Abstract

We investigate three main consequences of relying on financial data to conduct monetary policy: (i) its effects on the volatility of expected inflation, nominal and real short-term rates, (ii) its impact on the persistence of monetary shocks on macroeconomic variables, and (iii) its implications for the spread between the long-term and short-term nominal rate. Under a general equilibrium setting, we show how central banks conducting monetary policy based on both macroeconomic and financial data distort the term structure of interest rates relative to a benchmark economy where monetary authorities solely control the money supply by targeting output and inflation. We present closed form expressions for all equilibrium quantities and evaluate the impact of policy changes towards interest rate smoothing on nominal and real short-rates, expected inflation and yields.
1 Introduction

The 2007-2009 subprime mortgage crisis forced U.S. policymakers to rely on unconventional measures to prevent a spillover effect on the real economy. The severity of the crisis, immediately reflected in financial market data and corporate reports, led the Federal Reserve Bank (FED) to inject trillions of dollars into the financial system through the acquisition of mortgage-backed securities, commercial papers and direct lending to non-bank institutions in order to bring back stability to financial markets. \footnote{This set of measures are nowadays commonly referred to as Quantitative Easing (QE). In essence, the intention behind these operations is to ease loan conditions for non-financial firms by supplying banks with a quantity of new money equal to the value of the purchased assets. Some recent papers try to quantify the impact of QE on bond and stock market as well as its effectiveness in reducing long-term rates. Krishnamurthy and Vissing-Jorgensen (2011) show that QE was effective in reducing long-term nominal interest rates of safe assets, but much less successful in dropping riskier assets rates such as corporate bonds and mortgages securities. Nevertheless, the findings of Christensen and Krogstrup (2016) support the idea that the effectiveness in reducing long-term rates is independent of assets purchased by central banks.} Despite the relative success of these unorthodox measures in bringing down long-term interest rates, some critics raise concerns about the potential unintended consequences they might trigger, such as the rise of inflation and the loss of confidence in the bond market. To date, it is still unclear to what extent should central banks conduct monetary policy based on financial market data given their core mandate to tackle poor growth, high inflation and high unemployment.

Our paper investigates three main consequences of relying on financial data to conduct monetary policy: \( (i) \) its effects on the volatility of expected inflation, nominal and real short-term rates, \( (ii) \) its impact on the persistence of monetary shocks on macroeconomic variables, and \( (iii) \) its implications for the spread between the long-term and short-term nominal rate. In addition, we contribute to the understanding of how central banks conducting monetary policy based on both macroeconomic and financial data distort the term structure of interest rates and the term premium relative to a benchmark economy where monetary authorities solely control the money supply by targeting output and inflation. Under a general equilibrium setting, the additional objective of smoothing interest rates - our proxy for financial data - introduces an extra layer of complexity.\footnote{Interest rate smoothing may have different meanings in the literature. Here, we adopt the concept used in McCallum (1994) which is equivalent to reducing the interest rate spread between the long and short-term nominal rate.} The reason for the additional challenge is that all three variables targeted by the central bank are tied together through market clearing conditions in good and money markets. Therefore, it is not clear ex-ante that by engaging in a more aggressive interest rate smoothing policy, monetary authorities would necessarily succeed in narrowing the...
interest rate spread. To put it differently, depending on how the central bank balances the importance of yield curve smoothing, inflation and output, it could, in theory, increase the yield spread by attributing more weight to yield curve smoothing. We show that if the yield curve smoothing is “moderate”, this unintended consequence is prevented.

The idea that monetary policy could target financial indicators goes back to McCallum (1994). The author shows that a monetary policy aiming to smooth interest rates and the yield curve can explain the rejection of the expectation hypothesis. Despite his valuable insights on rationalizing Fama and Bliss (1987) empirical puzzle, the exogenous risk premium specification prevents one from understanding how factors driving the risk premium could also be related to interest rates movements. McCallum (1994) acknowledges in his concluding remarks the complexity of endogenizing the risk premium and simultaneously incorporating macroeconomic and financial data in the monetary policy.

More recently, Woodford (2012) argues that the importance of monetary policy for financial stability should not be neglected by central bankers. In his view, the traditional approach of targeting inflation does not take into account market imperfections such as excessive leverage and maturity transformation that could substantially affect the severity of risks to financial stability. Woodford (2012) illustrates his point by adding a loss function that measures the level of financial stability to the standard inflation and output gap targeting and showing that the important welfare consequences generated by credit market imperfections can be mitigated by the “flexible inflation targeting”.

To address the issues highlighted by McCallum (1994) and Woodford (2012), we present in Section 2 a macro-financial model that describes the behavior of households and firms using a real business cycle framework similar to Buraschi and Jiltsov (2005) and has an endogenous money supply equation that incorporates the central bank’s concern with the slope of the yield curve. Contrary to Buraschi and Jiltsov (2005), we adopt a simpler process to describe the evolution of the money supply persistent component, which is the state variable responsible for the time variability of the equilibrium quantities.

3 The simple version of the Expectation Hypothesis relates the forward rate with the expected future short-rate and a risk premium. When confronted with data, the regression coefficient on the term premium is significant smaller than predicted by the model.

4 In particular, the author highlights the limitation of a monetary policy seeking solely to smooth short-rate and the slope of the yield curve: “(It) represents a simplification relative to the actual behavior of the FED, which almost certainly responds to recent inflation and output or employment movements as well as the spread. So, if one were to attempt to econometrically estimate actual reaction functions, then measures of inflation and output gaps would need to be included. But in that case values of these variables would need to be explained endogenously, so the system of equations in the model would have to be expanded. (...) In short, this type of study would require specification and estimation of a complete dynamic macroeconometric model.”

5 The interpretation for such persistence component on the money supply growth varies according to authors. Andolfatto et al. (2004) argue that the persistent component reflects the long-run money
simplicity of the state variable dynamics allows us to solve the model in closed form and to run comparative statics on the volatility of the short-rate, expected inflation and yields with respect to the weight monetary authorities put on financial data. Unfortunately, simplicity comes at a cost. The main drawback is that prices of risk are constant in equilibrium. Our model can be adjusted to incorporate more complex state variables and utility functions such that the term structure empirical regularities are appropriately matched. Nevertheless, by compromising analytical tractability, we have to restrict our analysis to a specific numerical parametrization and lose the capability of deriving general statements about how yield curve smoothing affects equilibrium quantities.

Other papers also address McCallum’s challenge under a different perspective. Gallmeyer et al. (2005) explore the robustness of McCallum’s rule to endogenous risk premium and how this policy can be achieved through an interest rate targeting rule. The authors also focus on a manageable description of the state variables and impose no-arbitrage condition on bond prices to endogenize the risk premium. Gallmeyer et al. (2005) test two different specifications under the class of the affine term structure models to match empirical regularities: one with a stochastic volatility factor and another with a stochastic price of risk. Their analysis provides a theoretical set of restrictions on asset prices and macroeconomic variables such that the Taylor rule is equivalent to the McCallum’s rule. Rudebusch and Wu (2008) also explore the tractability of affine term structure models to analyze the impact of a monetary policy targeting inflation and output gap on the term structure dynamics. Under a New Keynesian framework, the authors assume that monetary policy is conducted such that the short-rate can be decomposed into a sum of two latent factors associated with output and inflation. However, the laws specifying the relationship between the latent factors and the macroeconomic variables are exogenous. Consequently, one cannot assure that the specification is consistent with a price-setting behavior once the link between the aggregate demand, output and inflation is absent.

expansions regime. Rudebusch and Wu (2008) argue that the persistent latent factor corresponds to the level of the macro-finance term structure and it can be interpreted as the central bank’s implicit inflation target. Atkeson and Kehoe (2009) argues emphatically in favor of models that are able to generate state-dependent prices of risk. The authors point out that, over the business cycle, the Federal Reserve Bank responds to shocks in real risk. Consequently, part of the short-rate movements has to be associated to changes in the conditional variance of the log marginal utility of consumption. Other recent studies in the macro-finance literature support the idea that the volatility of the pricing kernel is not constant. In particular, Boguth and Kuehn (2013) find empirically that the time varying conditional volatility of consumption is a priced risk factor. Bansal et al. (2014) also show that volatility risks carries a sizable positive risk premium, suggesting that volatility risk is an important factor to understand the mechanics of asset prices and macroeconomics dynamics.

These specifications are supported by the findings of Dai and Singleton (2003), who argue in favor of models with state-dependent prices of risk instead of state variables with stochastic volatility. The authors show that the former is equally tractable and tend to deliver better empirical fits.
The model we present in Section 2 does not have such inconsistencies because our general equilibrium formulation ties together interest rates, yield curve slope, prices of risk, inflation and output under a theoretical description of preferences and production technology, resulting in an asset pricing kernel which is consistent with the dynamics of macro-financial variables. Contrary to both models, our general equilibrium formulation allows for a clear exposition of the mechanism driving the co-movements among term structure, term premium and inflation. Therefore, our model serves as a theoretical foundation for the popular affine risk-price representation of Ang and Piazzesi (2003), Cochrane and Piazzesi (2005), Gallmeyer et al. (2005) and Rudebusch and Wu (2008) among others.

Our paper also investigates how yield curve smoothing impacts the propagation of monetary shocks to macro-financial variables. Similarly to Rudebusch and Wu (2008), we analyze the dynamics of output and price level using impulse response functions. The impulse response functions of monetary shocks has a transient and a permanent component. We show that a policy change towards yield curve smoothing increases the transient component of the output impulse response function, resulting into an amplification of the monetary shocks’ persistence to output, capital, consumption, money demand and investment. The effect on the transient component of the price level impulse response function is the opposite, meaning that monetary shocks’ persistence on price level is damped under a more intense yield curve smoothing policy. The permanent components are not affected by policy changes on interest rates smoothing.

1.1 A Brief Historical Perspective

For several years, academics and practitioners have been debating to what extent financial data should be taken into consideration by the Federal Reserve in the conduction of monetary policy. McCallum (1994) points out that one of the advantages of relying on this data to conduct monetary policy is that information on inflation and output gap are usually not available to policy makers at the moment of the short-rate setting. In addition, high-quality and reliable financial market data is usually readily available to monetary authorities. This can help the central bank to better assess the current economic conditions and rapidly adjust monetary policy accordingly. Clarida et al. (1999) raise similar concerns about the data reliability of inflation and output. They highlight that the imperfect information about macroeconomic variables poses several challenges.

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8 Other types of inconsistencies can emerge in this context. For example, Rudebusch and Wu (2008), in footnote 9, point out that their interpretation of the level factor as expected inflation is conflicting with their assumption on the inflation dynamics.
to the central bank such as the choice of the policy instrument and the specification for
targets in terms of forecasts rather than the ex-post values.

Rotemberg and Woodford (1997) present an important evidence that the central bank
had relied in the past on financial data to conduct monetary policy. The authors show
that the optimal interest rate predicted by the standard models in monetary economics
literature are generally much more volatile than the historical data for interest rates.
They conclude that the gradual adjustment of the interest rate allows the central bank
to reduce the volatility of the short rate helping it to achieve the stabilization goals.

This tendency of the central bank to adjust interest rates in a sluggish fashion is
generally referred to as “interest rate smoothing”. Bernanke (2004) and Bernanke and
Woodford (2007) also argue that a central bank with a larger preference for interest
rate smoothing rather than output stabilization can explain the persistence of short-
term interest rate, output and inflation. Amato and Laubach (1999) and Clarida et al.
(2000) obtain high estimates of the smoothing parameter and conclude that this is a clear
evidence of a monetary policy that induces “interest rate inertia”. Clarida et al. (2000)
survey some possible explanations for the interest rate smoothing behavior of the Federal
Reserve such as a possible way of manipulate the long-term rate and the fear of disruption
of financial markets (see, e.g., Goodfriend (1991) and Rotemberg and Woodford (1997)).
The argument made by Goodfriend (1991) is that output and prices only respond to long-
term interest rates. Thus, the only way the Federal Reserve can successfully stabilize
output and inflation is if its actions affect the long-term rate.

Woodford (2003) explores the idea of including the interest rate smoothing policy as
part of the central bank mandate. He also addresses the associated problem of “optimal
delegation”, which consists of choosing the appropriate objectives to target. The author
proves that, under a monetary rule that accounts for interest rate volatility, the central
bank can achieve a greater level of output and inflation stability.

The Federal Reserve concerns with the slope of the yield curve are clearly exposed
in two historical examples. In 1961, during the so called “Operation Twist”, the central
bank sold several short-term bonds and bought long-term bonds. Instead of lowering
short-term interest rate and potentially worsening the balance of payments, the central
bank decided to lower long-term rates to stimulate business investments through low
long-term rates. Swanson (2011) presents evidence that the program was successful in
lowering the long-term end of the yield curve. A similar attempt of flattening the term
structure was observed in late 2010. On 10th of November of 2010, the central bank
announced its intention of buying $600 billion of long-term securities in a window of three
quarters. Three months after the conclusion of this program, the Federal Reserve notified
the intention of purchasing $400 billion bonds with maturities between 6 and 30 years while selling the same amount of short-term bonds with maturities less than 3 years. This program, named “Maturity Extension Program”, also became known as “Operation Twist”.

Perhaps one of the best illustrations of how the Federal Reserve incorporates financial data to its monetary policy decision is presented by the Federal Open Market Committee transcripts of the series of emergency meetings held in 2008. The meetings start with Mr. William C. Dudley, manager of the System Open Market Account and President of the Federal Reserve Bank of New York, briefing the committee about financial market’s condition. One transcript, in particular, is very revealing. Two weeks after the collapse of Lehman Brothers, on September 29, 2008, the board decided to expand the swap lines in more than 300 billion dollars to reassure market participants they were prepared to take extraordinary steps to restore liquidity in the funding markets. During the meeting, the terms “unemployment”, “inflation” and “output” (or GDP) are not mentioned once by any of the members while “financial” and “market” are cited 10 and 30 times, respectively.

This paper is organized as follows. Section 2 describes the model and introduces the equilibrium definition. Section 3 presents the results. Section 5 concludes. Proofs are presented in the Appendix.

2 Model

Consider a continuous time economy on $[0, \infty)$ where the uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with augmented filtration $\mathbb{F} = (\mathcal{F}(t))_{t \geq 0}$ generated by two independent Brownian motions: $Z^k = (Z^k(t))_{t \geq 0}, Z^m = (Z^m(t))_{t \geq 0}$. These innovations are also referred to as technological and monetary shocks, respectively, throughout the text.

All the stochastic processes are assumed to be progressively measurable with respect to $\mathbb{F}$ and all the equalities of random variables presented are considered to hold $\mathbb{P}$-a.s.

2.1 Household

Consider an infinitely lived household that maximizes the expected life-time utility $J(t)$, with

$$J(t) = \mathbb{E} \left[ \int_t^\infty e^{-\beta(s-t)} (\alpha \log c(s) + (1-\alpha) \log m(s)) \, ds \, \bigg| \mathcal{F}(t) \right],$$

(1)
by choosing the flow of consumption good \( c(t) \) and real cash balance \( m(t) \). In this formulation, \( 0 < \beta \) is the subjective rate of time preference and \( 0 < \alpha < 1 \) is the expenditure share on consumption goods. All real quantities are normalized in terms of consumption good units.

The presence of money in the utility function is usually justified by the argument that cash reduces the associated costs of obtaining a higher net consumption good. These expenditures are commonly identified as transaction or liquidity costs. Examples of monetary asset pricing models that rely on this formulation are Sidrauski (1967), Danthine and Donaldson (1986), Stulz (1986), Bakshi and Chen (1996), Buraschi and Jiltsov (2005) and references therein.

### 2.2 Price Levels

In this economy, the capital stock \( K(t) \) is the only factor of production and the final good \( Y(t) \) is generated according to the following \( AK \) technology:

\[
Y(t) = AK(t).
\]  

(2)

\( A \) is the constant production technology factor that is strictly larger than \( \beta \).

The aggregate physical capital stock has its dynamics represented by the following stochastic differential equation\(^9\):

\[
dK(t) = (I(t) - \delta K(t))dt + \sigma K(t) dZ^k(t) - \frac{\tau K(t) dp(t)}{p(t)}. \]

(3)

The capital evolution has three components. The first term on the right hand side of (3) corresponds to the standard locally deterministic component of capital accumulation and it consists of investments net of depreciated capital. Whenever the firm’s investment \( I(t) \) surpasses depreciated capital \( \delta K(t) \), capital stock is expected to grow. The second term indicates that the accumulation of capital is risky and it is exposed to technological shocks \( Z^k \).\(^{10}\) The exposure of capital to technological shocks is set exogenously at the scalar \( \sigma > 0 \). While negative shocks have the natural interpretation of capital destruction,\(^9\) and Wälde (2011) highlight that this representation of capital accumulation can lead to negative capital stock. Nevertheless, we show in Proposition 3 that the equilibrium capital stock follows a geometric stochastic process.

