{\xi}_b$ as long as investor is not confident in the assets, that is $\varepsilon_a (\xi_a)$ is sufficiently high as required in (13). The lowest confidence level $\varepsilon_a$ consistent with asset exclusion is determined by the quadratic form on the right side of (23). If the GRS statistic is large then this quadratic form is going to be large as well because they both depend on alphas. In this case, the lowest admissible $\varepsilon_a$ consistent with asset exclusion would have to be high, corresponding to unreasonably wide confidence interval about the assets’ means. Therefore, we would conclude that ambiguity aversion is unlikely to account for violation of no-arbitrage restrictions of the factor model. However, the GRS statistic could be lower, although above conventional statistical levels, while $\varepsilon_a$ in (23) could be under conventional critical values. In such cases, assets could be excluded from the optimal portfolio under reasonable ambiguity aversion.

The lowest possible value of $\varepsilon_a$ consistent with asset exclusion is obtained from the right-hand side of (23):

$$\bar{\varepsilon}_a(\xi_b) = \frac{T(T-N)}{N(T-1)} g(\xi_b) ^\top \Sigma_{aa}^{-1} g(\xi_b)$$

$$= \frac{T(T-N)}{N(T-1)} \left( \hat{\alpha} + \sqrt{\frac{\xi_b}{\hat{\mu}_b ^\top \Sigma_{bb} ^{-1} \hat{\mu}_b}} \beta \hat{\mu}_b \right) ^\top \Sigma_{aa}^{-1} \left( \hat{\alpha} + \sqrt{\frac{\xi_b}{\hat{\mu}_b ^\top \Sigma_{bb} ^{-1} \hat{\mu}_b}} \beta \hat{\mu}_b \right)$$

(24)

which can be compared to the critical values of the $F$-distribution in (6).

To better see the intuitive explanation for the minimum exclusion threshold in (24) consider an investor fully confident in the factors (or ambiguity neutral with respect to factors) with $\xi_b = 0$. Then, we can rewrite (24) as follows:

$$\varepsilon_a(0) = \frac{T(T-N)}{N(T-1)} \hat{\alpha} ^\top \Sigma_{aa}^{-1} \hat{\alpha}$$

(25)

An interesting analogy exists here with the appraisal ratio known from the Treynor and Black (1973) active portfolio problem formulated for ambiguity-neutral mean-variance investor. The appraisal ratio for an active portfolio is defined as the ratio of absolute value of its alpha to the standard deviation of residual from the factor regression (1). For $N = 1$, the appraisal ratio is the square
Ambiguity Aversion as Limits to Arbitrage

The root of the GRS statistic $\sqrt{\hat{\alpha}^T \Omega^{-1} \hat{\alpha}}$ and is used as a performance metric comparing active portfolio alpha with portfolio-specific risk investor has to bear in order to take advantage of non-zero alpha.

The right hand side of (25) is related to an analogous ratio $\sqrt{\hat{\alpha}^T \Sigma_{aa}^{-1} \hat{\alpha}}$ which compares alpha with total risk of active portfolio. That is, ambiguity averse investor will exclude assets if his confidence is sufficiently wide relative to a function of this ratio. Intuitively, ambiguity averse investor is concerned with total asset risk because it determines how well he knows assets’ means, that is total risk affects the range of admissible beliefs for a given confidence level in (6). Higher alpha implies that wider range of beliefs (higher ambiguity aversion) is required to exclude assets, while highly volatile assets are easier to exclude (they require smaller range of beliefs for exclusion) because uncertainty about their mean returns is high.

In a general case when factors are uncertain ($\xi_b > 0$), from (24) we see that for positive alpha the effect from higher $\xi_b$ is like that of increasing alpha, it is harder to exclude the assets because they are beneficial as a hedge against factor ambiguity. For negative alpha it is the opposite because $g(\xi_b)$ absolute value is decreasing for lower values of $\xi_b$, as we already discussed.

Equation (24) allows us to evaluate empirically whether asset exclusion could occur at reasonable ambiguity critical levels. We first specify a degree of investor confidence in the factors $\xi_b$ via its corresponding critical level $\varepsilon_b$ as defined in (7). The case of $\varepsilon_b = \xi_b = 0$ presents a reference point when investor is fully confident in the factors or, equivalently, ambiguity neutral towards factors expected returns. As we widen the confidence area by increasing $\varepsilon_b$, factor mean returns become more uncertain to the investor. The minimum critical value of $\varepsilon_a(\xi_b)$ required for asset exclusion defined in (24) can be compared to the critical values of the appropriate $F$-distribution. We now apply this methodology to investigate empirical violations of the linear factor models.

