A Simple Measure of Default Risk Based on Endogenous Credit Risk Models

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Abstract

Both empirically and theoretically, corporate default is linked to depressed equity prices and poor financial conditions. I formalize this link/result in Leland-type endogenous models. That is, a low ratio of the equity price to the firm’s negative net cash flow (debt service and negative earnings) also describes the default event implicit in Leland credit-risk models—a sufficiently low asset value. While equity prices, debt, and the Merton model are used to infer a firm’s asset value and distance to default, this ratio directly approximates the latter, a volatility-adjusted measure of leverage, close to default. This result is appealing because equity prices and cash flows are readily available if only with lags. To track default risk, we just need to track this ratio; credit risk is the probability this ratio becomes small.

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1 Introduction

Both empirically and theoretically, corporate default is associated with zero equity prices. First, large empirical evidence suggests depressed equity prices are strongly correlated with measures of default risk;¹ second, most structural credit-risk models assume this property as a boundary condition (e.g., Merton, 1974, and Leland, 1994).² Further, default is also linked to poor financial conditions. Empirical results show firms in default have negative profitability and expired debt;³ likewise, endogenous credit-risk models predict net cash-flows (i.e., debt service and earnings) become large and negative at default.⁴

Thus, corporate default is linked to depressed equity prices and poor financial conditions. This paper formalizes this result in Leland-type endogenous models. We show the default event defined by endogenous credit-risk models, a sufficiently low asset value, can likewise be described by a low ratio of the equity price to the firm’s negative net cash flow.

Specifically, consider a Leland model, in which \( V \) is the asset value, \( \sigma \) is its volatility, \( V_B \) is the optimal default boundary that maximizes equity value \( E(V) \), and \( \frac{V-V_B}{\sigma V_B} \) is the firm’s distance to default (DD), a volatility-adjusted measure of leverage. From a second-order Taylor expansion of the time-value of equity, the following approximation of DD holds near the default point:

\[
\left( \frac{V-V_B}{\sigma V_B} \right)^2 \approx \frac{E}{b},
\]

where \( b(V) > 0 \) is the firm’s negative net cash-flows (debt service and negative earnings).

This result implies the ratio \( E/b \) shrinks when a firm’s asset value is close to the default boundary, and, conversely, this ratio becomes large (or negative) for a firm that is far away from default. Endogenous default is thus unambiguously associated with a small ratio, \( E/b \); credit risk is the probability this ratio becomes small.

Negative net cash flows \( b \) depend on the coupon and rollover cost of short-term debt minus dividends in Leland and Toft (1996) but include negative earnings in Garlappi and Yan (2011). By focusing our analysis on the default point, however, we do not have to specify a process for \( b \), which in a general setting equals debt service plus negative earnings minus dividends. Equity \( E \), like the default point \( V_B \) in DD (but also like \( b \) in Leland–Toft, because short-term

¹See, e.g., Gilson et al. (1990), Shumway (2001), Campbell et al. (2008), and Garlappi and Yan (2011).
²This boundary condition can be relaxed to accommodate strategic default; see Fan and Sundaresam (2000).
³For instance, Asquith et al. (1994) find default is linked to poor firm-specific performance, and Davydenko (2013) reports high financial costs and the clustering of firm defaults around coupon days.
⁴See Leland and Toft (1996), Manso et al. (2010), and He and Milbradt (2014).
debt is refinanced), directly depends on both: the firm’s debt profile (e.g., whether rollover risk is present) and the earnings dynamic.

Equity prices and cash flows, however, are partly used to explain (price or predict) credit risk. By contrast, many studies focus on asset value (Collin-Dufresne and Goldstein, 2001), or use equity prices, debt, and Merton’s model to infer asset value and DD (Duffie et al., 2007). This focus is likely due to the fact that structural models depend explicitly on asset value, and the lack of a result that precisely explains corporate default in terms of equity prices and cash flows. As we show, this ratio, $E/b$, directly approximated DD close to default.