\(^9\) Several studies such as Rebelo and Xie (1999), Buraschi and Jiltsov (2005), Posch (2011) and Wälde (2011) rely on a risky capital accumulation process similar to equation (3) rather than the usual locally deterministic capital evolution commonly adopted by neoclassical growth models. Remarkably, Rebelo and Xie (1999) and Wälde (2011) point out that both capital evolution formulations become indistinguishable once the market clearing condition and resources constraints are imposed.
tion, positive innovations can be interpreted as gains generated by the employment of new capital relative to the old capital in place. The third component in (3) captures how changes in the endogenous price level $p(t)$ impact capital accumulation. Assuming that the parameter $\tau$ is a positive scalar between zero and one, an increase in the price level reduces the current capital stock. Therefore, the third term can be interpreted as a proxy for capital depreciation generated by a constant capital gain tax $\tau$ on the nominal value of capital imposed by fiscal authorities. Stockman (1981) argues that the negative relationship between inflation and capital accumulation arises because inflation can be viewed as an investment tax even when taxes are not explicitly modeled. The author argues that the higher savings on real cash balances induced by high inflation reduces the net return on investment in real terms. Consequently, the equilibrium steady-state of capital stock is lower. This argument is supported by the empirical findings of De Gregorio (1993) and Fischer (1993). Both authors show a statistically significant negative relationship between inflation and capital accumulation when analyzing the panel regression for several countries with high inflation and poor economic growth. According to the authors, large fiscal imbalances and inefficient tax systems contribute with inflation persistence and amplify macroeconomic uncertainty. As a result, firms reduce investments and capital accumulates at a slower pace.

2.3 Policy and money supply

A central bank controls the money supply $M^s(t)$ and conduct monetary policy taking into account macroeconomic and financial data. The macroeconomic targets are the usual inflation and output growth. Similar to Buraschi and Jiltsov (2005) and Leippold and Matthys (2015), we set exogenously the long-term targets for output and inflation at $\bar{k}$ and $\bar{\pi}$, respectively. In the spirit of McCallum (1994), we assume that the financial data supporting the central bank’s decision to adjust the money supply is the slope of the yield curve, which is approximated by the difference between the nominal zero-coupon yield maturity in $\Delta$ years, denoted by $R^n(t, t + \Delta)$, and the nominal short-rate $i(t)$.

Under these assumptions, the money supply evolves according to the following stochastic differential equation:

$$\frac{dM^s(t)}{M^s(t)} = q_1 \left( R^n(t, t + \Delta) - i(t) \right) dt + q_2 \left( \frac{dp(t)}{p(t)} - \bar{\pi} dt \right) + q_3 \left( \frac{dK(t)}{K(t)} - \bar{k} dt \right) + dg(t),$$

where $g(t)$ is an exogenous persistent component that follows an Ornstein-Uhlenbeck
It is assumed that $\kappa_g$, $\bar{g}$ and $\sigma_g$ are positive constants. The parameters $q_1$, $q_2$ and $q_3$ correspond to the central bank’s weights on each of the mandates. The magnitude of each weight reflects the importance of the respective mandate relative to the other objectives. For example, if the magnitude of $q_1$ is high relative to $q_2$ and $q_3$, it means that financial data plays a major role in the central bank’s decision to tighten or ease the money supply.

The monetary policy described in (4) encompasses two of the most popular policy descriptions: Taylor’s rule and McCallum rule. For instance, consider the case that $q_1 = 0$. The resulting monetary policy is driven solely by concerns relative to inflation and output, embedding the characteristics of the widely adopted Taylor’s rule. As pointed out by Leippold and Matthys (2015), under this configuration, if $q_3 > 0$ and the economy is expanding with current growth rate above $\bar{k}$, the central bank increases the supply of money to the economy. A similar monetary expansion is observed if $q_2 < 0$ and the economy is contracting, with current inflation below the aimed target $\bar{\pi}$.

Alternatively, consider the situation in which $q_2 = q_3 = 0$. In this instance, the central bank adjusts the money supply plainly by considering the slope of the yield curve, accommodating a monetary policy in the spirit of McCallum’s rule. If $0 < q_1$ and the curve is upward sloping, the yield curve targeting contributes to an expansion of the monetary base. In fact, the description of how the central bank takes into consideration the slope of the yield curve shown in (4) can be interpreted as if monetary authorities were trying to flat the yield curve, or simply targeting a zero slope. For this reason, we refer to $q_1$ as the smoothing parameter or smoothing intensity interchangeably throughout the text. We assume that $q_1$ and $q_3$ are positive weights while $q_2$ is negative.

Next, we present the equilibrium concept in this continuous-time production based monetary economy. It consists of the following:

**Definition 1.** The representative agent equilibrium is defined as a set of prices (interest rate, prices of risk and price level) given by the functions $r(t), \theta_k, \theta_m, p(t)$, respectively, a value function $J(K(t), g(t))$ and a set of decision rules on consumption good, real cash balances and investment, represented by the functions $\{c(t), m(t), I(t)\}$, such that the representative agent maximizes expected utility in (1) over consumption and cash goods subject to the resource constraints (3) and (2), money and good markets clear.

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11 Under a New Keynesian model, $g(t)$ would be interpreted as the degree of forward-looking behavior in the private sector. Our modeling choice of introducing persistence only through serially correlated shocks is also found in Rotemberg and Woodford (1997) and Ireland (2001).
3 Asset Pricing

This section presents the characterization of the equilibrium. We derive closed form expressions for state price density, bonds and claims over consumption. We present the optimal policies and discuss in detail the term structure emerging from our model. We start describing the equilibrium state price density in real and nominal terms in our economy.

**Proposition 2.** The real state pricing density is expressed as

$$\xi(t) = e^{-\beta t} \frac{K(0)}{K(t)}.$$  

The nominal state price density is the quotient between the real state price density $\xi(t)$ and the price level $p(t)$ and it is given by

$$\zeta(t) = \frac{\xi(t)}{p(t)}.$$  

The price level, the real and the nominal state price density satisfy the following stochastic differential equations

\[
\frac{dp(t)}{p(t)} = \pi(t) dt + \sigma_k dZ^k(t) + \sigma_m dZ^m(t),
\]

\[
\frac{d\xi(t)}{\xi(t)} = -r(t) dt - \theta_k dZ^k(t) - \theta_m dZ^m(t),
\]

\[
\frac{d\zeta(t)}{\zeta(t)} = -i(t) dt - \theta^n_k dZ^k(t) - \theta^n_m dZ^m(t),
\]

where the expected inflation $\pi(t)$, the real short-term rate $r(t)$, the real market prices of risk $\theta_k$ and $\theta_m$, the nominal short-term rate $i(t)$ and the nominal market prices of risk $\theta^n_k$ and $\theta^n_m$ are, respectively,

$$\pi(t) = \pi_0 + \pi_1 g(t), \quad \sigma_k = (q_3 - 1) \frac{\sigma}{\psi}, \quad \sigma_m = \frac{\sigma_g}{\psi},$$

$$r(t) = r_0 + r_1 g(t), \quad \theta_k = (1 - q_2) \frac{\sigma}{\psi}, \quad \theta_m = -\frac{\tau \sigma_g}{\psi},$$

$$i(t) = i_0 + i_1 g(t), \quad \theta^n_k = \sigma_k \sigma + (1 - \tau) \sigma^2_k, \quad \theta^n_m = (1 - \tau) \sigma^2_m,$$
with

\[
\begin{align*}
\pi_1 &= \frac{\Delta \kappa_g^2}{q_1 (1 - \tau)(1 - e^{-\kappa g \Delta} - \Delta \kappa_g) - \Delta \kappa_g \psi}, \\
i_1 &= (1 - \tau) \pi_1, \\
r_1 &= -\tau \pi_1,
\end{align*}
\]

\[
\begin{align*}
r_0 &= A - \delta - \tau \pi_0 - (\sigma - \tau \sigma_k)^2 - (\tau \sigma_m)^2, \\
\psi &= 1 - q_2 + \tau (q_3 - 1),
\end{align*}
\]

\[
\begin{align*}
i_0 &= r_0 + \pi_0 - \sigma_k (\sigma_k + \theta_k) - \sigma_m (\sigma_m + \theta_m), \\
\pi_0 &= \frac{q_1 (1 - \tau) \pi_1}{\psi} \left( \bar{g} - \frac{(1 - \tau) \sigma_k^2}{\psi \kappa_g} \left( e^{-\kappa g \Delta} - 1 \right) + 1 \right) \right. \\
&\quad - \frac{q_1 (\sigma_g (1 - \tau) \pi_1)^2}{2 \psi \kappa_g} \left( 1 + \frac{4 e^{-\kappa g \Delta} - 3 - e^{-2 \kappa g \Delta}}{2 \kappa g \Delta} \right) \\
&\quad + \frac{\kappa \bar{g} - q_2 \bar{\pi} - q_3 \bar{k} + \tau \sigma_m^2 - \sigma_k (\sigma - \tau \sigma_k) + (q_3 - 1)(A - \delta - \beta)}{\psi}.
\end{align*}
\]

Nominal and real market prices of risk presented in Proposition 2 are constant. Moreover, the market prices of risk are only affected by the weights relative to the macroeconomic quantities. Notice that monetary shocks carry a negative risk premium since the assumptions on the weights implies that $0 < \psi$.

Monetary shocks have an additional effect on the pricing kernel’s evolution through their impact on the expected growth rate of the state price density. As it is indicated in Proposition 2, both nominal and real short-rate are affine functions of the money supply persistent component $g(t)$. However, the exposure of these rates to $g(t)$ have opposite signs. Since we assume $0 < q_1$ and $0 < \psi$, it follows from the expression presented in Proposition 2 that $\pi_1 < 0$. Consequently, the nominal short-rate is negatively related to positive monetary shocks while the real short-term rate is positively related to it. Contrary to the market prices of risk, the nominal and the real short-term interest rate are affected by macro and financial weights.

A natural question that emerges from this analysis is to what extent the short-term interest rates are affected by a change in policy aiming to smooth the slope of the yield curve. Does the central bank reduce or boost the interest rates’ volatility as it pursue a more intense interest rate smoothing policy? What are the effects on the long-term yields’ volatility under this new regime? We postpone these exciting questions and address all related policy implication inquiries in Section 4.

With the complete characterization of the state price density, we turn to the description of the optimal policies and the endogenous evolution of capital. The next proposition presents the results.

**Proposition 3.** The optimal consumption, cash holdings and investment policies are,
respectively,

c(t) = \alpha \beta K(t), \quad m(t) = (1 - \alpha) \beta K(t), \quad I(t) = (A - \beta) K(t).

The equilibrium capital accumulation satisfies the following stochastic differential equation:

\[ \frac{dK(t)}{K(t)} = (A - \beta - \delta - \tau \pi_0 - \tau \pi_1 g(t)) dt + (\sigma - \tau \sigma_k) dZ^k(t) - \tau \sigma_m dZ^m(t). \]

The endogenous capital stock plays a central role in the characterization of the equilibrium. Firstly, the optimal policies presented in Proposition 3 are all linear in the capital level. The result arises from the scaling property of the value function. Secondly, the stochastic differential equation representing the capital evolution illustrates the propagation mechanism through which monetary shocks affect expected capital growth. Nominal price changes directly impact the level of capital stock while changes in expected inflation affect the expected growth rate of capital accumulation. Since the expected inflation is perfectly negatively correlated with monetary shocks (\( \pi_1 < 0 \)), positive monetary innovations increase the expected capital growth and decrease the current capital stock level. Consequently, consumption, money demand and investment decrease with positive monetary shocks.

The characterization of the state price density and the price level allows us to determine bond prices as well as the price of claims on consumption goods, both in nominal and real terms. We start defining the zero-coupon yield and term premium that are explicitly determined in our equilibrium model.

**Definition 4.** Let nominal and real quantities be indexed by \( j \in \{n, r\} \), respectively. Define the nominal (real) \((t + u)\)-year zero-coupon yield as the discount rate that makes the present value of one unit of money (consumption) equal to the arbitrage-free price of the nominal (real) zero-coupon bond, i.e.,

\[ R^j(t, t + u) = -\frac{1}{u} \ln B^j(t, t + u). \]

The next proposition shows the expression for bond prices and the yield curve.

**Proposition 5.** The zero-coupon bond price maturing at date \( t + u \) is

\[ B^j(t, t + u) = e^{-\rho^j_0 u + \eta^j_0 (u) + (\eta^j_1 (u) - \omega_j) g(t)} \] (6)
where

\[ \eta_1(u) = \frac{\pi_1}{\kappa_g} \left( e^{-\kappa_g u} - 1 \right) - \frac{1}{\psi}, \quad \eta_1^n(u) = (1 - \tau) \eta_1(u), \quad \eta_1^r(u) = -\tau \eta_1(u), \]

\[ \omega_n = -\frac{1 - \tau}{\psi}, \quad \omega_r = \frac{\tau}{\psi}, \]

\[ \rho_0^n = A - \delta + (1 - \tau) \pi_0 - \frac{(1 - \tau) \kappa_g \bar{g}}{\psi} - \frac{(1 + \tau^2) \sigma_n^2}{2} - \sigma^2 - \sigma_k^2 (1 - \tau + \tau^2), \]

\[ -\sigma \sigma_k (1 - 2 \tau), \]

\[ \rho_0^r = A - \delta - \tau \pi_0 + \frac{\tau \kappa_g \bar{g}}{\psi} - (\sigma - \tau \sigma_k)^2 - \frac{(\tau \sigma_m)^2}{2}, \]

\[ \eta_0^n(u) = (1 - \tau) \left( \eta_1(u) + \frac{1}{\psi} + \left( \pi_1 + \frac{\kappa_g}{\psi} \right) u \right) \left( \frac{(1 - \tau) \sigma_n^2}{2 \kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) - \bar{g} \right), \]

\[ + \frac{\sigma_g^2 (1 - \tau)^2}{4 \kappa_g} \left( \frac{1}{\psi^2} - (\eta_1(u))^2 \right) \]

\[ \eta_0^r(u) = \tau \left( \eta_1(u) + \frac{1}{\psi} + \left( \pi_1 + \frac{\kappa_g}{\psi} \right) u \right) \left( \frac{\tau \sigma_n^2}{2 \kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) + \bar{g} \right), \]

\[ + \frac{\sigma_g^2 \tau^2}{4 \kappa_g} \left( \frac{1}{\psi^2} - (\eta_1(u))^2 \right). \]

The nominal and the real zero-coupon yields are, respectively,

\[ R^n(u) = \rho_0^n - \frac{\eta_0^n(u)}{u} + (1 - \tau) \pi_1 \left( \frac{1 - e^{-\kappa_g u}}{\kappa_g u} \right) g(t), \]

\[ R^r(u) = \rho_0^r - \frac{\eta_0^r(u)}{u} - \tau \pi_1 \left( \frac{1 - e^{-\kappa_g u}}{\kappa_g u} \right) g(t). \]

The bond prices presented in Proposition 5 belong to the class of one factor models introduced by Vasicek (1977). The sole driving factor moving bond prices in this monetary based production economy is the money supply persistent component \( g(t) \). Consequently, all yield rates are perfectly correlated across maturities.

The equilibrium nominal and real yield curve belong to the class of the arbitrage-free Nelson-Siegel model proposed by Christensen et al. (2011). In addition, the exposure of nominal and real yields to the persistent component \( g(t) \) have opposite signs. In particular, positive monetary innovations increase real yields and decrease nominal rates.
4 Policy Implications

One of the main objectives of our study is to understand how changes in the interest rate smoothing policy affect the term structure of interest rates, expected inflation, prices of risk and the volatility of interest rates. In fact, this objective can be achieved by analyzing the response of such equilibrium variables characterized in Section 3 to variations of $q_1$.

We start investigating what happens with the market prices of risk when the central bank changes its yield curve smoothing policy. The expressions for $\theta_m$ and $\theta_k$ derived in Proposition 2 show that both market prices of risk are independent of $q_1$. Consequently, the only policy changes affecting the prices of risk are those affecting the macroeconomic targets, i.e., $q_2$ and $q_3$.

The next sections characterize the quantities that are affected by changes in the smoothing policy.