4 Empirical limits to arbitrage from ambiguity

A feasible portfolio strategy with statistically significant alpha typically is referred to as “anomaly.” We use data on 11 anomalies to evaluate the strength of violations of asset pricing models’ no-arbitrage restrictions. The anomaly data is described in Appendix B.\textsuperscript{13} Each anomaly consists of long and short leg portfolio. We analyze each portfolio leg separately because the strength of factor model violations could vary across long and short portfolios and, as discussed earlier, the sign of portfolio alpha also matters for exclusion conditions.

We add one anomaly portfolio at a time to the factors and analyze portfolio exclusion criteria. Throughout this exercise we consider three linear factor models: single market factor CAPM; Fama-French three-factor model which adds SMB and HML factors for size and value premium,\textsuperscript{13}We are grateful to Jianfeng Yu and Long Chen for sharing their data on anomalies. This is the same data that was used in Stambaugh, Yu, and Yuan (2011) and a more detailed summary is provided in Table 1 of their paper.
respectively; and a four-factor model which adds momentum factor to the Fama-French three-factor model. Because momentum is one of our anomaly portfolios we exclude it from the analysis when applying the four-factor model. Thus, we consider 22 portfolios when we apply CAPM or Fama-French three-factor model and 20 portfolios for the four-factor model.

Table 1 reports Jensen’s alphas and the GRS statistics with corresponding $p$-values for each of 22 anomaly portfolios and each of the three models. Portfolios show sizable alphas and almost all GRS statistics are significant at least at 10% significance level and most are significant at least at 5% level. Short legs of the anomalies typically have larger alphas and stronger statistical significance. As more factors are included, the size of alphas tend to be smaller, but most alphas are still significant and large for three- and four-factor models.

We now apply our analysis to anomaly portfolios. We state exclusion criteria in terms of the probability associated with implied confidence areas. The uncertainty about factors is expressed through confidence area parameter $\xi_b > 0$ in (5). The range where benchmark assets are included in the optimal portfolio is bounded by $\xi_b < \hat{\mu}_b^\top \Sigma^{-1}_{bb} \hat{\mu}_b$ in (14). We can convert $\xi_b$ to $\varepsilon_b$ as defined in (7) and compute probability associated with the confidence area using corresponding cdf of the $F$ distribution: $P(\varepsilon_b) = F(\varepsilon_b(\xi_b); K, T - K)$. For each $\xi_b$ we construct the minimum exclusion threshold $\bar{\varepsilon}_a(\xi_b)$ defined in equation (24) and compute its corresponding confidence interval probability $P(\bar{\varepsilon}_a) = F(\bar{\varepsilon}_a(\xi_b); N, T - N)$. Since we analyze one anomaly portfolio at a time, $N = 1$. Thus, we can map uncertainty in the factors measured as probability of the confidence area about their mean $P(\varepsilon_b)$ into the minimum uncertainty in the anomaly portfolio mean required for exclusion $P(\bar{\varepsilon}_a)$. If $P(\bar{\varepsilon}_a)$ is low we can say that anomaly can be excluded under reasonable ambiguity aversion and vice versa.

Figures 2 and 3 show the exclusion criteria for long-side portfolios and Figures 4 and 5 show them for short-side portfolios. The exact values for exclusion confidence area for each portfolio and model are reported in Table 2 for select values of $P(\varepsilon_b) = \{0, 0.5, 0.9, 0.95\}$.

4.1 Long-side anomaly portfolios

We discuss one long-side anomaly as an example to illustrate the methodology and follow with overall discussion of the patterns among the long-side anomaly portfolios. Short-side portfolios are discussed separately in the next section.