4.1 Policy change effect on the spread between the long and short-term nominal rate

The money supply equation in (4) explicitly states that monetary authorities try to smooth the yield curve. Initially, one may think that a policy change that pursues a more aggressive yield curve smoothing would necessarily be successful in reducing the spread between the short and long-term interest rate. However, in our model, both rates are endogenously determined. Consequently, it is not clear a priori that, even if the central bank explicitly adopts a more strict yield curve smoothing policy, it would automatically generate the intended outcome of reducing the spread. The next theorem shows that when $q_1$ is not “very large”, a policy change aiming to further smooth the yield curve is successful in reducing the spread and the unintended consequence of increasing the term structure slope is prevented.

**Proposition 6.** Define the nominal spread between the nominal long-term rate and the nominal short-term rate as

$$s(t, T) = R^n(t, T) - i(t).$$

If

(i) $0 < g(t)$,

(ii) $0 < \frac{\sigma_g^2 (1 - \tau)}{\psi \kappa g} - \bar{g}$,
(iii) \( 0 < q_1 < \bar{q} \), where \( \bar{q} \) is defined in Appendix A,

then,

(i) \( 0 < s(t, T), \)

(ii) \( \frac{ds(t, T)}{dq_1} < 0. \)

Proposition 6 contains two important results. Firstly, the positive spread indicates that the nominal yield curve is upward sloping. Secondly, the spread is a decreasing function of the weight \( q_1 \). Thus, a policy change that pursues a more aggressive yield curve smoothing is successful in reducing the spread. In essence, the theorem ensures that for “moderate” values of \( q_1 \) a policy change aiming to further smooth interest rate can be effective in achieving the initial objective of reducing the spread.

4.2 Policy change effect on equity volatility

Our model allows us to investigate what is the impact of policy changes on the volatility of contingent claims on consumption. Similar to Abel (1999) and Bansal and Yaron (2004), we denote by equity a claim on leveraged consumption. The price and the volatility of such claim is presented next.

**Proposition 7.** Suppose dividends are leveraged consumption, i.e.,

\[ D(t) = c(t)^\ell, \]

where \( \ell \) is the leverage ratio. Let \( P(t,T) \) represent the price of contingent claim over dividends paid between \( t \) and \( T \).

The fair price of this claim is

\[
P(t, T) = \mathbb{E}_t \left[ \int_t^T \frac{\xi(s)}{\xi(t)} c(s)^\ell ds \right] = (\alpha \beta K(t))^{\ell} h(t, T, g(t)),
\]

where \( h(\cdot, \cdot, \cdot) \) is a nonlinear function of the state variable \( g(t) \) described in Appendix A.

The equity volatility can be expressed as

\[
\sigma_E = \sqrt{(\sigma - \tau \sigma_k)^2 + \left( \frac{h_g}{h} \sigma_g - \tau \sigma_m \right)^2},
\]
where \( h_g \) is the partial derivative of \( h(\cdot, \cdot, \cdot) \) with respect to the variable \( g \).

The functional form of the equity price displayed in Proposition 7 illustrates that the expression is nonlinear on both state variables. To understand the effects of yield curve smoothing on equity volatility, we need to rely on a numerical exercise since comparative statics is not an option anymore. Figure 1 shows how equity volatility responds to a policy change towards yield curve smoothing for the calibration presented in Table 3. The graph shows that equity volatility is reduce when the central bank takes into consideration financial data to adjust the money stock relative to the benchmark model where the central bank only relies on macroeconomic data to conduct monetary policy.

**Figure 1**: The figure illustrates how the equity volatility responds to changes in \( q_1 \). The solid line illustrates the behavior of equity volatility as we increase the smoothing intensity. The dashed line is the benchmark model where \( q_1 = 0 \). The calibration is presented in Table 3.

Contrary to the findings of Bakshi and Chen (1996), monetary shocks impact not only the short-rate and the term structure of interest rates but also the equity dynamics. The mechanisms driving these results are the following: (i) the explicit impact of price level variation in the capital evolution and, (ii) monetary authorities targeting output and inflation. Both mechanisms establish the link between the real sector and the monetary policy, making real output and money supply indissociable. Consequently, the real pricing kernel, which is fully characterized by capital stock, is also impacted by monetary shocks as well as any claim priced in real terms.
4.3 Policy change effect on interest rates volatility

The policy change on the yield curve smoothing has also an impact on nominal and real short-rate, expected inflation as well as on nominal and real yield rates. To assess its impact on the volatility of these quantities, we exploit the tractability of the equilibrium expressions obtained in Appendix A and calculate the sensitivity of the quadratic variation of nominal short-rate, real short-rate, expected inflation, nominal and real long-term rate relative to $q_1$. Corollary 8 summarizes the result.

**Proposition 8.** The quadratic variation of expected inflation, nominal long-term and short-term rates are decreasing functions of the weight $q_1$, while the quadratic variation of the real long-term and short-term rates are increasing functions of $q_1$.

<table>
<thead>
<tr>
<th>Quadratic Variation</th>
<th>$<a href="t">\pi, \pi</a>$</th>
<th>$<a href="t">i, i</a>$</th>
<th>$<a href="t">r, r</a>$</th>
<th>$[R^n, R^n](t, T)$</th>
<th>$[R^r, R^r](t, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dq_1}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

**Table 1:** The table displays the sensitivities of the quadratic variation of macro-financial variables relative to changes in the weight $q_1$.

A policy change aiming to further flatten the yield curve reduces the exposure of the expected inflation and the nominal rates to monetary shocks, while it increases the exposure of real rates to these shocks.

4.4 Policy change effect on shocks’ persistence

Another effect of interest rates smoothing investigated by us is how a change in monetary policy impacts the propagation of shocks to macroeconomic variables. As the central bank changes its yield curve smoothing policy, it affects the persistence of monetary shocks on capital and price level, perpetuating the long-lasting effects to output, consumption, real cash balances and investment. To investigate such effect, we need first to establish the linkage between the current policy and the shock propagation to capital and price level. Once we understand this link, we can analyze its sensitive with respect to variations in the yield curve smoothing policy. The natural approach to unveil the relationship is to perform an analysis of dynamics of capital and price, and explore the associated impulse response functions.
Impulse response functions are extensively used in the macroeconomic literature to assess how shocks propagate through a system with several variables and the duration of the shocks’ effect on these equilibrium quantities.

The next proposition presents the impulse response functions of output and price level with respect to monetary and technological shocks. It helps to illustrate under which conditions a policy change could make monetary shocks more or less persistent.

**Proposition 9.** Define the impulse response functions of output and price level with respect to monetary and technological shocks as the conditional expectation of their normalized Malliavin derivatives $D$ in the direction of the each shock, i.e,

$$
\varepsilon_{t,T}^{Y,m} = \frac{E_t[D^m_t Y(T)]}{E_t[Y(T)]}, \quad \varepsilon_{t,T}^{Y,k} = \frac{E_t[D^k_t Y(T)]}{E_t[Y(T)]}
$$

$$
\varepsilon_{t,T}^{p,m} = \frac{E_t[D^m_t p(T)]}{E_t[p(T)]}, \quad \varepsilon_{t,T}^{p,k} = \frac{E_t[D^k_t p(T)]}{E_t[p(T)]}
$$

Thus, the impulse response functions of output and price level with respect to monetary and technological shocks are, respectively,

$$
\varepsilon_{t,T}^{Y,m} = -\frac{\pi_1 \sigma_g}{\kappa_g} (1 - e^{-\kappa_g(T-t)}) - \frac{\tau \sigma_g}{\psi}, \quad \varepsilon_{t,T}^{Y,k} = \sigma - \tau \sigma_k,
$$

$$
\varepsilon_{t,T}^{p,m} = \frac{\pi_1 \sigma_g}{\kappa_g} (1 - e^{-\kappa_g(T-t)}) + \frac{\sigma_g}{\psi}, \quad \varepsilon_{t,T}^{p,k} = \sigma_k.
$$

The impulse response sensitivities with respect to the yield curve slope weight $q_1$ are summarized in the following table:

<table>
<thead>
<tr>
<th>Impulse Response</th>
<th>$d$</th>
<th>$dq_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{t,T}^{Y,m}$</td>
<td>$\varepsilon_{t,T}^{Y,k}$</td>
<td>$\varepsilon_{t,T}^{p,m}$</td>
</tr>
<tr>
<td>$-\pi_1 \sigma_g$</td>
<td>$\tau \sigma_g$</td>
<td>$\pi_1 \sigma_g$</td>
</tr>
</tbody>
</table>

Table 2: The table displays the sensitivities of the impulse response functions relative to changes in the weight $q_1$.

The expressions in Proposition 9 illustrate that technological shocks are time independent. Consequently, technological innovations have a permanent impact on output.
and price level. In addition, policy changes affecting the weight of financial data do not impact the persistence of technological shocks on output and price level.

Monetary shocks have a different impact on macroeconomic quantities. The impulse response functions relative to monetary shocks have two components: (i) a transient and (ii) a permanent component. The transient component is an exponentially decreasing function of time that converges to zero in the limit. The impulse response function of output has a positive transient component since $\pi_1 < 0$, while the permanent component is negative and independent of the smoothing intensity $q_1$. Note that the impulse response function of price level has the signs reverse. While the permanent component is positive indicating that positive monetary shocks has a persistent effect on the price level, the transient has a negative effect that decays to zero as $T \to \infty$.

The impulse response sensitivities with respect to the interest rate smoothing parameter are presented in Table 2. The sensitivity of output and price with respect to $q_1$ have opposite signs. The analysis indicates that a more intense yield smoothing policy amplifies the decreases the impulse response function of output while it decreases the impulse response function of price level. Thus, monetary shocks have their persistence effect on output dampened under a more vigorous interest rate smoothing regime.

5 Conclusion

We present a macro-finance model in which monetary authorities adjust the money supply by targeting not only output and inflation but also the slope of the yield curve. Under a continuous-time production based monetary economy, the money supply persistent component becomes the driving factor of the long and short-term rate as well as expected inflation movements. Moreover, the endogenous characterization of these quantities in closed form allow us to study how the decision to incorporate financial data into monetary decision affect the volatility of interest rates, the spread between long and short-term rate and the persistence of monetary shocks on macroeconomic variables.

Our findings support the Federal Reserve’s interventions in 2007-2009 to stabilize financial markets and restore economic confidence. By incorporating financial data in their decision to expand the money base can result into a smaller interest rate spread.

There are several venues that can be explored departing from our paper. The most obvious one is an empirical analysis to understand what are the regularities that the model is able to reproduce. Second, one can explore what are the consequences of introducing other mandates such as the volatility of the nominal rate, the volatility of the stock market such as the VIX or even a house price index to control for credit bubbles on the housing
markets. Third, questions related to the optimal delegation and the implications for the welfare of the representative agent can also be investigated under our framework.

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A Equilibrium Characterization

The methodology we use to characterize the equilibrium can be summarized in the following steps:

1. Given the affine structure of the problem, conjecture an affine structure for the dynamics of the price level.

2. Using the evolution of the price level, obtain the dynamics for capital that only depends on the capital itself and the state variable $g(t)$.

3. Set the Hamilton-Jacobi-Bellman equation for the representative agent using the capital accumulation equation (2) and the resources constraint.

4. Using the first order conditions, obtain the optimal policies for consumption and money demand.

5. Plugging the controls back into the Hamilton-Jacobi-Bellman equation and using the traditional guess for the value function:

$$J(K(t), g(t)) = \frac{1}{\beta} \log(\beta K(t)) + \mu_1 g(t) + \mu_0,$$

obtain a system that it is linear on the state variable $g(t)$.

6. Using the method of undetermined coefficients, solve for $\mu_0$ and $\mu_1$.

7. Substitute the conjectured solution back into the Hamilton-Jacobi-Bellman equation and verified that it is in fact the true solution.

8. Obtain the optimal policies for consumption, money and investment as a linear function on capital.
9. Conjecture an affine structure in the state variables for the short-rate and use it with the market clearing for money market to obtain a new equation representing the dynamics of the price level \( p(t) \).

10. Use the method of undetermined coefficients to pin down the price level exposures to each shock.

11. Using household preferences and the capital accumulation equation, derive the pricing kernel expression for the economy.

12. Apply Ito’s lemma to recover the short-rate and solve the fixed point problem for the exposure of \( r(t) \) to \( g(t) \).

13. Derive the expression for the nominal bond using the pricing kernel.

14. Derive the explicit solution for the real yield and take the limit on the maturity to obtain the long-term rate and conclude the characterization of the equilibrium.

**Optimal Policies.** There are two state variables in the problem: capital \( K(t) \) and the money supply persistent component \( g(t) \). Capital accumulates according to the equation:

\[
dK(t) = (I(t) - \delta K(t))dt + \sigma K(t) dZ^k(t) - \tau K(t) \frac{dp(t)}{p(t)}. \tag{8}
\]

Combining the firm’s output equation

\[
Y(t) = AK(t),
\]

with the resource

\[
Y(t) = I(t) + c(t) + m(t),
\]

the investment policy can be rewritten as

\[
I(t) = AK(t) - c(t) - m(t). \tag{9}
\]

Substituting (9) into (8), we have that capital evolves according to the following stochastic differential equation:

\[
dK(t) = (A - \delta)K(t) - c(t) - m(t))dt + \sigma K(t) dZ^k(t) - \tau K(t) \frac{dp(t)}{p(t)}. \tag{10}
\]
Following Buraschi and Jiltsov (2005), we assume there exist equilibrium processes for expected inflation $\pi(t)$, the nominal short-rate $i(t)$, the real short-rate $r(t)$, the nominal long-term interest rate $R^*(t,t+\Delta)$ and the real long-term interest rate $R^r(t,t+\Delta)$ that are affine functions of the state variable $g(t)$,

$$
\begin{align*}
\pi(t) &= \pi_0 + \pi_1 g(t), \\
i(t) &= i_0 + i_1 g(t), \\
r(t) &= r_0 + r_1 g(t), \\
R^r(t,t+\Delta) &= R^r_0(\Delta) + R^r_1(\Delta) g(t), \\
R^*(t,t+\Delta) &= R^*_0(\Delta) + R^*_1(\Delta) g(t),
\end{align*}
$$

while that the price level dynamics $p(t)$ can be expressed as:

$$
\frac{dp(t)}{p(t)} = \pi(t) dt + \sigma_k dZ^k(t) + \sigma_m dZ^m(t).
$$

The constants $\sigma_k, \sigma_m, \pi_0, \pi_1, i_0, i_1, r_0, r_1, R^*_0(\Delta), R^*_1(\Delta), R^r_0(\Delta)$ and $R^r_1(\Delta)$ are determined later in the proof when the money market clearing condition is imposed. Once these quantities are characterized, the conjectured structure can be verified to be the true solution of the optimization problem.

By substituting the evolution of $p(t)$ described in (69) and the expression for $\pi(t)$ displayed in (68) into the evolution of capital in (10), it follows that

$$
\begin{align*}
dK(t) &= ((A - \delta - \tau(\pi_0 + \pi_1 g(t)))K(t) - c(t) - m(t))dt + (\sigma - \tau\sigma_k)K(t)dZ^k(t) \\
&\quad - \tau\sigma_m K(t)dZ^m(t).
\end{align*}
$$

The representative agent solves the problem of maximizing expected utility in (1), subject to (13) and the evolution of the state variables $g(t)$:

$$
dg(t) = \kappa_g(\bar{g} - g(t))dt + \sigma_g dZ^m(t).
$$
The associated Hamilton-Jacobi-Bellman (HJB) equation is

\[ 0 = \sup_{c(t), m(t)} \left\{ -\beta J + \alpha \log c(t) + (1 - \alpha) \log m(t) \right. \]

\[ + J_K \left( (A - \delta - \tau (\pi_0 + \pi_1 g(t)))K(t) - c(t) - m(t) \right) + K(t)^2 J_{KK} \frac{(\sigma - \tau \sigma_k)^2 + (\tau \sigma_m)^2}{2} \]

\[ + J_g \kappa_g (\bar{g} - g(t)) + \frac{\sigma_g^2}{2} J_{gg} - \tau \sigma_m \sigma_g K(t) J_{Kg} \left. \right\}. \]  

(14)

The first order condition with respect to \( c(t) \) and \( m(t) \) give

\[ c(t) = \frac{\alpha}{J_K}, \]

\[ m(t) = \frac{1 - \alpha}{J_K}. \]

Plugging the optimal controls and the conjecture

\[ J(K(t), g(t)) = \frac{1}{\beta} \log(\beta K(t)) + \mu_1 g(t) + \mu_0 \]

back into (14), we obtain

\[ 0 = -2A + 2\beta (\beta \mu_0 - \bar{g} \kappa_g \mu_1 + 1) + 2\delta + 2\tau \pi_0 + (\sigma - \tau \sigma_k)^2 + (\tau \sigma_m)^2 \]

\[ + (1 - \alpha) \log(1 - \alpha) + \alpha \log \alpha + g(t) \left( 2\beta \mu_1 (\beta + \kappa_g) + 2\pi_1 \tau \right). \]

For this equation to be satisfied at all time, we use the method of undetermined coefficients, set all coefficients to zero and solve for \( \mu_0 \) and \( \mu_1 \). It follows that

\[ \mu_1 = \frac{-\pi_1 \tau}{\beta (\beta + \kappa_g)}, \]

\[ \mu_0 = \frac{A - \beta - \delta + \beta \bar{g} \kappa_g \mu_1 - \tau \pi_0 - (1 - \alpha) \log(1 - \alpha) - \alpha \log \alpha - (\sigma - \tau \sigma_k)^2 + (\tau \sigma_m)^2}{\beta^2}. \]

With the conjectured solution completely characterized, a substitution of (72) into (14) verifies that it is indeed the true solution of the dynamic programming problem. At this stage, we showed that, given constants \( \sigma_k, \sigma_m, \pi_0, \pi_1, r_0 \) and \( r_1 \), (72) is the true solution associated with the Hamilton-Jacobi-Bellman equation (14). The next steps demonstrate how to pin down these constants.

First, we use the expression derived for (72) to obtain the following expressions for
the optimal policies on consumption goods, real cash balance and investment, respectively,

\[ c(t) = \alpha \beta K(t), \]
\[ m(t) = (1 - \alpha) \beta K(t), \]
\[ I(t) = (A - \beta) K(t). \]

By replacing the optimal policies into (13), the endogenous capital evolution becomes:

\[
\frac{dK(t)}{K(t)} = (A - \delta - \beta - \tau(\pi_0 + \pi_1 g(t))) dt + (\sigma - \tau \sigma_k) dZ^k(t) - \tau \sigma_m dZ^m(t),
\]

(16)

with solution given by

\[
K(T) = K(t) \exp \left\{ (A - \delta - \beta - \tau \pi_0 - (\sigma - \tau \sigma_k) \frac{\sigma_m^2}{2} - (\tau \sigma_m^2) \frac{\tau}{2} ) (T - t) - \tau \pi_1 \int_t^T g(v) dv 
\right. 
\]
\[
+ (\sigma - \tau \sigma_k)(Z^k(T) - Z^k(t)) - \tau \sigma_m(Z^m(T) - Z^m(t)) \right\}. 
\]

(17)

Next, we turn to the money market. By imposing that money demand equals to money supply, it follows that:

\[ M^s(t) = m(t)p(t) = p(t)(1 - \alpha) \beta K(t). \]

An application of Ito’s lemma leads to the following expression for the price level process:

\[
\frac{dp(t)}{p(t)} = \frac{dM^s(t)}{M^s(t)} - \frac{dK(t)}{K(t)} - \left[ \frac{dp}{p}, \frac{dK}{K} \right]_t dt.
\]

(18)

From (69) and (74), the quadratic covariation between \( dp(t)/p(t) \) and \( dK(t)/K(t) \) is:

\[
\left[ \frac{dp}{p}, \frac{dK}{K} \right]_t = \sigma_k (\sigma - \tau \sigma_k) - \tau \sigma_m^2.
\]

(19)

Substituting the money supply equation

\[
\frac{dM^s(t)}{M^s(t)} = q_1 \left( R^n(t, t + \Delta) - i(t) \right) dt + q_2 \left( \frac{dp(t)}{p(t)} - \bar{\pi} dt \right) + q_3 \left( \frac{dK(t)}{K(t)} - \bar{k} dt \right) + dg(t),
\]

(20)

the nominal short-rate (68) and the quadratic covariation (77) into (76), the dynamics for
the price level \( p(t) \) can be represented by the following stochastic differential equation:

\[
\frac{dp(t)}{p(t)} = \frac{1}{1 - q_2} \left( q_1(R^n_0(\Delta) - i_0) - (q_2\bar{\pi} + q_3\bar{\kappa}) + \kappa_g\bar{g} - \sigma_k(\sigma - \tau\sigma_k) \\
+ \tau\sigma_m^2 + (q_3 - 1)(A - \delta - \beta - \tau\pi_0) + (q_1(R^n_1(\Delta) - i_1) - \tau(q_3 - 1)\pi_1 - \kappa_g)g(t) \right) dt \\
+ \frac{q_3 - 1}{1 - q_2}(\sigma - \tau\sigma_k)dZ^k(t) + \frac{\sigma_g - \tau(q_3 - 1)\sigma_m}{1 - q_2}dZ^m(t).
\]

By defining

\[
\psi = 1 - q_2 + \tau(q_3 - 1),
\]

and matching the coefficients of (79) with the ones in (68), it follows that

\[
\sigma_k = \frac{(q_3 - 1)\sigma}{\psi}, \quad \sigma_m = \frac{\sigma_2}{\psi}, \quad \pi_1 = \frac{q_1(R^n_1(\Delta) - i_1) - \kappa_g}{\psi}, \\
\pi_0 = \frac{q_1(R^n_0(\Delta) - i_0) - q_2\bar{\pi} - q_3\bar{\kappa} + \kappa_g\bar{g} + \tau\sigma_m^2 - \sigma_k(\sigma - \tau\sigma_k) + (q_3 - 1)(A - \delta - \beta)}{\psi}
\]

In order to complete the equilibrium characterization, we still need to determine \( i_0, i_1, r_0, r_1, R^n_0(\Delta), R^n_1(\Delta) \) and \( R^n(\Delta) \). For this reason, we turn to the characterization of the real and nominal state price density in this economy. We start characterizing the real state price density, which is given by

\[
\xi(t) = e^{-\beta t}\frac{K(0)}{K(t)}.
\]

An application of Ito’s lemma leads to

\[
\frac{d\xi(t)}{\xi(t)} = -\left( A - \delta - \tau\pi_0 - (\sigma - \tau\sigma_k)^2 - (\tau\sigma_m)^2 - \tau\pi_1g(t) \right) dt \\
- (\sigma - \tau\sigma_k)dZ^k(t) + \tau\sigma_mdZ^m(t).
\]
The structure above implies that the prices of risk are

$$\theta_k = \sigma - \tau \sigma_k,$$
$$\theta_m = - \tau \sigma_m,$$

while the real short-rate is

$$r(t) = A - \delta - \tau \pi_0 - (\sigma - \tau \sigma_k)^2 - (\tau \sigma_m)^2 - \tau \pi_1 g(t).$$

We note that the expression for the real short-rate is an affine function of $g(t)$ and can be written as

$$r(t) = r_0 + r_1 g(t),$$

where

$$r_1 = - \tau \pi_1, \quad r_0 = A - \delta - \tau \pi_0 - (\sigma - \tau \sigma_k)^2 - (\tau \sigma_m)^2.$$  \hspace{1cm} (24)

Next, we characterize the nominal state price density $\zeta(t)$, given by

$$\zeta(t) = \frac{\xi(t)}{p(t)}.$$ \hspace{1cm} (25)

An application of Ito’s lemma gives the following stochastic differential equation for the nominal state price density:

$$\frac{d\zeta(t)}{\zeta(t)} = - (r(t) + \pi(t) - \sigma_k (\sigma_k + \theta_k) - \sigma_m (\sigma_m + \theta_m)) dt$$
$$- (\sigma_k + \theta_k) dZ^k(t) - (\sigma_m + \theta_m) dZ^m(t).$$ \hspace{1cm} (26)

Thus, the nominal short-term interest rate, represented by the drift of (26), is

$$i(t) = r(t) + \pi(t) - \sigma_k (\sigma_k + \theta_k) - \sigma_m (\sigma_m + \theta_m) = i_0 + i_1 g(t),$$

where

$$i_1 = r_1 + \pi_1,$$
$$i_0 = r_0 + \pi_0 - \sigma_k (\sigma_k + \theta_k) - \sigma_m (\sigma_m + \theta_m).$$ \hspace{1cm} (27)

Substituting the expressions for $r_0$ and $r_1$ in (24), and the expressions for $\theta_m$ and $\theta_k$
in (22) into (90), we have that

\[ i_0 = A - \delta + (1 - \tau)\pi_0 - \sigma^2 - \sigma\kappa(1 - 2\tau) - (1 - \tau + \tau^2)(\sigma_k^2 + \sigma_m^2), \]

\[ i_1 = (1 - \tau)\pi_1. \] (28)

Substituting the expression for \( i_1 \) presented in (28) into (22), we conclude that

\[ \pi_1 = \frac{q_1 R^n_1(\Delta) - \kappa g}{\psi + q_1(1 - \tau)} \] (29)

The only remaining quantity to be determined are the nominal and real yield rate for the bond maturing in \( \Delta \) years. For this reason, we turn to the characterization of the nominal and real bond price, and their respective yields. Once these quantities are determined, the equilibrium characterization is completed.

**Proof of Real and Nominal Bonds.** Using the expression for the nominal state price density in (25), the nominal zero-coupon bond price with maturity at time \( T \) can be expressed as

\[ B^n(t, T) = \mathbb{E}_t \left[ \frac{\xi(T) p(t)}{\xi(t) p(T)} \right] = \mathbb{E}_t \left[ e^{-\beta(T-t)} \frac{K(t) p(t)}{K(T) p(T)} \right], \] (30)

The solution for the stochastic differential equation in (69) is

\[ p(T) = p(t) \exp \left\{ \int_t^T \left( \pi_0 + \pi_1 g(s) - \frac{\sigma_k^2}{2} - \frac{\sigma_m^2}{2} \right) ds + \sigma_k (Z^k(T) - Z^k(t)) + \sigma_m (Z^m(T) - Z^m(t)) \right\}. \] (31)

Rewriting the solution of \( g(t) \) in (5) as

\[ \sigma_g (Z^m(T) - Z^m(t)) = g(T) - g(t) - \kappa_g \bar{g}(T - t) + \kappa_g \int_t^T g(s)ds, \]

and substituting it back into (31), the ratio \( p(t)/p(T) \) becomes

\[ \frac{p(t)}{p(T)} = \exp \left\{ \left( \frac{\sigma_k^2 + \sigma_m^2}{2} + \frac{\sigma_m}{\sigma_g} \kappa_g \bar{g} - \pi_0 \right) (T - t) - \left( \pi_1 + \kappa_g \frac{\sigma_m}{\sigma_g} \right) \int_t^T g(s)ds \right. \]

\[ \left. - \sigma_k (Z^k(T) - Z^k(t)) - \frac{\sigma_m}{\sigma_g} (g(T) - g(t)) \right\}. \] (32)
Similarly, the ratio $K(t)/K(T)$ becomes

$$
\frac{K(t)}{K(T)} = \exp \left\{ \left( \delta + \beta - A + \tau \pi_0 - \frac{\sigma_m}{\sigma_g} \kappa_g \bar{g} - \frac{(\sigma - \sigma_k)^2 + (\tau \sigma_m)^2}{2} \right) (T - t) \\
- (\sigma - \tau \sigma_k)(Z^k(T) - Z^k(t)) + \tau \frac{\sigma_m}{\sigma_g} (g(T) - g(t)) + \tau \left( \pi_1 + \kappa_g \frac{\sigma_m}{\sigma_g} \right) \int_t^T g(s)ds \right\}.
$$

(33)

Using that the shocks are independent and that $\frac{\sigma_g}{\sigma_m} = \frac{1}{\psi}$, a substitution of (95) and (33) into (30) gives

$$
B^n(t, T) = \mathbb{E}_t \left[ \exp \left\{ \left( \delta - A + (\tau - 1)\pi_0 + \frac{(1 - \tau)\kappa_g \bar{g}}{\psi} + \frac{(1 + \tau^2)\sigma_m^2 + \sigma_k^2 + (\sigma - \tau \sigma_k)^2}{2} \right) (T - t) \\
- (\sigma + (1 - \tau)\sigma_k)(Z^k(T) - Z^k(t)) - \frac{1 - \tau}{\psi} (g(T) - g(t)) - (1 - \tau) \left( \pi_1 + \kappa_g \frac{\sigma_m}{\psi} \right) \int_t^T g(v)dv \right\} \right] \\
= e^{\frac{1 - \tau}{\psi} g(t) - \rho^n_0(T - t)} \mathbb{E}_t \left[ e^{-\frac{1 - \tau}{\psi} g(T) - (1 - \tau)(\pi_1 + \frac{\sigma_k}{\psi}) \int_t^T g(v)dv} \right].
$$

(34)

where

$$
\rho^n_0 = A - \delta + (1 - \tau)\pi_0 - \frac{(1 - \tau)\kappa_g \bar{g}}{\psi} - \frac{(1 + \tau^2)\sigma_m^2 + \sigma_k^2}{2} - \sigma^2 - \sigma_k^2 (1 - \tau + \tau^2) - \sigma \sigma_k (1 - 2\tau).
$$

$$
i_0 = A - \delta + (1 - \tau)\pi_0 - \sigma^2 - \sigma \sigma_k (1 - 2\tau) - (1 - \tau + \tau^2)(\sigma_k^2 + \sigma_m^2),
$$

(35)

We turn to the characterization of the conditional expectation in (96). Defining

$$
f(t, g(t)) = \mathbb{E}_t \left[ e^{-\frac{1 - \tau}{\psi} g(T) - (1 - \tau)(\pi_1 + \frac{\sigma_k}{\psi}) \int_t^T g(v)dv - \frac{1 - \tau}{\psi} g(t)} \right],
$$

(36)

we have that

$$
f(t, g(t)) e^{-\frac{1 - \tau}{\psi} \int_t^T g(v)dv} \right]
$$
is a martingale. An application of Ito’s lemma gives:

\[
\frac{d}{e^{-\tau} f_w g(v)dv} = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial g} \kappa g (g - g(t)) - (1 - \tau) \left( \frac{\pi_1}{\psi} \right) g(t) f \right. \\
+ \left. \frac{1}{2} \frac{\partial^2 f}{g^2 \sigma^2} \right) dt + \frac{\partial f}{\partial g} \sigma g dZ_t.
\]

Setting the drift to zero, we obtain the following partial differential equation for \( f \):

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial g} \kappa g (g - g(t)) - (1 - \tau) \left( \frac{\pi_1}{\psi} \right) g(t) f + \frac{1}{2} \frac{\partial^2 f}{g^2 \sigma^2} = 0.
\]

Conjecture that

\[
f(t, g(t)) = e^{\eta_0 (T-t) + \eta_1 (T-t) g(t)},
\]

and plug it back into the PDE. Using the method of undetermined coefficients, we have the following system of ODEs:

\[
(\eta_0') = \eta_0 \kappa g + \frac{\sigma^2}{2} (\eta_1)^2, \\
- (\eta_1') = \eta_1 \kappa g + (1 - \tau) \left( \pi_1 + \frac{\kappa g}{\psi} \right),
\]

with boundary conditions given by \( \eta_0(0) = 0 \) and \( \eta_1(0) = -\frac{1 - \tau}{\psi} \).