Consider top left panel in Figure 2 which shows the results for long AG (Asset Growth) anomaly portfolio. Low values of $P(\varepsilon_b)$ on the X-axis correspond to high confidence in the factors and in particular $P(\varepsilon_b) = 0$ corresponds to investor fully confident in the factor means or investor who is not concerned about factor uncertainty for other reasons such as benchmarking. If such an investor would use CAPM (solid line) as a benchmark model, then it would take at least 95%
Ambiguity Aversion as Limits to Arbitrage

confidence interval uncertainty around the mean AG portfolio return in order to exclude it. Thus, when investor is confident in factor mean under CAPM, long AG portfolio can be excluded only if ambiguity aversion is relatively high. If investor is less confident in market index mean return and considers a 50% confidence interval then the required exclusion threshold is even higher, at about 98% confidence interval. Therefore, for CAPM, long AG portfolio constitutes quite strong anomaly.

Consider now the exclusion thresholds for Fama-French three factor model applied to the long AG portfolio shown in the same panel by the dashed line. An investor fully confident in the factors would exclude long AG anomaly when uncertainty about the mean is at least as large as 60%-tile confidence interval. Thus, for investor who uses three-factor model as a benchmark and is confident in benchmark means, long AG anomaly is not that strong and a moderate ambiguity about its mean return is sufficient to exclude it from the optimal portfolio. When the ambiguity of factor means raises, exceeding 50%-tile interval, then exclusion of the long AG anomaly requires much higher levels of uncertainty.

Finally, consider four-factor model shown by the dotted line. For high confidence in the factor means investor excludes long AG anomaly if his ambiguity about its mean is in approximately 80%-tile confidence interval or more. This is more stringent exclusion than under three-factor model because long AG anomaly has a higher and more significant alpha and GRS statistic (Table 1) in the four-factor model than in the three factor model. In general, there is no monotonic decline in the lowest exclusion criteria from models with lower to higher number of factors just like there is no monotonic reduction of alphas and GRS statistic. For less uncertain factor means though (higher \( P(\xi_b) \)) the exclusion threshold of the four-factor model is below that of the three-factor model but it is still above 90%-tile level.

The analysis for the long AG anomaly suggest that using multi-factor model with high confidence about factor means results in relatively easy exclusion of the anomaly from the optimal portfolio of benchmarks. For higher ambiguity about benchmark means, the anomaly is much stronger. Which level of ambiguity about benchmarks to accept is going to depend on the situation and investor personal preference. In cases when benchmarks own ambiguity is not essential while deviations from benchmark returns is costly, investors may prefer to exclude such weaker active portfolios.

The rest of the panels on Figures 2 and 3 show similar analyses for the remaining 10 anomaly portfolios. To better observe patterns across portfolios we consolidate information from the figures and present exclusion confidence probability at particular thresholds of \( P(\xi_b) = \{0, 0.5, 0.9, 0.95\} \) in Table 2.

Consider top panel of Table 2 showing the results for the long-side anomalies. We observe that CAPM “passes” the most portfolios (defined as exclusions requiring at least 90%-tile, or higher, uncertainty about means shown in bold or italic). This is not surprising, as alphas are particularly high and significant under one-factor CAPM. When confidence about factors is high (columns with
there are only few sufficiently strong anomalies that could not be excluded for ambiguity aversion under 90%-tile confidence interval. For CAPM there are three such cases (AG, INV, and MOM), for three-factor model there are only two (GPA and MOM), and only one (GPA) for the four-factor model. Thus, using three- or four-factor model, out of 11 anomalies only two (GPA and MOM) are strong enough to be included in the optimal portfolio when investor has high confidence in factor means. Relatedly, we can see that these two portfolios have some of the highest alphas and GRS statistics among the long-side portfolios shown in Table 1.

Note, however, that GRS statistic or alpha are not sufficient to determine which portfolio would be easier to exclude under ambiguity aversion. Consider, for example, the TA (Total accruals) portfolio which under the four-factor model has the second lowest alpha of 0.376% (highly significant) after the GPA portfolio alpha of 0.428%. Yet, when uncertainty about factor means is low, the TA portfolio is excluded for moderate ambiguity aversion thresholds within 80-90%-tiles.

When uncertainty about factors is high, defined by at least 90% confidence areas, many of the long anomalies under three- or four-factor model are not excluded for conventional levels of confidence. This is particularly apparent in the case of three-factor model when none of the long-side portfolios can be excluded under 95%-tile confidence in means. If the confidence in factors is moderate, at 50%-tile confidence areas, then many of the long anomalies could be excluded for confidence area for the means under 90%-tile. For example, for four-factor model and \( P(\varepsilon_b) = 0.5 \) six portfolios out of ten are excluded using less than 90%-tile confidence area for mean returns (CEI, INV, O, ROA, TA and FAIL). Figures 2 and 3 confirm that in the middle of the confidence range for factors around 50%-tile many anomalies could be excluded for moderate ambiguity aversion about their mean returns.