Solving for \( \eta_1 \), we have:

\[
\eta_1(T-t) = (1 - \tau) \left( \frac{\pi_1}{\kappa g} \left( e^{-\kappa g (T-t)} - 1 \right) - \frac{1}{\psi} \right) = (1 - \tau) \eta_1(T-t),
\]

where we define

\[
\eta_1(T-t) = \frac{\pi_1}{\kappa g} \left( e^{-\kappa g (T-t)} - 1 \right) - \frac{1}{\psi}.
\]

30
Thus, the expression for $\eta^0_0(T - t)$ becomes

$$
\eta^0_0(T - t) = \kappa_g \bar{g} \int_0^{T-t} \eta^1_1(v) dv + \frac{\sigma^2_g}{2} \int_0^{T-t} (\eta^1_1)^2(v) dv
$$

$$
\eta^0_0(T - t) = (1 - \tau) \left( \eta_1(T - t) + \left( \pi_1 + \frac{\kappa_g}{\psi} \right) (T - t) + \frac{1}{\psi} \right) \left( \frac{\sigma^2_g (1 - \tau)}{2\kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) - \bar{g} \right)
$$

$$
+ \frac{\sigma^2_g (1 - \tau)^2}{4\kappa_g} \left( \frac{1}{\psi^2} - (\eta_1(T - t))^2 \right),
$$

(39)

where the second line follows from the fact that

$$
\int_0^{T-t} (\eta^1_1(v))^2 dv = \frac{(1 - \tau)^2}{2\kappa_g} \left( \frac{1}{\psi^2} - (\eta_1(T - t))^2 \right)
$$

$$
+ \frac{(1 - \tau)^2}{\kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) \left( \eta_1(T - t) + \left( \pi_1 + \frac{\kappa_g}{\psi} \right) (T - t) + \frac{1}{\psi} \right)
$$

$$
\int_0^{T-t} \eta^1_1(v) dv = - \frac{1 - \tau}{\kappa_g} \left( \eta_1(T - t) + \left( \pi_1 + \frac{\kappa_g}{\psi} \right) (T - t) + \frac{1}{\psi} \right).
$$

Thus, the nominal bond is explicitly characterized by the expression

$$
B^n(t, T) = e^{-\rho_0(T-t) + \eta^0_0(T-t) + (\eta^1_1(T-t) + \frac{1 - \tau}{\psi}) g(t)},
$$

(40)

and the nominal yield rate becomes

$$
R^n(t, T) = - \frac{1}{T - t} \log B^n(t, T) = \rho^n_0 - \frac{\eta^0_0(T - t)}{T - t} - \frac{(1 - \tau)/\psi + \eta^1_1(T - t)}{T - t} g(t).
$$
Thus, for a bond maturing in \( t + \Delta \) years, we have

\[
R^0_n(T - t) = \rho^0_n + (1 - \tau) \left( \pi_1 \left( \frac{e^{-\kappa_g(T-t)} - 1}{\kappa_g(T-t)} + 1 \right) + \frac{\kappa_g}{\psi} \right) \left( \bar{g} - \frac{\sigma^2_g(1 - \tau)}{2\kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) \right)
\]

\[
\quad + \frac{\sigma^2_g(1 - \tau)^2}{4\kappa_g(T-t)} \left( (\eta_1(T-t))^2 - \frac{1}{\psi^2} \right),
\]

\[
= \rho^0_n + (1 - \tau) \kappa_g \left( \bar{g} - \frac{\sigma^2_g(1 - \tau)}{2\kappa_g\psi} \right) + (1 - \tau) \pi_1 \left( \frac{e^{-\kappa_g(T-t)} - 1}{\kappa_g(T-t)} + 1 \right) \left( \bar{g} - \frac{\sigma^2_g(1 - \tau)}{\kappa_g\psi} \right)
\]

\[
\quad - \frac{(\sigma_g(1 - \tau)\pi_1)^2}{2\kappa_g^2} \left( 1 + \frac{4e^{-\kappa_g(T-t)} - 3 - e^{-2\kappa_g(T-t)}}{2\kappa_g} \right)
\]

\[
R^1_n(T - t) = (1 - \tau) \pi_1 \left( \frac{1 - e^{-\kappa_g(T-t)}}{\kappa_g(T-t)} \right).
\]

(41)

Thus, for a bond maturing in \( t + \Delta \) years, we have

\[
R^0_n(\Delta) = \rho^0_n + (1 - \tau) \kappa_g \left( \bar{g} - \frac{\sigma^2_g(1 - \tau)}{2\kappa_g\psi} \right) + (1 - \tau) \pi_1 \left( \frac{e^{-\kappa_g\Delta} - 1}{\kappa_g\Delta} + 1 \right) \left( \bar{g} - \frac{\sigma^2_g(1 - \tau)}{\kappa_g\psi} \right)
\]

\[
\quad - \frac{(\sigma_g(1 - \tau)\pi_1)^2}{2\kappa_g^2} \left( 1 + \frac{4e^{-\kappa_g\Delta} - 3 - e^{-2\kappa_g\Delta}}{2\kappa_g\Delta} \right)
\]

\[
R^1_n(\Delta) = (1 - \tau) \pi_1 \left( \frac{1 - e^{-\kappa_g\Delta}}{\kappa_g\Delta} \right).
\]

(42)

Substituting the expression for \( \pi_1 \) in (29) into (42), and solving for \( R^1_n(\Delta) \), it follows that

\[
R^1_n(\Delta) = \frac{\kappa_g(1 - \tau)(1 - e^{-\kappa_g\Delta})}{q_1(1 - \tau)(1 - e^{-\kappa_g\Delta} - \Delta\kappa_g) - \kappa_g\Delta\psi}
\]

(43)

Note that this expression is characterize only in terms of the primitives of the model. Plugging (43) in (29), we have

\[
\pi_1 = \frac{\kappa_g^2\Delta}{q_1(1 - \tau)(1 - e^{-\kappa_g\Delta} - \Delta\kappa_g) - \kappa_g\Delta\psi}
\]

\[
= \frac{\kappa_g}{q_1(1 - \tau) \left( \frac{1 - e^{-\kappa_g\Delta}}{\kappa_g\Delta} - 1 \right) - \psi}
\]

(44)
The final expression for $r_1$ and $i_1$ is obtained by substituting (44) into (24) and (28):

$$r_1 = \frac{-\tau \kappa_g^2 \Delta}{q_1(1 - \tau)(1 - e^{-\kappa_g \Delta} - \Delta \kappa_g) - \kappa_g \Delta \psi},$$

$$i_1 = \frac{(1 - \tau) \kappa_g^2 \Delta}{q_1(1 - \tau)(1 - e^{-\kappa_g \Delta} - \Delta \kappa_g) - \kappa_g \Delta \psi}. \quad (45)$$

Next, we subtract the expression for $R_0^n(\Delta)$ in (42) from $i_0$ in (28) and obtain:

$$R_0^n(\Delta) - i_0 = (1 - \tau) \pi_1 \left( \bar{g} - \frac{\sigma_g^2(1 - \tau)}{\kappa_g \psi} \right) \left( \frac{e^{-\kappa_g \Delta} - 1}{\kappa_g \Delta} + 1 \right) - \frac{(\sigma_g(1 - \tau) \pi_1)^2}{2 \kappa_g^2} \left( 1 + \frac{4e^{-\kappa_g \Delta} - 3 - e^{-2\kappa_g \Delta}}{2 \kappa_g \Delta} \right) \quad (46)$$

$$= (1 - \tau) \pi_1 \left( \gamma_0(\Delta) - \frac{\pi_1}{2} \gamma_1(\Delta) \right),$$

where we define the functions $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$ to be

$$\gamma_0(\Delta) = \left( \bar{g} - \frac{\sigma_g^2(1 - \tau)}{\kappa_g \psi} \right) \left( \frac{e^{-\kappa_g \Delta} - 1}{\kappa_g \Delta} + 1 \right),$$

$$\gamma_1(\Delta) = \frac{\sigma_g^2(1 - \tau)}{\kappa_g} \left( 1 + \frac{4e^{-\kappa_g \Delta} - 3 - e^{-2\kappa_g \Delta}}{2 \kappa_g \Delta} \right). \quad (47)$$

Substituting (46) into (22), the expression for $\pi_0$ becomes

$$\pi_0 = \frac{(1 - \tau) \pi_1 q_1}{\psi} \left( \gamma_0(\Delta) - \frac{\pi_1}{2} \gamma_1(\Delta) \right) \nonumber$$

$$+ \frac{\kappa_g \bar{g} - q_2 \bar{\pi} - q_3 \bar{k} + \tau \sigma_m^2 - \sigma_k (\sigma - \tau \sigma_k) + (q_3 - 1)(A - \delta - \beta)}{\psi}. \quad (48)$$

Note that the expression for $\pi_0$ above is completely characterized in terms of the primitives of the model since $\pi_1$ and $i_1$ are determined in (44) and (45), respectively. Consequently, the expressions for $r_0$ and $i_0$ in (24) and (28) are also pinned down with the characterization of $\pi_0$ in (48).

Next, we derive the nominal bond dynamics and the associated term premium. An
application of Ito’s lemma on (40) gives the following dynamics of $B^n(t, T)$:

$$\frac{dB^n(t, T)}{B^n(t, T)} = (\rho^n_0 - \eta^n_0 - (1 - \tau)\eta^n_1 g(t))dt + (1 - \tau) \left( \eta_1 + \frac{1}{\psi} \right) (\kappa_g (\bar{g} - g(t)))dt$$

$$+ \sigma_g dZ^m(t) + \frac{(1 - \tau)^2 \sigma_g^2}{2} \left( \eta_1 + \frac{1}{\psi} \right)^2 dt$$

$$= (\rho^n_0 - \eta^n_0 + (1 - \tau) \left( \eta_1 + \frac{1}{\psi} \right) \left( \kappa_g \bar{g} + \left( \eta_1 + \frac{1}{\psi} \right) \frac{\sigma_g^2}{2} \right)$$

$$- (1 - \tau) \left( \kappa_g \left( \eta_1 + \frac{1}{\psi} \right) + \eta' \right) g(t) dt + (1 - \tau) \left( \eta_1 + \frac{1}{\psi} \right) \sigma_g dZ^m(t).$$

The nominal term premium $TP^n(t, T)$ can be expressed as:

$$TP^n(t, T) = \rho^n_0 - \eta^n_0 + (1 - \tau) \left( \eta_1 + \frac{1}{\psi} \right) \left( \kappa_g \bar{g} + \left( \eta_1 + \frac{1}{\psi} \right) \frac{\sigma_g^2}{2} \right) - i_0$$

$$- (1 - \tau) \left( \kappa_g \left( \eta_1 + \frac{1}{\psi} \right) + \eta' + \pi_1 \right) g(t).$$

To conclude the equilibrium proof, we turn to the characterization of the real bond price, real yield curve and real term premium. Using the expression for the state price density in (23), the real zero-coupon bond price with maturity at time $T$ can be expressed as

$$B^n(t, T) = \mathbb{E}_t \left[ \frac{\xi(T)}{\xi(t)} \right] = \mathbb{E}_t \left[ e^{-\beta(T-t) K(t)} K(T) \right]$$

$$= \mathbb{E}_t \left[ \exp \left\{ \left( \delta - A + \tau \pi_0 - \frac{\tau \kappa_g \bar{g}}{\psi} + \frac{(\sigma - \tau \sigma_k)^2 + (\tau \sigma_m)^2}{2} \right) (T - t) \right. \right.$$

$$\left. \left. - (\sigma - \tau \sigma_k) (Z^k(T) - Z^k(t)) + \frac{\tau}{\psi} (g(T) - g(t)) + \tau \left( \pi_1 + \frac{\kappa_g}{\psi} \right) \int_t^T g(s) ds \right\} \right]$$

$$= e^{-\rho^n_0 (T-t) + \eta^n_0 (T-t) + (\eta_1 (T-t) - \frac{\psi}{2}) g(t)},$$

(50)
where

\[ \rho_0^* = A - \delta - \tau \pi_0 + \frac{\tau \kappa_g \bar{g}}{\psi} - (\sigma - \tau \sigma_k)^2 - \frac{(\tau \sigma_m)^2}{2}, \]

\[ \eta_1^*(T - t) = -\tau \frac{\pi_1}{\kappa_g} \left( e^{-\kappa_g(T-t)} - 1 \right) + \frac{\tau}{\psi} = -\tau \eta_1(T - t) \]

\[ \eta_0^*(T - t) = \tau \left( \eta_1(T - t) + \left( \frac{\pi_1}{\kappa_g} + \frac{\kappa_g}{\psi} \right) (T - t) + \frac{1}{\psi} \right) \left( \frac{\tau \sigma_g^2}{2\kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) + \bar{g} \right) \]

\[ + \frac{\sigma_g^2 \tau^2}{4\kappa_g} \left( \frac{1}{\psi^2} - (\eta_1(T - t))^2 \right) \] (51)

An application Ito's lemma in (50) gives the following dynamics of \( B^r(t, T) \):

\[ \frac{dB^r(t, T)}{B^r(t, T)} = (\rho^r_0 - \eta_0^{r'} + \tau \eta_1^r g(t)) dt - \tau \left( \eta_1 + \frac{1}{\psi} \right) (\kappa_g (\bar{g} - g(t))) dt + \sigma_g dZ^m(t) \]

\[ + \frac{\sigma_g^2 \tau^2}{2} \left( \eta_1 + \frac{1}{\psi} \right)^2 dt \]

\[ = \left( \rho^r_0 - \eta_0^{r'} - \tau \left( \eta_1 + \frac{1}{\psi} \right) \left( \kappa_g \bar{g} - \frac{\sigma_g^2 \tau}{2} \left( \eta_1 + \frac{1}{\psi} \right) \right) + \tau \left( \kappa_g \left( \eta_1 + \frac{1}{\psi} \right) + \eta_1^r \right) g(t) \right) dt \]

\[ - \tau \left( \eta_1 + \frac{1}{\psi} \right) \sigma_g dZ^m(t). \]

Thus, the real term premium \( TP^r(t, T) \) is expressed as

\[ TP^r(t, T) = \rho^r_0 - \eta_0^{r'} - \tau \left( \eta_1 + \frac{1}{\psi} \right) \left( \kappa_g \bar{g} - \frac{\sigma_g^2 \tau}{2} \left( \eta_1 + \frac{1}{\psi} \right) \right) - r_0 \]

\[ + \tau \left( \kappa_g \left( \eta_1 + \frac{1}{\psi} \right) + \eta_1^r + \pi_1 \right) g(t). \] (52)

The real yield rate becomes

\[ R^r(t, T) = -\frac{1}{T-t} \log B^r(t, T) = \rho^r_0 - \frac{\eta_0^r(T - t)}{T - t} + \frac{\tau / \psi - \eta_1^r(T - t)}{T - t} g(t). \] (53)
Matching the coefficients in (53) with the ones in (68), it follows that:

\[
R^r_r(t, T) = -\gamma_0(T - t) = A - \delta - \tau \pi_0 + \frac{\tau^2 \kappa_g \bar{g}}{\psi} - (\sigma - \tau \sigma_k)^2 - \frac{(\tau \sigma_m)^2}{2}
\]

\[-\tau \left( \frac{\pi_1 \left( e^{-\kappa_g(T-t)} - 1 \right)}{\kappa_g (T - t)} + 1 \right) + \frac{\kappa_g}{\psi} \left( \bar{g} + \frac{\tau \sigma_g^2}{2 \kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) \right)
\]

\[+ \frac{\sigma_g^2 \tau^2}{4 \kappa_g (T - t)} \left( \eta_1 (T - t) \right)^2 - \frac{1}{\psi^2},
\]

\[= A - \delta - \tau \pi_0 + \frac{\tau^2 \kappa_g \bar{g}}{\psi} - (\sigma - \tau \sigma_k)^2 - \frac{(\tau \sigma_m)^2}{2} - \tau \pi_1 \left( \frac{e^{-\kappa_g(T-t)} - 1}{\kappa_g (T - t)} + 1 \right) \left( \bar{g} + \frac{\tau \sigma_g^2}{\kappa_g \psi} \right)
\]

\[-\frac{(\tau \sigma_g)^2}{2 \kappa_g^2} \left( \frac{1}{1 - \frac{4 e^{-\kappa_g(T-t)} - 3 - e^{-2\kappa_g(T-t)}}{2 \kappa_g (T - t)}} \right)
\]

\[R^r_1(t, T) = \frac{\tau/\psi - \eta_1(T - t)}{T - t} = -\tau \pi_1 \left( \frac{1 - e^{-\kappa_g(T-t)}}{\kappa_g (T - t)} \right),
\]

which concludes the characterization of the equilibrium.

Proof of Proposition 6. Define \( \bar{q} \) as

\[
\bar{q} = -\frac{\kappa_g \gamma_1 (T - t) + \gamma_0 (T - t) \psi}{(1 - \tau) \left( 1 + \frac{1 - e^{-\kappa_g \Delta \sigma}}{\Delta \kappa_g} \right) \gamma_0 (T - t)},
\]

where \( \gamma_0(\cdot) \) and \( \gamma_1(\cdot) \) are defined in (47).

The assumption that \( q_1 < \bar{q} \) can be rewritten in terms of \( \pi_1 \) using expression (44). It follows that \( \pi_1 \) satisfies

\[
\frac{\gamma_0 (T - t)}{\gamma_1 (T - t)} < \pi_1.
\]

Since \( \gamma_0 (T - t) < 0 \) and \( 0 < \gamma_1 (T - t) \), it follows that

\[
2 \frac{\gamma_0 (T - t)}{\gamma_1 (T - t)} < \frac{\gamma_0 (T - t)}{\gamma_1 (T - t)}.
\]

The inequality implies that

\[
\gamma_0 (T - t) - \frac{\pi_1}{2} \gamma_1 (T - t) < \gamma_0 (T - t) - \pi_1 \gamma_1 (T - t) < 0.
\]

From (46) and (42), the expression for the spread between the long-term and short-
term nominal rate becomes

\[ s(t, T) = -\frac{\sigma^2}{\kappa_g} (1 - \tau)^2 \pi_1 \left\{ \frac{1}{\psi} \left( e^{-\kappa_g(T-t)} - 1 \right) + 1 \right\} + \frac{\pi_1}{2} \left( 1 + \frac{4e^{-\kappa_g(T-t)} - 3 - e^{-2\kappa_g(T-t)}}{2\kappa_g(T-t)} \right) \]

\[ + (1 - \tau) \pi_1 \left( \bar{g} - g(t) \right) \left( e^{-\kappa_g(T-t)} - 1 \right) \frac{\kappa_g(T-t)}{1} \]

\[ s(t, T) = (1 - \tau) \pi_1 \left( \gamma_0(T-t) - \frac{\pi_1}{2} \gamma_1(T-t) \right) - (1 - \tau) \pi_1 \left( e^{-\kappa_g(T-t)} - 1 \right) \frac{\kappa_g(T-t)}{1} g(t). \]

(56)

From (55) and (44), we have that the first term on the right hand side of (56) is positive since \( \pi_1 \) and \( \gamma_0 - \pi_1 \gamma_1 / 2 \) are both negative. Similarly, the second term is positive whenever \( 0 < g(t) \). Thus,

\[ 0 < s(t, T). \]

Next, we differentiate (56) with respect to \( q_1 \) and obtain

\[ \frac{ds(t, T)}{dq_1} = (1 - \tau) \frac{d\pi_1}{dq_1} \left( \gamma_0(T-t) - \pi_1 \gamma_1(T-t) - \left( e^{-\kappa_g(T-t)} - 1 \right) \frac{\kappa_g(T-t)}{1} g(t) \right). \]

(57)

From inequality (55), we know \( \gamma_0(T-t) - \pi_1 \gamma_1(T-t) < 0 \). Thus, the term in the brackets is negative as long as \( 0 < g(t) \). In addition, as it is shown in Proposition 8 in equation (60), \( d\pi_1/dq_1 \) is positive. Thus,

\[ \frac{ds(t, T)}{dq_1} < 0. \]

\[ \Box \]

**Proof of Proposition 7.** Assume that the equity is a leverage claim on consumption, i.e.,

\[ P(t, T) = \mathbb{E}_t \left[ \int_t^T \frac{\xi(s)}{\xi(t)} (c(s))^\ell ds \right] = \mathbb{E}_t \left[ \int_t^T e^{-\beta(s-t)} \frac{K(t)}{K(s)} (\alpha \beta K(s))^\ell ds \right] \]

\[ = (\alpha \beta K(t))^\ell \int_t^T \mathbb{E}_t \left[ e^{-\beta(s-t)} \left( \frac{K(s)}{K(t)} \right)^{\ell-1} \right] ds \]

\[ = (\alpha \beta K(t))^\ell \int_t^T e^{-\tilde{\beta}_0(s-t) + \tilde{\eta}_0(s-t) + \tau(t-1)(\eta_1(s-t)+1/\psi)g(t)} ds, \]

37
where
\[ \tilde{\rho}_0 = -(\ell - 1) \left( A - \delta - \tau \pi_0 + \frac{\tau \kappa_g \bar{g}}{\psi} - \frac{\tau \sigma_m^2}{2} - \frac{(2 - \ell)(\sigma - \tau \sigma_k)^2}{2} \right) - \ell \beta, \]
\[ \tilde{\eta}_0(s - t) = \tau(\ell - 1) \left( \eta_1(s - t) + \left( \pi_1 + \kappa_g \right)(s - t) + \frac{1}{\psi} \right) \left( \frac{\sigma_g^2 \tau (\ell - 1)}{2 \kappa_g} \left( \frac{\pi_1}{\kappa_g} + \frac{1}{\psi} \right) - \bar{g} \right) \]
\[ + \frac{\sigma_g^2 \tau^2 (\ell - 1)^2}{4 \kappa_g} \left( \frac{1}{\psi^2} - (\eta_1(s - t))^2 \right), \]
Defining
\[ h(t, T, \sigma(t)) = \int_t^T e^{-\tilde{\rho}_0(s-t)+\tilde{\eta}_0(s-t)+\tau(\ell-1)(\eta_1(s-t)+1/\psi)\sigma(t)} ds, \]
the nominal stock price can be rewritten as
\[ P(t, T) = (\alpha \beta K(t))^\ell h(t, T, \sigma(t)). \quad (58) \]
To obtain the evolution of the nominal stock price, apply Ito’s lemma in (58) to obtain the following dynamics:
\[ \frac{dP(t, T)}{P(t, T)} = \frac{dh(t, T, \sigma(t))}{h(t, T, \sigma(t))} + \frac{d(K(t))^\ell}{(K(t))^\ell} + \frac{d \left[h(t, T, \sigma(t)), (K(t))^\ell \right]_t}{h(t, T, \sigma(t))(K(t))^\ell}, \]
\[ \frac{dP(t, T)}{P(t, T)} = \left( \frac{\partial h}{h} + \frac{\partial g h}{h} \kappa_g (\bar{g} - \sigma(t)) + \frac{\sigma_g^2 \partial_g h}{2} \right) dt + \frac{\partial g h}{h} \sigma_g dZ^m(t) \]
\[ + \ell \left( A - \delta - \beta - \tau \pi_0 + \frac{\ell - 1}{2} \left( (\sigma - \tau \sigma_k)^2 + (\tau \sigma_m)^2 \right) \right) dt + \ell (\sigma - \tau \sigma_k) dZ^k(t) \]
\[ - \ell \tau \sigma_m dZ^m(t) - \tau \sigma_m \sigma_g \frac{\partial h}{h} dt \]
\[ \frac{dP(t, T)}{P(t, T)} = \ell \left( A - \delta - \beta - \tau \pi_0 + \frac{\ell - 1}{2} \left( (\sigma - \tau \sigma_k)^2 + (\tau \sigma_m)^2 \right) \right) \]
\[ + \frac{\partial h}{h} (\kappa_g \bar{g} - \tau \sigma_m \sigma_g) + \frac{\sigma_g^2 \partial_g h}{2} \frac{\partial h}{h} - \kappa_g \frac{\partial h}{h} + \ell \tau \pi_1 \right) \sigma(t) \right) dt \]
\[ + \ell (\sigma - \tau \sigma_k) dZ^k(t) + \left( \frac{\partial h}{h} \sigma_g - \ell \tau \sigma_m \right) dZ^m(t). \]
The equity volatility can be expressed as
\[
\sigma_E = \sqrt{(\sigma - \tau \sigma_k)^2 + \left(\frac{h_g}{h} \sigma_g - \tau \sigma_m\right)^2}.
\]
Differentiating with respect to \(q_1\) leads to
\[
\frac{d\sigma_E}{dq_1} = \frac{\sigma_g}{2\sigma_E} \frac{d}{dq_1} \left(\frac{h_g}{h}\right).
\]
Thus, the effect of a policy change is determined by the sign of the term \(\frac{d}{dq_1} \left(\frac{h_g}{h}\right)\), which has to be evaluated numerically.

**Proof of Proposition 8.** Applying Ito’s lemma to the expression for the expected inflation \(\pi(t)\), the nominal short-term interest rate \(i(t)\) and the real short-term interest rate \(r(t)\) presented in Propositions 2, we obtain the following stochastic differential equations:

\[
\begin{align*}
\frac{d\pi(t)}{dt} &= \kappa_g (\pi_0 + \bar{\pi} \pi_1 - \pi(t)) dt + \pi_1 \sigma_g dZ^m(t), \\
\frac{di(t)}{dt} &= \kappa_g (i_0 + \bar{i} i_1 - i(t)) dt + \pi_1 \sigma_g dZ^m(t), \\
\frac{dr(t)}{dt} &= \kappa_g (r_0 + \bar{r} r_1 - r(t)) dt + \pi_1 \sigma_g dZ^m(t), \\
\frac{dR^n(t,T)}{dt} &= \kappa_g (R^n_0(t,T) + \bar{R}^{n-1}(t,T) - R^n(t,T)) dt + R^n_1(t,T) \sigma_g dZ^m(t), \\
\frac{dR^r(t,T)}{dt} &= \kappa_g (R^r_0(t,T) + \bar{R}^{r-1}(t,T) - R^r(t,T)) dt + R^r_1(t,T) \sigma_g dZ^m(t).
\end{align*}
\]
Thus, all five variables follow an Ornstein-Uhlenbeck process. Given the assumptions that \(0 < \psi \) and \(0 < q_1\), it follows from (44) and (45) that \(\pi_1 < 0\), \(i_1 < 0\) and \(0 < r_1\).
Differentiating (44) and (45) with respect to $q_1$, we have that

$$
\frac{d\pi_1}{dq_1} = \frac{\kappa_g (1 - \tau) \left( \frac{e^{-\kappa g \Delta}}{\kappa g \Delta} - 1 \right)}{q_1 (1 - \tau) \left( \frac{1 - e^{-\kappa g \Delta}}{\kappa g \Delta} - 1 \right) - \psi} > 0,
$$

$$
\frac{d\pi_1}{dq_1} = \frac{\kappa_g (1 - \tau) \left( \frac{e^{-\kappa g \Delta}}{\kappa g \Delta} - 1 \right)}{q_1 (1 - \tau) \left( \frac{1 - e^{-\kappa g \Delta}}{\kappa g \Delta} - 1 \right) - \psi} > 0,
$$

$$
\frac{dr_1}{dq_1} = \frac{\tau \kappa_g (1 - \tau) \left( \frac{1 - e^{-\kappa g \Delta}}{\kappa g \Delta} - 1 \right)}{q_1 (1 - \tau) \left( \frac{1 - e^{-\kappa g \Delta}}{\kappa g \Delta} - 1 \right) - \psi} < 0,
$$

$$
\frac{dR_1^n(T - t)}{dq_1} \times (1 - \tau) \left( \frac{1 - e^{-\kappa g (T - t)}}{\kappa g (T - t)} \right) \frac{d\pi_1}{dq_1} > 0,
$$

$$
\frac{dR_1^n(t, T)}{dq_1} = -\tau \left( \frac{1 - e^{-\kappa g (T - t)}}{\kappa g (T - t)} \right) \frac{d\pi_1}{dq_1} < 0.
$$

Consequently, the quadratic variation sensitivity to changes in $q_1$ can be evaluated using the implicit function theorem:

$$
\frac{d[\pi, \pi](t)}{dq_1} = 2\pi_1 \sigma_g^2 \frac{d\pi_1}{dq_1} < 0,
$$

$$
\frac{d[i, i](t)}{dq_1} = 2i_1 \sigma_g^2 \frac{di_1}{dq_1} < 0,
$$

$$
\frac{d[r, r](t)}{dq_1} = 2r_1 \sigma_g^2 \frac{dr_1}{dq_1} > 0,
$$

$$
\frac{dR_1^n(T - t)}{dq_1} \times (1 - \tau) \left( \frac{1 - e^{-\kappa g (T - t)}}{\kappa g (T - t)} \right) \frac{d\pi_1}{dq_1} > 0,
$$

$$
\frac{dR_1^n(t, T)}{dq_1} = -\tau \left( \frac{1 - e^{-\kappa g (T - t)}}{\kappa g (T - t)} \right) \frac{d\pi_1}{dq_1} < 0.
$$

Proof of Proposition 9. To calculate the output and price level impulse response function, we follow Detemple et al. (2003). First, at time $t < T$, we calculate the Malliavin derivative of output $Y_T$ in the direction of the persistent monetary shock $Z_m$. Given that output is an AK model with constant productivity factor $A$, we can focus on the Malliavin derivative of capital and we denote this quantity by $\mathcal{D}_t^{m} K(t)$. By Malliavin differentiating
with respect to $Z^m$ the expression in (75), we have

$$\mathcal{D}_t^m K(T) = K(T) \left( -\tau \pi_1 \int_t^T \mathcal{D}_t^m g(v) dv - \tau \sigma_m \right). \quad (61)$$

The Malliavin derivative of the state variable $g(t)$, $\mathcal{D}_t^m g(v)$, can be solved in closed form. By Malliavin differentiating the dynamics of $g(t)$ with respect to $Z^m$, we have

$$d\mathcal{D}_t^m g(u) = -\kappa g \mathcal{D}_t^m g(u) du, \quad \mathcal{D}_t^m g(t) = \sigma_g.$$ Integrating it from $t$ to $T$, it follows that:

$$\mathcal{D}_t^m g(v) = \sigma_g e^{-\kappa g (v-t)} \to \int_t^T \mathcal{D}_t^m g(v) dv = \frac{\sigma_g}{\kappa g} (1 - e^{-\kappa g (T-t)}). \quad (62)$$

Using Clark-Hausmann-Ocone representation and the expression in (62) for $\mathcal{D}_t^m g(t)$, we can write the monetary shock impulse response function of output as:

$$\varepsilon_{t,T}^n = \mathbb{E}_t \left[ \mathcal{D}_t^m K(T) \right] = \mathbb{E}_t \left[ K(T) \left( -\tau \pi_1 \int_t^T \mathcal{D}_t^m g(v) dv - \tau \sigma_m \right) \right] \mathbb{E}_t[K(T)]$$

$$= \frac{\tau \pi_1 \sigma_g}{\kappa g} (e^{-\kappa g (T-t)} - 1) - \tau \sigma_m.$$

Differentiating the impulse response function with respect to $q_1$, we have

$$\frac{d\varepsilon_{t,T}^n}{dq_1} = \tau \sigma_g (e^{-\kappa g (T-t)} - 1) \frac{d\pi_1}{dq_1} < 0.$$

A similar calculation gives the expression for the monetary shock impulse response function of price level. By Malliavin differentiating the solution of (69) with respect to $Z^m$, we obtain:

$$\mathcal{D}_t^m p(T) = p(T) \left( \pi_1 \int_t^T \mathcal{D}_t^m g(v) dv + \sigma_m \right) = p(T) \left( \pi_1 \frac{\sigma_g}{\kappa g} (1 - e^{-\kappa g (T-t)}) + \sigma_m \right), \quad (63)$$

which results into the following monetary shock impulse response function of price:

$$\varepsilon_{t,T}^{p,n} = \mathbb{E}_t \left[ \mathcal{D}_t^m p(T) \right] = \mathbb{E}_t \left[ p(T) \right] \frac{\pi_1 \sigma_g}{\kappa g} (1 - e^{-\kappa g (T-t)}) + \sigma_m.$$
Differentiating this impulse response function with respect to $q_1$, we have

$$\frac{d\varepsilon_{p,m}^{t,T}}{dq_1} = \frac{\sigma_1}{\kappa_1} (1 - e^{-\kappa_g(T-t)}) \frac{d\pi_1}{dq_1} > 0.$$  

To find the impulse response functions with respect to capital shocks, we Malliavin differentiate $K(t)$ and $p(t)$ with respect to $Z^k(t)$. It follows that

$$\mathcal{D}^k_t K(T) = K(T) (\sigma - \tau \sigma_k),$$
$$\mathcal{D}^k_t p(T) = p(T) \sigma_k.$$  

Thus,

$$\varepsilon_{Y,k}^{t,T} = \frac{\mathbb{E}_t [\mathcal{D}^t K(T)]}{\mathbb{E}_t [K(T)]} = \sigma - \tau \sigma_k,$$
$$\varepsilon_{p,k}^{t,T} = \frac{\mathbb{E}_t [\mathcal{D}^t p(T)]}{\mathbb{E}_t [p(T)]} = \sigma_k.$$

\section*{B Integrability of the Expected Utility}

We demonstrate next that the expected utility is well-defined. Substituting the optimal controls in (1), it follows that

$$J(K(0), g(0)) = \mathbb{E}_0 \left[ \int_0^\infty e^{-\beta t} [\alpha \log(c(t)) + (1 - \alpha) \log(m(t))] dt \right]$$
$$= \mathbb{E}_0 \left[ \int_0^\infty e^{-\beta t} [\alpha \log(\alpha \beta K(t)) + (1 - \alpha) \log((1 - \alpha) \beta K(t))] dt \right]$$
$$= \mathbb{E}_0 \left[ \int_0^\infty e^{-\beta t} [\alpha \log(\alpha \beta K(t)) + (1 - \alpha) \log((1 - \alpha) \beta K(t))] dt \right]$$
$$= \frac{\alpha \log(\alpha \beta) + (1 - \alpha) \log((1 - \alpha) \beta)}{\beta} + \int_0^\infty e^{-\beta t} \mathbb{E}_0 [\log(K(t))] dt$$
Calculating the expectation inside integral, we have

$$
\mathbb{E}_0[\log(K(t))] = \log(K(0)) + \left(A - \delta - \beta - \tau\pi_0 - \frac{(\sigma - \tau\sigma_k)^2}{2} - \frac{(\tau\sigma_m)^2}{2}\right) t - \tau\pi_1\mathbb{E}_0 \left[ \int_0^t g(v)dv \right]
$$

$$
= \log(K(0)) + \left(A - \delta - \beta - \tau\pi_0 - \frac{(\sigma - \tau\sigma_k)^2}{2} - \frac{(\tau\sigma_m)^2}{2}\right) t
$$

$$
- \tau\pi_1 \left( (g(0) - \bar{g}) \left( \frac{1 - e^{-\kappa t}}{\kappa}\right) + \bar{g}t \right).
$$

Substituting the expected value above in (64) and evaluating the integral, we have

$$
J(K(0), g(0)) = \frac{\alpha \log(\alpha\beta) + (1 - \alpha) \log((1 - \alpha)\beta) + \log(K(0))}{\beta}
$$

$$
+ \frac{2(A - \delta - \beta - \tau\pi_0 - (\sigma - \tau\sigma_k)^2 - (\tau\sigma_m)^2)}{2\beta^2}
$$

$$
- \tau\pi_1 \left( \frac{1}{\beta^2} + \frac{\bar{g} - g(0)}{\kappa} \left( \frac{1}{\beta} - \frac{1}{\kappa + \beta} \right) \right).
$$

\section*{C \ A model with Duffie-Epstein-Zin Preference and Cox-Ingersoll-Ross persistent component}

We introduce two layers of complexity in the baseline model presented in Section 2:

(i) the persistent component $g(t)$ follows a Cox-Ingersoll-Ross process:

$$
dg(t) = \kappa_g(\bar{g} - g(t))dt + \sigma_g\sqrt{g(t)}dZ^m(t), \quad (65)
$$

(ii) the representative household has Duffie-Epstein-Zin preference with elasticity of intertemporal substitution equals to one.