To summarize, the long-side anomaly portfolios are attractive to ambiguity averse investors when the investors have also high ambiguity about benchmark mean returns. If investors have high-to-moderate confidence in factor means, then many anomalies do not appear very strong to be combined with benchmarks, especially in the case of three- and four-factor model. These results provide different view on the strength of the anomalies. Traditional efficiency test focused on alpha or GRS statistic does not predict which anomalies are easier to exclude on the grounds of ambiguity aversion. Some portfolios with highly significant GRS statistic turn out to be weak candidates when ambiguity about their means is factored in the portfolio analysis.

### 4.2 Short-side anomaly portfolios

We now consider the results for the short-side anomaly portfolios with negative alphas shown on Figures 4 and 5 and in the lower panel of Table 2. Consider first Figures 4 and 5. When alpha
is negative, the minimum exclusion threshold in (24) is non-monotone in factor confidence $\xi_b$. For lower values of $\xi_b$ the exclusion threshold is declining and it is increasing as confidence in factors deteriorates and $\xi_b$ becomes larger. As already discussed, intuitively this is associated with the fact that taking advantage of short anomaly portfolio requires increasing optimal allocation to factors portfolio. As factor means become more ambiguous, negative alpha assets become less attractive and easier to exclude. At some point, when factor uncertainty is relatively high, assets are excluded for any level of confidence when $g(\xi_b) = 0$. After that point, further increase in factor uncertainty makes short-side portfolios more attractive because they can hedge large uncertainty in the factors.15

Most short portfolios are relatively easy to exclude even when factor confidence is high. For higher levels of factor uncertainty than 50% confidence area, most of the short portfolios in our sample are excluded. There are a couple of notable exceptions: O (Olson O-score sort) and momentum. FAIL, failure probability portfolio, also appears strong anomaly under CAPM and three-factor model but not under four-factor model. The reason why short portfolios are so different from long ones is because they require investor to increase position in factors in order to take advantage of their alphas. While their alphas are high and attractive, the additional risk and ambiguity of factor benchmark has an offsetting effect.

Table 2 shows that when factor confidence is high ($P(\varepsilon_b) = 0$) short portfolios are very attractive under CAPM and three factor model. If we again regard portfolios that are hard to exclude as those that require at least 90% uncertainty for exclusion, we see that CAPM has 6 such portfolios and three-factor model has seven. Under four-factor model only two portfolios cannot be excluded (NSA and ROA) at levels below 90%-tile. When factors are more uncertain at 90% confidence, then only one portfolio (momentum) cannot be excluded for three factor model benchmark. And for factor uncertainty of 95%-tile neither three- or four-factor model has any short portfolios that could not be excluded for ambiguity below 90%-tile level.

In the data short-side anomalies typically have larger and more statistically significant alphas compared with long-side anomalies. However, short portfolios are generally regarded as harder to implement and subject to more frictions-based arbitrage limits due to short selling restrictions and higher associated transactions costs. Our analysis shows that for ambiguity averse investors there is an additional important consideration related to factor ambiguity because factors are used to hedge short anomalies. As a result, short portfolios are typically much easier to exclude using smaller confidence areas for their mean returns.

15For some portfolios this point is never achieved because of additional constraint on $\xi_b < \hat{\mu}_b^T \Sigma_{bb}^{-1} \hat{\mu}_b$ in (14). That is, the level of $\xi_b$ where $g(\xi_b) = 0$ is too high so that at this point both factors and assets are excluded and investor only holds risk-free asset.
5 Conclusion

We propose to use asset exclusion from the mean-variance optimal portfolio of the ambiguity averse investor as a criterion for the strength of violation of no-arbitrage restrictions in linear factor models. For weak violations of the factor model, investor concerned with ambiguity will exclude non-zero alpha assets from the optimal portfolio. We show that, for a given confidence in the benchmark assets’ means, the exclusion condition can be converted to a standard $F$-distribution confidence area and is easy to implement empirically.

We apply our methodology to a number of well known anomaly portfolios and show that some anomalies with significant alphas do not pass the exclusion test when investor is reasonably ambiguity averse. For investors who are confident in the factor means, stronger violations tend to be on the short side while the opposite is the case when investors have high ambiguity about factors portfolio means.