The expected utility function is defined recursively as

$$
J(K(0), g(0)) = \mathbb{E} \left[ \int_0^\infty k(c(t), m(t), J(K(t), g(t)))dt \right] \quad (66)
$$
where $h(\cdot, \cdot, \cdot)$ is the Duffie-Epstein aggregator given by

$$h(c, m, J) = \beta (1 - \gamma) J \left[ \alpha \log c + (1 - \alpha) \log m - \frac{1}{1 - \gamma} \log((1 - \gamma)J) \right].$$  \hspace{1cm} (67)

Similar to the previous case, we assume there exist equilibrium processes for expected inflation $\pi(t)$, the nominal short-rate $i(t)$, the real short-rate $r(t)$, the nominal long-term interest rate $R^n(t, t + \Delta)$ and the real long-term interest rate $R^r(t, t + \Delta)$ that are affine functions of the state variable $g(t)$,

$$\pi(t) = \pi_0 + \pi_1 g(t),$$

$$i(t) = i_0 + i_1 g(t),$$

$$r(t) = r_0 + r_1 g(t),$$

$$R^r(t, t + \Delta) = R^r_0(\Delta) + R^r_1(\Delta) g(t),$$

$$R^n(t, t + \Delta) = R^n_0(\Delta) + R^n_1(\Delta) g(t),$$

while the price level dynamics $p(t)$ is now conjectured to be:

$$\frac{dp(t)}{p(t)} = \pi(t) dt + \sigma_k dZ^k(t) + \sigma_m \sqrt{g(t)} dZ^m(t).$$ \hspace{1cm} (69)

The constants $\sigma_k, \sigma_m, \pi_0, \pi_1, i_0, i_1, r_0, r_1, R^r_0(\Delta), R^r_1(\Delta), R^n_0(\Delta)$ and $R^n_1(\Delta)$ are determined later in the proof when the money market clearing condition is imposed. Once these quantities are characterized, the conjectured structure can be verified to be the true solution of the optimization problem.

By substituting the evolution of $p(t)$ described in (69) and the expression for $\pi(t)$ displayed in (68) into the evolution of capital in (10), it follows that

$$dK(t) = ((A - \delta - \tau(\pi_0 + \pi_1 g(t)))K(t) - c(t) - m(t)) dt + (\sigma - \tau \sigma_k) K(t) \sqrt{g(t)} dZ^k(t)$$

$$- \tau \sigma_m K(t) dZ^m(t).$$ \hspace{1cm} (70)
Thus, the Hamilton-Jacobi-Bellman equation for the social planner becomes

\[
0 = \sup_{c(t), m(t)} \left\{ \beta (1 - \gamma) J[\alpha \log(c(t)) + (1 - \alpha) \log(m(t)) - \frac{1}{1 - \gamma} \log((1 - \gamma)J)] + J_K((A - \delta)K(t) - c(t) - m(t)) + K(t)^2 J_{KK} \left( \frac{(\sigma - \tau \sigma_k)^2}{2} + \frac{(\tau \sigma_m)^2}{2}g(t) \right) + J_g \kappa_g (\bar{g} - g) + \frac{\sigma_g^2}{2}g(t)J_{gg} - \tau \sigma_m \sigma_g g(t)K(t)J_{kg} \right\}.
\]

The first order condition with respect to \(c(t)\) and \(m(t)\) gives

\[
c(t) = \alpha \beta (1 - \gamma) \frac{J}{J_K}, \]
\[
m(t) = (1 - \alpha) \beta (1 - \gamma) \frac{J}{J_K}.
\]

Plugging the optimal controls back into (71) and substituting the conjectured value function

\[
J(K(t), g(t)) = \frac{K(t)^{1-\gamma}}{1-\gamma} e^{\mu_0 + \mu_1 g(t)},
\]

where \(\mu_0\) and \(\mu_1\) are constants to be determined, we have

\[
0 = 2(A - \beta - \delta)(1 - \gamma) - 2\beta \mu_0 + 2 \bar{g} \kappa_g \mu_1 - \gamma(1 - \gamma)(\sigma - \tau \sigma_k)^2 - 2\tau \pi_0 (1 - \gamma)
\]
\[
+ 2(1 - \gamma)\beta ((1 - \alpha) \log(1 - \alpha) + \alpha \log \alpha + \log \beta)
\]
\[
+ \left( \mu_1^2 \sigma_g^2 - 2 \mu_1 (\kappa_g + \beta + \sigma_g \sigma_m \tau(1 - \gamma)) - 2\tau \pi_1 (1 - \gamma) - \gamma(1 - \gamma)\sigma_m^2 \tau^2 \right) g(t).
\]

For this equation to be satisfied at all time, we set both coefficients to zero and solve for \(\mu_0\) and \(\mu_1\). It follows that

\[
\mu_1 = \frac{2(\kappa_g + \beta + \sigma_g \sigma_m \tau(1 - \gamma)) \pm \sqrt{(\kappa_g + \beta + \sigma_g \sigma_m \tau(1 - \gamma))^2 + 4\sigma_g^2 \tau(1 - \gamma)(2\pi_1 - \gamma \tau \sigma_m^2)}}{2\sigma_g^2}.
\]

(73)
The expression for \( \mu_0 \) is

\[
\mu_0 = \frac{(1 - \gamma)}{\beta} \left( A - \beta - \delta - \pi_0 \tau \right) + \frac{g \mu_1}{1 - \gamma} - \frac{\gamma (\sigma - \tau \sigma_k)^2}{2} + (1 - \alpha) \log(1 - \alpha) + \alpha \log \alpha + \log \beta
\]

Note that given the existence of two possible values for \( \mu_1 \) may indicate the possibility of multiple solutions. Nevertheless, we invoke the same reasoning used by Wachter (2013) to argue that only one of these values make economic sense. In essence, the uncertainty generated by monetary shocks should decrease the expected utility of the representative agent which leads to the conclusion that only the negative root is economically plausible.

With the conjectured solution completely characterized, a substitution of (72) into (71) verifies that it is indeed the true solution of the dynamic programming problem. At this stage, we showed that, given constants \( \sigma_k, \sigma_m, \pi_0, \pi_1, i_0, i_1, r_0, r_1, R_0^n(\Delta), R_1^n(\Delta), R_0^r(\Delta) \) and \( R_1^r(\Delta), (72) \) is the true solution associated with the Hamilton-Jacobi-Bellman equation (71). The next steps demonstrate how to pin down these constants.

First, we use the expression derived for (72) to obtain the following expressions for the optimal policies on consumption goods, real cash balance and investment, respectively,

\[
c(t) = \alpha \beta K(t), \\
m(t) = (1 - \alpha) \beta K(t), \\
I(t) = (A - \beta) K(t).
\]

By replacing the optimal policies into (70), the endogenous capital evolution becomes:

\[
\frac{dK(t)}{K(t)} = (A - \delta - \beta - \tau (\pi_0 + \pi_1 g(t))) dt + (\sigma - \tau \sigma_k) dZ^k(t) - \tau \sigma_m \sqrt{g(t)} dZ^m(t), \quad (74)
\]

with solution given by

\[
K(T) = K(t) \exp \left\{ \left( A - \delta - \beta - \tau \pi_0 - \frac{(\sigma - \tau \sigma_k)^2}{2} \right) (T - t) - \left( \tau \pi_1 + \frac{(\tau \sigma_m)^2}{2} \right) \int_t^T g(v) dv 
\right.
\]

\[
\left. + (\sigma - \tau \sigma_k) (Z^k(T) - Z^k(t)) - \tau \sigma_m \int_t^T \sqrt{g(s)} Z^m(s) ds \right\}, \quad (75)
\]

Next, we turn to the money market. By imposing that money demand equals to money...
supply, it follows that:

\[ M^*(t) = m(t)p(t) = p(t)(1 - \alpha)\beta K(t). \]

An application of Ito’s lemma leads to the following expression for the price level process:

\[
\frac{dp(t)}{p(t)} = \frac{dM^*(t)}{M^*(t)} - \frac{dK(t)}{K(t)} - \left[ \frac{dp}{p}, \frac{dK}{K} \right](t)dt. \tag{76}
\]

From (69) and (74), the quadratic covariation between \(dp(t)/p(t)\) and \(dK(t)/K(t)\) is:

\[
\left[ \frac{dp}{p}, \frac{dK}{K} \right](t) = \sigma_k(\sigma - \tau \sigma_k) - \tau \sigma_m^2 g(t). \tag{77}
\]

Substituting the money supply equation

\[
\frac{dM^*(t)}{M^*(t)} = q_1 \left( R^n(t, t + \Delta) - i(t) - i_0 \right) dt + q_2 \left( \frac{dp(t)}{p(t)} - \bar{\pi} dt \right) + q_3 \left( \frac{dK(t)}{K(t)} - \bar{k} dt \right) + dg(t), \tag{78}
\]

the nominal short-rate (68) and the quadratic covariation (77) into (76), the dynamics for the price level \(p(t)\) can be represented by the following stochastic differential equation:

\[
\frac{dp(t)}{p(t)} = \frac{1}{1 - q_2} \left[ q_1 (R^n_0(\Delta) - \bar{i}_0) - (q_2 \bar{\pi} + q_3 \bar{k}) + \kappa_g \bar{g} - \sigma_k(\sigma - \tau \sigma_k) \right. \\
\left. + (q_3 - 1)(A - \delta - \beta - \tau \pi_0) + (q_1 (R^n_1(\Delta) - \bar{i}_1) - \tau(q_3 - 1)\pi_1 - \kappa_g + \tau \sigma_m^2 g(t) \right) dt \\
+ \frac{q_3 - 1}{1 - q_2} (\sigma - \tau \sigma_k) dZ^k(t) + \frac{\sigma_g - \tau(q_3 - 1)\sigma_m}{1 - q_2} \sqrt{g(t)} dZ^m(t). \tag{79}
\]

By defining

\[ \psi = 1 - q_2 + \tau(q_3 - 1), \]

\[ 47 \]
and matching the coefficients of (79) with the ones in (68), it follows that
\[ \sigma_k = \frac{(q_3 - 1)\sigma}{\psi}, \quad \sigma_m = \frac{\sigma_g}{\psi}, \quad \pi_1 = \frac{q_1(R^0_1(\Delta) - i_1) - \kappa g + \tau\sigma^2_m}{\psi}, \]
\[ \pi_0 = \frac{1}{\psi} \left( q_1(R^0_0(\Delta) - i_0) - q_2\pi - q_3\bar{k} + \kappa g - \sigma_k(\sigma - \tau\sigma_k) + (q_3 - 1)(A - \delta - \beta) \right). \] (80)

In order to complete the equilibrium characterization, we still need to determine \(i_0, i_1, r_0, r_1, R^0_0(\Delta), R^1_1(\Delta), R^0_1(\Delta)\) and \(R^1_0(\Delta)\). For this reason, we turn to the characterization of the real and nominal state price density in this economy. We start characterizing the real state price density, which is given by
\[ \xi(t) = e^{\int_0^t h_J(c(s),m(s),J(K(s),g(s)))ds} h_c(c(t),m(t),J(K(t),g(t))). \] (81)
The partial derivatives of (67) give
\[ h_c(c(t),m(t),J(K(t),g(t))) = K(t)^{-\gamma}e^{\mu_0 + \mu_1 g(t)}, \]
\[ h_J(c(s),m(s),J(K(s),g(s))) = \bar{\mu}_0 - \beta\mu_1 g(s), \] (82)
where
\[ \bar{\mu}_0 = \beta((1 - \gamma)(\alpha \log \alpha + (1 - \alpha) \log(1 - \alpha) + \beta \log \beta) - \mu_0 - 1) \]
Thus, the state price density can be rewritten as
\[ \xi(t) = K(t)^{-\gamma}e^{\mu_0 + \bar{\mu}_0 t - \beta\mu_1 \int_0^t g(s)ds + \mu_1 g(t)} = K(t)^{-\gamma}X(t), \] (83)
where we defined
\[ X(t) = e^{\mu_0 + \bar{\mu}_0 t - \beta\mu_1 \int_0^t g(s)ds + \mu_1 g(t)}. \]
An application of Ito’s Lemma gives
\[ dX(t) = e^{\mu_0 + \bar{\mu}_0 t + \mu_1 g(t)} d\left( e^{-\beta \mu_1 \int_0^t g(s)ds} \right) + e^{\mu_0 + \bar{\mu}_0 t - \beta\mu_1 \int_0^t g(s)ds} d\left( e^{\mu_1 g(t)} \right) + \bar{\mu}_0 X(t) dt. \] (84)
Evaluating the terms
\[ d\left( e^{-\beta \mu_1 \int_0^t g(s)ds} \right) = -\beta \mu_1 e^{-\beta \mu_1 \int_0^t g(s)ds} g(t) dt, \]
\[ d\left( e^{\mu_1 g(t)} \right) = \mu_1 e^{\mu_1 g(t)} , \]
we have that
\[
\frac{dX(t)}{X(t)} = \left( \frac{\mu^2 \sigma^2}{2} - (\beta + \kappa_g) \mu_1 \right) g(t) + \bar{\mu_0} \right) dt + \mu_1 \sigma_g \sqrt{g(t)} dZ(t)^m. \tag{85}
\]

In addition,
\[
\frac{dK(t)^{-\gamma}}{K(t)^{-\gamma}} = -\gamma \left( A - \delta - \gamma - \tau \pi_0 - \frac{1 + \gamma}{2} (\sigma - \tau \sigma_k)^2 - \left( \tau \pi_1 + \frac{1 + \gamma}{2} (\tau \sigma_m)^2 \right) g(t) \right) dt
\]
\[
- \gamma (\sigma - \tau \sigma_k) dZ(t)^k + \gamma \tau \sigma_m \sqrt{g(t)} dZ(t)^m. \tag{86}
\]

Thus, an application of Ito’s lemma in (83) gives the following dynamics of the state price density:
\[
\frac{d\xi(t)}{\xi(t)} = -r(t) dt - \theta_k dZ^k(t) - \theta_m \sqrt{g(t)} dZ^m(t),
\]

where the real short rate and real prices of risk are, respectively,
\[
r(t) = \gamma \left( A - \delta - \gamma - \tau \pi_0 - \frac{1 + \gamma}{2} (\sigma - \tau \sigma_k)^2 \right) - \bar{\mu_0}
\]
\[
- \gamma \left( \tau \pi_1 + \frac{1 + \gamma}{2} (\tau \sigma_m)^2 + \mu_1 \sigma_g \sigma_m \tau - \frac{\mu_1}{\gamma} (\kappa_g + \beta) + \frac{\mu_1^2 \sigma^2}{2\gamma} \right) g(t)
\]
\[
\theta_k = \gamma (\sigma - \tau \sigma_k) = \gamma \sigma \frac{1 - q_2}{\psi},
\]
\[
\theta_m = - (\mu_1 \sigma_g + \gamma \tau \sigma_m) = -\sigma_g \left( \frac{\mu_1 + \tau \gamma}{\psi} \right).
\]

Contrary to the model presented in Section 2, note that now the market price of monetary risk is a function of \( q_1 \). Thus, a monetary policy that changes the smoothing intensity of the yield curve will also have consequences to the monetary price of risk.

Similar to the baseline model, the expression for the real short-rate is an affine function of \( g(t) \) and can be written as
\[
r(t) = r_0 + r_1 g(t),
\]
where
\[
\begin{align*}
\tilde{r}_0 &= \gamma \left( A - \delta - \gamma - \tau \pi_0 - \frac{1+\gamma}{2} \left( \sigma - \tau \sigma_k \right)^2 \right) - \bar{\mu}_0, \\
\tilde{r}_1 &= -\gamma \left( \tau \pi_1 + \frac{1+\gamma}{2} (\tau \sigma_m)^2 + \mu_1 \sigma_g \sigma_m \tau - \frac{\mu_1}{\gamma} (\kappa_g + \beta) + \frac{\mu^2_1 \sigma_g^2}{2\gamma} \right).
\end{align*}
\] (87)

Next, we characterize the nominal state price density \( \zeta(t) \), given by
\[
\zeta(t) = \frac{\xi(t)}{p(t)}.
\] (88)

An application of Ito’s lemma gives the following stochastic differential equation for the nominal state price density:
\[
\frac{d\zeta(t)}{\zeta(t)} = -\left( r(t) + \pi(t) - \sigma_k (\sigma_k + \theta_k) - \sigma_m (\sigma_m + \theta_m) g(t) \right) dt
- (\sigma_k + \theta_k) dZ^k(t) - (\sigma_m + \theta_m) \sqrt{g(t)} dZ^m(t).
\] (89)

Thus, the nominal risk premiums and the nominal short-term interest rate are, respectively,
\[
\begin{align*}
\theta^n_k &= \sigma_k + \theta_k = \frac{\sigma}{\psi} \left( q_3 - 1 + \gamma (1 - q_2) \right), \\
\theta^n_m &= \sigma_m + \theta_m = \frac{\sigma_g}{\psi} \left( 1 - \tau \gamma - \mu_1 \psi \right), \\
i(t) &= r(t) + \pi(t) - \sigma_k \theta^n_k - \sigma_m \theta^n_m g(t) = i_0 + i_1 g(t),
\end{align*}
\]

where
\[
\begin{align*}
i_0 &= r_0 + \pi_0 - \sigma_k \theta^n_k, \\
i_1 &= r_1 + \pi_1 - \sigma_m \theta^n_m.
\end{align*}
\] (90)

Note that the nominal market price of monetary risk is also a function of the smoothing intensity through \( \mu_1 \).

Substituting the expressions for \( r_0 \) and \( r_1 \) in (87), and the expressions for \( \theta_m \) and \( \theta_k \)
in (80) into (90), we have that
\[ i_0 = \gamma \left( A - \delta - \gamma - \frac{(1 + \gamma)(1 - q_2)^2 \sigma^2}{2 \psi^2} \right) + (1 - \tau \gamma) \pi_0 - \bar{\mu}_0 \]
\[ + \frac{\sigma^2 (1 - q_3)(q_3 - 1 + \gamma(1 - q_2))}{\psi^2}, \]  
\[ i_1 = -\gamma \left( \frac{(1 + \gamma)^2 \sigma_g^2}{2 \psi^2} + \frac{\mu_1 \tau \sigma_g^2}{\psi} - \frac{\mu_1}{\gamma} (\kappa_g + \beta) + \frac{\mu_1^2 \sigma_g^2}{2 \gamma} \right) + (1 - \tau \gamma) \pi_1 \]
\[ + \frac{\sigma_g^2 (1 - \tau \gamma - \mu_1 \psi)}{\psi^2}. \]  

Rewriting the expression for \( \pi_1 \) in (80) as
\[ R^n_1(\Delta) = \frac{\psi}{q_1} \pi_1 + \frac{\kappa_g}{q_1} - \frac{\tau \sigma_g^2}{q_1 \psi^2}, \]
and substituting the expression for \( i_1 \) presented in (91), we conclude that
\[ R^n_1 = \psi + q_1 (1 - \tau \gamma) \pi_1 - \gamma \left( \frac{(1 + \gamma)(\tau \sigma_g)^2}{2 \psi^2} + \frac{\mu_1 \tau \sigma_g^2}{\psi} - \frac{\mu_1}{\gamma} (\kappa_g + \beta) + \frac{\mu_1^2 \sigma_g^2}{2 \gamma} \right) \]
\[ + \frac{\sigma_g^2 (1 - \tau \gamma - \mu_1 \psi)}{\psi^2} + \frac{\kappa_g}{q_1} - \frac{\tau \sigma_g^2}{q_1 \psi^2}. \]  

The only remaining quantity to be determined are the nominal and real yield rate for the bond maturing in \( \Delta \) years. For this reason, we turn to the characterization of the nominal and real bond price, and their respective yields. Once these quantities are determined, the characterization of the equilibrium for the recursive utility case is completed.

**Proof of Real and Nominal Bonds.** Using the expression for the nominal state price density in (88), the nominal zero-coupon bond price with maturity at time \( T \) can be expressed as
\[ B^n(t,T) = \mathbb{E}_t \left[ \frac{\xi(T)}{\xi(t)} \frac{p(t)}{p(T)} \right] = \mathbb{E}_t \left[ \frac{\zeta(T)}{\zeta(t)} \right], \]  
\[ (93) \]
The solution for the stochastic differential equation in (89) is

\[ \zeta(T) = \zeta(t) \exp \left\{ - \int_t^T \left( i_0 + \frac{(\theta_k^n)^2}{2} + \left( i_1 + \frac{(\theta_m^n)^2}{2} \right) g(s) \right) ds - \theta_k^n (Z^k(T) - Z^k(t)) \right\} \]

(94)

Rewriting the solution of \( g(t) \) in (65) as

\[ \int_t^T \sqrt{g(s)} dZ^m(s) = \frac{g(T) - g(t)}{\sigma_g} - \frac{\kappa_g \bar{g}}{\sigma_g} (T - t) + \frac{\kappa_g}{\sigma_g} \int_t^T g(s) ds, \]

and substituting it back into (94), the ratio \( p(t)/p(T) \) becomes

\[ \frac{\zeta(T)}{\zeta(t)} = \exp \left\{ \left( \frac{\kappa_g \bar{g} \theta_m^n}{\sigma_g} - i_0 - \frac{(\theta_k^n)^2}{2} \right) (T - t) - \int_t^T \left( i_1 + \frac{(\theta_m^n)^2}{2} + \frac{\kappa_g \theta_m^n}{\sigma_g} \right) g(s) ds \right\} \]

(95)

Using that the shocks are independent, by substituting (95) into (30) we have

\[ B^n(t, T) = \mathbb{E}_t \left[ \exp \left\{ \left( \frac{\kappa_g \bar{g} \theta_m^n}{\sigma_g} - i_0 - \frac{(\theta_k^n)^2}{2} \right) (T - t) - \int_t^T \left( i_1 + \frac{(\theta_m^n)^2}{2} + \frac{\kappa_g \theta_m^n}{\sigma_g} \right) g(s) ds \right\} \right] \]

\[ - \theta_k^n (Z^k(T) - Z^k(t)) - \theta_m^n \int_t^T \sqrt{g(s)} dZ^m(s) - \frac{\theta_m^n}{\sigma_g} (g(T) - g(t)) \right\} \]

\]

(96)

where

\[ \rho_0^n = i_0 - \frac{\kappa_g \bar{g} \theta_m^n}{\sigma_g}, \]

\[ \rho_1^n = i_1 + \frac{(\theta_m^n)^2}{2} + \frac{\kappa_g \theta_m^n}{\sigma_g}. \]

(97)
Substituting the expressions for $i_0$ and $i_1$ in (91) into (97), it follows that

$$
\rho_0^n = \gamma \left( A - \delta - \gamma - \frac{(1 + \gamma)(1 - q_2^2)\sigma^2}{2\psi^2} \right) + (1 - \tau \gamma)\pi_0 - \bar{\mu}_0
$$

$$
+ \frac{\sigma^2(1 - q_3)(q_3 - 1 + \gamma(1 - q_2))}{\psi^2} + \frac{\kappa_g \bar{g}(1 - \tau \gamma - \mu_1\psi)}{\psi},
$$

$$
\rho_1^n = \gamma \left( \frac{\mu_1(\kappa_g + \beta)}{\gamma} - \frac{(1 + \gamma)\tau^2\sigma^2_g}{2\psi^2} - \frac{\mu_1 \tau \sigma^2_g}{\psi} - \frac{\mu_1^2 \sigma^2_g}{2\gamma} \right) + (1 - \tau \gamma)\pi_1
$$

$$
+ \left( \frac{\sigma^2_g(1 + \tau \gamma + \mu_1\psi)}{2\psi} - \kappa_g \right) (1 - \tau \gamma - \mu_1\psi). \tag{98}
$$

We turn to the characterization of the conditional expectation in (96). Defining

$$
f(t, g(t)) = \mathbb{E}_t \left[ e^{-\rho_0^n_{T-t} \int_t^T g(v)dv} \right], \tag{99}
$$

we have that

$$
f(t, g(t))e^{-\rho_1^n_{T-t} \int_t^T g(v)dv}
$$

is a martingale. An application of Ito’s lemma gives:

$$
\frac{d \left( f e^{-\rho_1^n_{T-t} \int_t^T g(v)dv} \right)}{e^{-\rho_1^n_{T-t} \int_t^T g(v)dv}} = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial g} \kappa_g (\bar{g} - g(t)) - \rho_1^n g(t)f + \frac{1}{2} \frac{\partial^2 f}{\partial g^2} \sigma^2_g g(t) \right) dt
$$

$$
+ \frac{\partial f}{\partial g} \sigma_g \sqrt{g(t)} dZ^m_t.
$$

Setting the drift to zero, we obtain the following partial differential equation for $f$:

$$
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial g} \kappa_g (\bar{g} - g(t)) - \rho_1^n g(t)f + \frac{1}{2} \frac{\partial^2 f}{\partial g^2} \sigma^2_g g(t) = 0.
$$

Conjecture that

$$
f(t, g(t)) = e^{\eta_0^n (T-t) + \eta_1^n (T-t)g(t)},
$$

and plug it back into the partial differential equation. Using the method of undetermined coefficients, we have the following system of ordinary differential equations:

$$
(\eta_0^n)' = \eta_1^n \kappa_g \bar{g},
$$

$$
(\eta_1^n)' = -\eta_1^n \kappa_g - \rho_1^n + \frac{\sigma^2_g}{2} (\eta_1^n)^2. \tag{100}
$$

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with boundary conditions given by \( \eta^0_m(0) = 0 \) and \( \eta^1_m(0) = -\theta^n_m/\sigma_g \).

The solution for \( \eta^0_1 \) and \( \eta^2_1 \) are, respectively,

\[
\eta^0_1(u) = \frac{\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2(2\rho_1^0\sigma_g - \kappa_g^0\theta^n_m)} \sin \left( \frac{u\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2}}{2} \right) - \kappa_g^0\theta^n_m \left( \kappa_g^2 + 2\rho_1^0\sigma_g^2 \right) \cos \left( \frac{u\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2}}{2} \right)}{\sigma_g \left( \kappa_g^2 + 2\rho_1^0\sigma_g^2 \right) \cos \left( \frac{u\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2}}{2} \right) - \sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2} \left( \sigma_g^2 \theta^n_m + \sigma_g \right) \sin \left( \frac{u\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2}}{2} \right)},
\]

\[
\eta^2_0(u) = \frac{2i\kappa_g^0 \tan^{-1} \left( \frac{(\kappa_g + \sigma_g^0\theta^n_m) \tan \left( \frac{u\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2}}{2} \right)}{\sqrt{\kappa_g^2 + 2\rho_1^0\sigma_g^2}} \right) + \kappa_g^0(u\kappa_g + \log \left( \frac{\kappa_g^2 + 2\rho_1^0\sigma_g^2}{\sigma_g^2} \right))}{\frac{\kappa_g^2}{\sigma_g} \log \left( (2\sigma_g(\rho_1^0 - \kappa_g^0\theta^n_m) - (\sigma_g^0\theta^n_m)^2) \cos \left( u\sqrt{-\kappa_g^2 - 2\rho_1^0\sigma_g^2} \right) + 2\sigma_g(\rho_1^0 + \kappa_g^0\theta^n_m) + (\sigma_g^0\theta^n_m)^2 + 2\kappa_g^2 \right)},
\]

(101)

where \( i = \sqrt{-1} \) and \( \tan^{-1} \) is the arctangent function, conditioned on \( 0 < \kappa_g^2 + 2\rho_1^0\sigma_g^2 \).

Thus, the nominal bond is explicitly characterized by the expression

\[
B^0(t, T) = e^{-\rho_0^0(T-t) + \eta^0_0(T-t) + (\eta^0_0(T-t) + \frac{\theta^n_m}{\sigma_g})g(t)},
\]

(102)

and the nominal yield rate becomes

\[
R^n(t, T) = -\frac{1}{T-t} \log B^n(t, T) = \rho_0^n - \frac{\eta^0_0(T-t)}{T-t} = \frac{-\theta^n_m/\sigma_g - \eta^1_0(T-t)}{T-t} = R^n_0(T-t),
\]

\[
R^n(t, T) = -\frac{1}{T-t} \log B^n(t, T) = \rho_0^n - \frac{\eta^0_0(T-t)}{T-t} = \frac{-\theta^n_m/\sigma_g - \eta^1_0(T-t)}{T-t} = R^n_0(T-t),
\]

Thus, for a bond maturing in \( t + \Delta \) years, we have

\[
R^n_0(\Delta) = \rho_0^n - \frac{\eta^0_0(\Delta)}{\Delta},
\]

\[
R^n_1(\Delta) = \frac{1 - \tau/\gamma - \mu_1\psi}{\psi\Delta} - \frac{\eta^1_0(\Delta)}{\Delta},
\]

(103)

where the second line follows from the fact that \( -\theta^n_m/\sigma_g = (1 - \tau/\gamma - \mu_1\psi)/\psi \).

Matching (103) with (92), we have an equation that is only a function of \( \pi_1 \), given \( \mu_1 \) in (73) and \( \rho_1^n \) in (98) are also functions of \( \pi_1 \). Solving the fix point problem, we pin down \( \pi_1 \). Consequently, \( \mu_1 \) and \( \rho_1^n \) are determined as well as \( i_1 \) in (91), \( r_1 \) in (87), \( R^n_0(\Delta) \) in (92) and \( \eta^0_0(\Delta) \) and \( \eta^1_0(\Delta) \) in (101).
The last step is to substitute the expression for $\rho^n_0$ in (97) into (103) and obtain the relationship:

$$R_n^i(\Delta) - i_0 = \frac{\kappa_0 \bar{g}}{\psi} \left(1 - \tau \gamma - \mu_1 \psi \right) - \frac{\eta^n_0(\Delta)}{\Delta}. \quad (104)$$

Substituting (104) in the expression for $\pi_0$ into (80), it becomes a function of $\pi_1$ and, consequently, it is also pinned down. Thus, the expressions for $r_0$ in (87), $i_0$ in (91) and $R_0^i(\Delta)$ in (103) are also completely characterized and the equilibrium is determined. ■

**D Calibration**

**Table 3: Table of Parameters**

<table>
<thead>
<tr>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = 0.1$</td>
</tr>
<tr>
<td>$\delta = 0.05$</td>
</tr>
<tr>
<td>$\kappa_g = 0.15$</td>
</tr>
<tr>
<td>$\bar{\pi} = 0.04$</td>
</tr>
<tr>
<td>$\Delta = 15$</td>
</tr>
</tbody>
</table>

**References**


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