Our methodology can be used to enhance the tests of asset pricing models and inform researchers about the strength of model violations beyond the standard tests designed for ambiguity-neutral setting. This is useful for identifying relatively strong anomalies in the data which warrant further research about sources of their risk premiums.
References


Figure 1: **Solution structure**
Boundaries of the four solution regions are indicated by the solid lines: (i) Both factors and assets are held; (ii) Only factors are held; (iii) Only assets are held; (iv) No risky assets are held.
Figure 2: Exclusion confidence probability as a function of factor confidence probability

Each panel shows the probability $P(\bar{\epsilon}_a)$ of the confidence interval corresponding to the minimum ambiguity threshold $\bar{\epsilon}_a$ required for asset exclusion from equation (24). On the horizontal axis is the probability corresponding to the confidence area in the benchmark assets’ means $P(\epsilon_b)$. Solid, dashed, and dotted lines represent, respectively, the graphs for CAPM, Fama-French three-factor model, and four factor model (Fama-French three factor plus momentum).
Figure 3: **Exclusion confidence probability as a function of factor confidence probability**
Each panel shows the probability $P(\bar{\varepsilon}_a)$ of the confidence interval corresponding to the minimum ambiguity threshold $\bar{\varepsilon}_a$ required for asset exclusion from equation (24). On the horizontal axis is the probability corresponding to the confidence area in the benchmark assets’ means $P(\varepsilon_b)$. Solid, dashed, and dotted lines represent, respectively, the graphs for CAPM, Fama-French three-factor model and four factor model (Fama-French three factor plus momentum).
Figure 4: Exclusion confidence probability as a function of factor confidence probability. Each panel shows the probability $P(\bar{\epsilon}_a)$ of the confidence interval corresponding to the minimum ambiguity threshold $\bar{\epsilon}_a$ required for asset exclusion from equation (24). On the horizontal axis is the probability corresponding to the confidence area in the benchmark assets' means $P(\epsilon_b)$. Solid, dashed, and dotted lines represent, respectively, the graphs for CAPM, Fama-French three-factor model and four-factor model (Fama-French three factor plus momentum).
Figure 5: **Exclusion confidence probability as a function of factor confidence probability**

Each panel shows the probability $P(\bar{\varepsilon}_a)$ of the confidence interval corresponding to the minimum ambiguity threshold $\bar{\varepsilon}_a$ required for asset exclusion from equation (24). On the horizontal axis is the probability corresponding to the confidence area in the benchmark assets’ means $P(\varepsilon_b)$. Solid, dashed, and dotted lines represent, respectively, the graphs for CAPM, Fama-French three-factor model and four factor model (Fama-French three factor plus momentum).
Table 1: **Anomaly portfolios characteristics**
Table reports Jensen’s alpha and the GRS statistic with corresponding \( p \)-value for each factor model and anomaly portfolio short and long leg separately. Anomaly portfolios are described in Appendix B.

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<th>Anomaly</th>
<th>Long ( T )</th>
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<th>( p )-val.</th>
<th>( \alpha )</th>
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<td>0.011</td>
<td><strong>0.02</strong></td>
<td>-0.088</td>
<td>0.001</td>
<td>0.54</td>
</tr>
<tr>
<td>FAIL</td>
<td>398</td>
<td>-1.037</td>
<td>0.029</td>
<td><strong>0.00</strong></td>
<td>-1.170</td>
<td>0.056</td>
<td><strong>0.00</strong></td>
<td>-0.580</td>
<td>0.019</td>
<td><strong>0.01</strong></td>
</tr>
</tbody>
</table>
Table 2: Exclusion confidence interval probability by factor model

Table reports confidence interval corresponding of the minimum exclusion threshold \( P(\bar{\varepsilon}_a) \) for select values of confidence in factor means \( P(\varepsilon_b) \). \( P = 0.0 \) corresponds to full confidence and \( P = 0.95 \) corresponds to 95% confidence area around sample estimates. Bold entries indicate at least 95%-tile interval and italicized entries indicate at least 90%-tile interval. Anomaly portfolios are added individually to each factor model, the long-side portfolios (with positive alpha) are shown in the top panel and the short-side portfolios (with negative alpha) are in the bottom panel.

<table>
<thead>
<tr>
<th>Anomaly</th>
<th>CAPM ( P(\varepsilon_b) )</th>
<th>FF 3-factor ( P(\varepsilon_b) )</th>
<th>4-factor ( P(\varepsilon_b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Long</td>
<td>Short</td>
<td>Long</td>
</tr>
<tr>
<td>AG</td>
<td>0.00 0.50 0.90 0.95</td>
<td>0.00 0.50 0.90 0.95</td>
<td>0.00 0.50 0.90 0.95</td>
</tr>
<tr>
<td>CEI</td>
<td>0.94 0.98 1.00 1.00</td>
<td>0.59 0.91 1.00 1.00</td>
<td>0.79 0.92 0.98 0.99</td>
</tr>
<tr>
<td>GPA</td>
<td>0.88 0.96 0.99 1.00</td>
<td>0.10 0.75 0.99 1.00</td>
<td>0.95 0.98 0.99 1.00</td>
</tr>
<tr>
<td>INV</td>
<td>0.76 0.91 0.98 0.99</td>
<td>0.95 0.99 1.00 1.00</td>
<td>0.48 0.88 1.00 1.00</td>
</tr>
<tr>
<td>MOM</td>
<td>0.90 0.97 1.00 1.00</td>
<td>0.97 0.99 1.00 1.00</td>
<td>0.97 0.99 1.00 1.00</td>
</tr>
<tr>
<td>NOA</td>
<td>0.00 0.50 0.90 0.95</td>
<td>0.00 0.50 0.90 0.95</td>
<td>0.00 0.50 0.90 0.95</td>
</tr>
<tr>
<td>O</td>
<td>0.65 0.85 0.97 0.98</td>
<td>0.68 0.91 0.99 1.00</td>
<td>0.87 0.94 0.98 0.99</td>
</tr>
<tr>
<td>ROA</td>
<td>0.87 0.96 0.99 1.00</td>
<td>0.69 0.94 1.00 1.00</td>
<td>0.95 0.98 0.99 1.00</td>
</tr>
<tr>
<td>MOM</td>
<td>0.79 0.55 0.01 0.18</td>
<td>0.90 0.59 0.24 0.50</td>
<td>0.31 0.01 0.41 0.53</td>
</tr>
<tr>
<td>NOA</td>
<td>0.44 0.71 0.92 0.95</td>
<td>0.63 0.86 0.98 0.99</td>
<td>0.81 0.89 0.96 0.97</td>
</tr>
<tr>
<td>NSA</td>
<td>0.93 0.81 0.44 0.27</td>
<td>0.95 0.79 0.15 0.12</td>
<td>0.73 0.56 0.24 0.12</td>
</tr>
<tr>
<td>O</td>
<td>0.98 0.94 0.81 0.73</td>
<td>0.99 0.94 0.66 0.49</td>
<td>0.91 0.86 0.75 0.70</td>
</tr>
<tr>
<td>ROA</td>
<td>0.99 0.97 0.90 0.85</td>
<td>0.98 0.93 0.73 0.62</td>
<td>0.91 0.86 0.75 0.70</td>
</tr>
<tr>
<td>TA</td>
<td>0.88 0.71 0.28 0.11</td>
<td>0.71 0.40 0.19 0.37</td>
<td>0.21 0.04 0.21 0.29</td>
</tr>
<tr>
<td>FAIL</td>
<td>0.97 0.94 0.81 0.74</td>
<td>0.99 0.95 0.74 0.61</td>
<td>0.97 0.95 0.74 0.61</td>
</tr>
</tbody>
</table>
Appendix A Derivations and Proofs

To derive multi factor analog of the decomposition of the squared maximum Sharpe ratio from Gibbons, Ross and Shanken (1989) we apply the formula for partitioned matrix inverse to the variance-covariance matrix of assets and factors defined in (3) and we have that:

\[
\Sigma^{-1} = \begin{bmatrix}
\Omega^{-1} & -\Omega^{-1}\beta \\
-\beta^T\Omega^{-1} & \Sigma_{bb} + \beta^T\Omega^{-1}\beta
\end{bmatrix}
\]  

(A.1)

Substituting \(\mu\) from (2) and using (A.1), the maximum squared Sharpe ratio of the assets and factors can be written as:

\[
\hat{\mu}^T\Sigma^{-1}\hat{\mu} = \hat{\mu}^T\Sigma_{bb}^{-1}\hat{\mu} + \hat{\alpha}^T\Omega^{-1}\hat{\alpha}
\]

To obtain conditions for excluding factors rather than assets start with solutions from (9) and (10). Setting optimal portfolio weight of the factors \(w_b = 0_{K \times 1}\) results in the following:

\[
\xi_a < g(w_b)^T\Sigma_{aa}^{-1}g(w_b) = \hat{\mu}_b^T\Sigma_{bb}^{-1}\hat{\mu}_b
\]  

(A.2)

\[
\xi_b \geq h(w_a)^T\Sigma_{bb}^{-1}h(w_a)
\]  

(A.3)

\[
g(w_b) = \hat{\mu}_a
\]  

(A.4)

\[
h(w_a) = \hat{\mu}_b - \gamma\Sigma_{ba}w_a
\]  

(A.5)

Using (10) gives the optimal assets' portfolio weight \(w_a\):

\[
w_a = \left(1 - \sqrt{\frac{\xi_a}{\hat{\mu}_a^T\Sigma_{aa}^{-1}\hat{\mu}_a}}\right)\frac{1}{\gamma}\Sigma_{aa}^{-1}\hat{\mu}_a
\]  

(A.6)

A.1 Proof of Proposition 1

First consider \(\xi_b = 0\).

\[
g(\xi_b)^T\Sigma_{aa}^{-1}g(\xi_b) = \hat{\alpha}^T\Sigma_{aa}^{-1}\hat{\alpha}
\]

\[
= \hat{\alpha}^T(\beta\Sigma_{bb}\beta^T + \Omega)^{-1}\hat{\alpha}
\]

\[
= \hat{\alpha}^T[\Omega^{-1} - \Omega^{-1}\beta(\Sigma_{bb}^{-1} + \beta^T\Omega^{-1}\beta)^{-1}\beta^T\Omega^{-1}]\hat{\alpha}
\]

\[
= \hat{\alpha}^T\Omega^{-1}\hat{\alpha} - \hat{\alpha}^T\Omega^{-1}\beta(\Sigma_{bb}^{-1} + \beta^T\Omega^{-1}\beta)^{-1}\beta^T\Omega^{-1}\hat{\alpha}
\]  

(A.7)

The second equality relies on \(\Sigma_{aa} = \beta\Sigma_{bb}\beta^T + \Omega\). The third one is based on the formula of inverse of sum of matrices by H. V. Henderson and S. R. Searle (1981). Both \(\Omega\) and \(\Omega^{-1}\) are positive definite given their eigenvalues are the reciprocals of each other; then definition of positive definite implies \(x^T\beta^T\Omega^{-1}\beta x > 0\) for every non-zero column vector \(x\) of \(n\) real numbers, which in turn implies \(\beta^T\Omega^{-1}\beta\) is positive definite. Then, \(\Sigma_{bb}^{-1} + \beta^T\Omega^{-1}\beta\) is positive definite since \(\Sigma_{bb}\) and \(\Sigma_{bb}^{-1}\) are positive.
definite and the sum of positive definite matrices is positive definite; equivalently, \((\Sigma_{bb}^{-1} + \beta^T \Omega^{-1} \beta)^{-1}\) is positive definite and thus

\[
\hat{\alpha}^T \Omega^{-1} \beta (\Sigma_{bb}^{-1} + \beta^T \Omega^{-1} \beta)^{-1} \beta^T \Omega^{-1} \hat{\alpha} > 0
\]

(A.8)

As a result,

\[
\hat{\alpha}^T \Omega^{-1} \hat{\alpha} - \hat{\alpha}^T \Omega^{-1} \beta (\Sigma_{bb}^{-1} + \beta^T \Omega^{-1} \beta)^{-1} \beta^T \Omega^{-1} \hat{\alpha} < \hat{\alpha}^T \Omega^{-1} \hat{\alpha}
\]

(A.9)

Now consider \(\xi_b > 0\). The quadratic form \(g(\xi_b)^T \Sigma_{aa}^{-1} g(\xi_b)\) is defined by a positive definite matrix \(\Sigma_{aa}^{-1}\) and is a continuous function such that:

\[
\lim_{\xi_b \to +\infty} g(\xi_b)^T \Sigma_{aa}^{-1} g(\xi_b) = +\infty
\]

(A.10)

Therefore, by continuity of the quadratic form there exist a threshold \(\bar{\xi}_b > 0\) such that

\[
g(\bar{\xi}_b)^T \Sigma_{aa}^{-1} g(\bar{\xi}_b) = \hat{\alpha}^T \Omega^{-1} \hat{\alpha}
\]

(A.11)

For every value of \(\xi_b\) for which inequality (22) holds we can select critical levels \(\varepsilon_\alpha\) and \(\varepsilon_a\) defined by the outer boundaries in the inequality (22):

\[
\varepsilon_\alpha \frac{(1 + \hat{\mu}_b^T \Sigma_{bb}^{-1} \hat{\mu}_b) N}{(T - N - L)} = \hat{\alpha}^T \Omega^{-1} \hat{\alpha}
\]

(A.12)

\[
\varepsilon_a \frac{N(T - 1)}{T(T - N)} \equiv \xi_a = g(\xi_b)^T \Sigma_{aa}^{-1} g(\xi_b)
\]

(A.13)

This completes the proof.

**Appendix B  Anomaly Data Description**

**B.1 Anomaly Data Description**

As described by Stambaugh, Yu and Yuan (2012), the 11 anomalies in this study are based on sorts on measures including financial distress, net stock issues, composite equity issues, total accruals, net operating assets, momentum, gross profit-to-assets, asset growth, return-on-assets (ROA), and investment-to assets. The long and short position for each anomaly takes on the stocks that are in the highest-performing decile and lowest-performing decile, respectively. For each anomaly we use a short abbreviated name (indicated in capital letters in parenthesis after each anomaly number) to label them in the figures and tables throughout the paper.

Anomaly 1 (FAIL) and 2 (O) are both distress-oriented with different measures: 1 is failure probability and 2 is the Ohlson (1980) O-score. As documented by Campbell, Hilscher, and
Szilagyi (2008), firms on the high tail of financial distress distribution have lower, not higher, excess returns than firms on the low tail. For both anomalies, the long-short strategy is to go long position in firms with low value of distress measure and short position in firms with high value of distress measure.

Anomaly 3 (NSA) and 4 (CEI), net stock issues and composite equity issues, both rely on the argument that managers issue shares when stocks are overvalued and thus firms with high stock issues have relatively low subsequent returns. Ritter (1991), Loughran and Ritter (1995) find equity issuers under-perform matching nonissuers based on net stock issues and Daniel and Titman (2006) find similar result based on composite equity issues. The long-short strategy consists of going long in firms with low stock issues and going short in firms with high stock issues.

Anomaly 5 (TA), total accruals, is suggested by Sloan (1996) with the reasoning that investors might neglect the distinction between cash flow and accrual components of earnings and thus become overly optimistic about the future prospects of high accrual firms. The long-short strategy of this anomaly is to go long position in firms with low total accruals and go short position in firms with high total accruals.

Anomaly 6 (NOA) is net operating assets. Hirshleifer, Hou, Teoh and Zhang (2004) find a strong negative relation between net operating assets scaled by lagged total asset and future stock returns and suggest investors’ focus on accounting profitability with neglection of cash profitability as an explanation for this phenomenon. The long-short strategy is to go long in firms with low net operating assets and go short in firms with high net operating assets.

Anomaly 7 (MOM), momentum, is found by Jegadeesh and Titman (1993), referring to the phenomenon that stock price is more likely to move in the same direction. The implied arbitrage profit comes from going long in stocks with high past returns and going short in stocks with low past returns.

Anomaly 8 (GPA), gross profitability premium, is implied by the abnormal benchmark-adjusted returns found by Novy-Marx (2010) after sorting firms on gross profit-to-assets. More profitable firms have higher returns.

Anomaly 9 (AG), asset growth, is discovered by Cooper, Gulen, and Schill (2008). They suggest investors initially overreact to firm’s asset expansion and thus firms with higher asset growth have lower subsequent returns.

Anomaly 11 (INV), investment-to-asset, is backed by the strong negative relation between firm past investment and future returns, documented by Titman, Wei, and Xie (2004) and Xing (2008). As suggested by Titman, Wei, and Xie (2004), it is investors’ underreaction to firm’s overinvestment due to manager’s empire building behavior that drives the anomalous return associated with investment-to-asset. Finally, an equal combination of these 11 anomalies is created to examine the average behavior of these anomalies.