The ratio $E/b$ has two clear implications. First, although asset value and its endogenous default point are not observable and require a model to be recovered (e.g., Merton), equity prices and cash flows are readily available if only with lags. Further, we show equation (1) is robust to a multifactor setting (e.g., stochastic volatility), which implies that although the default boundary $V_B$ is a function that depends on the other state variables, a small (equity-price-to-negative-net-cash-flow) ratio is a sufficient statistic to describe default. In both cases, the focus on the ratio $E/b$ substantially simplifies the analysis of default risk.\footnote{For instance, Duffie and Lando (2001) show how a structural model becomes a default-intensity reduced-form model simply because the exogenous state variable (i.e., the asset value) is not fully observable.}

Thus, if we control for $E$, a firm becomes closer to default if $b$ rises, which implies default will rarely happen between coupon or refinancing dates; if $b$ is small, the ratio $E/b$ spikes between these dates. $b$ depends on negative earnings; a simple way to understate default risk is to hide/delay losses.\footnote{The dividend yield, a positive cash flow for shareholders, enters $b$ negatively, delaying the default decision.} Conversely, default concerns are lessened by easing refinancing costs in $b$, such as by lowering interest rates or extending maturities on existing loans, or by lowering the principal, which increases equity value $E$ (e.g., underwater mortgages).

A second implication of $E/b$ is that credit risk, or the probability of default, is the probability the ratio $E/b$ becomes small. Assume no cash outflows between 0 and $T$; $T$ is the next coupon/payment day. This assumption allows us to map the continuous-time model to a discrete-time model; if net cash flows are nonnegative, defaulting is never optimal. The event $\{E_T/b_T \leq \alpha\}$, which approximates (the event of) default for small $\alpha \geq 0$, depends on the paths of equity prices and net cash flows, which yield a new DD once log-equity prices are mean-variance normalized. That is, the ratio $E_0/b_0$ is neither associated with default today (if $t = 0$ is not a coupon day, $b_0 \leq 0$ ) nor is it a sufficient statistic to predict default, which requires this new DD based on the event $\{E_T/b_T \leq \alpha\}$.  

\[\text{Equation (1)}\]
This new DD is similar to Merton’s DD, but (instead of asset values and a zero-coupon bond notional) is cast in terms of equity prices and net cash flows. This new DD depends on and therefore explains why price per share, past (or expected) equity returns, equity volatility, a firm’s debt, and profitability are five important covariates in reduced-form models that predict corporate default. See Campbell et al. (2008) and Bharath and Shumway (2008), who also report Merton’s DD becomes insignificant if controlling for these covariates.\(^7\)

Merton’s DD, however, is the main variable for assessing and predicting financial distress (Duffie et al., 2007, 2009). Campbell et al. and Duffie et al. implicitly use the default event associated with the normalized ratio \(E/b\) and asset value \(V \sigma V_B\), respectively. From equation (1), if we control for the former, the latter will necessarily become redundant.\(^8\)

The preceding analysis of one-period default enables us to link our results to discrete hazard models, which are the basis of empirical work. The five covariates associated with the new DD offer more flexibility than a single DD statistic, and a one-period logit model is rich enough to fit the associated default probability.\(^9\) Further, the same binary function \(1\{E_t \leq \alpha \times b_t\}\) is the payoff of an European digital put, \(\alpha \times b_t\) is a low strike price; out-of-the-money equity puts and credit risk are thus closely linked (as in Carr and Wu, 2011).\(^10\)

In sum, this paper provides a rationale to study credit risk based on equity prices, the debt-service calendar, and negative earnings, instead of unobservable asset value and their endogenous default boundary. Endogenous default is triggered by low equity prices and large negative net cash flows, where the next two events are approximately equivalent for small \(\alpha\):

\[
\{V \leq V_B \times (1 + \sigma \sqrt{\alpha})\} \approx \{E (V)/b (V) \leq \alpha\},
\]

which immediately follows from equation (1). This result enables us to bridge the gap between Merton’s asset-value view and reduced-form econometric models of default, which focus on equity prices and cash flows, and is related to several strands of credit risk.\(^11\)

\(^7\)We follow Campbell et al.’s (2008, p.2912) static comparative analysis of their failure probability estimates (moving each covariate one-standard-deviation from the sample mean values) to sum up their empirical results.

\(^8\)For loan portfolios, Gordy (2000) drives the analogy between two industry credit-risk models, JP Morgan’s CreditMetrics and Credit Suisse’s CreditRisk\(^+\), by also linking the default events underlying each model.

\(^9\)Multi-period default is just given by compounding one-period default and surviving events. Computing the term structure of default probabilities is beyond the scope of this paper; see Duffie et al. (2007), who numerically address multi-period default in a reduced-form default-intensity setting.

\(^10\)Carr and Wu show a single deep out-of-the-money American put, scaled by the strike price, becomes a digital put in a default corridor model, which replicates a pure credit contract.

\(^11\)Merton’s DD is used to explain equity returns (Vassalou and Xing, 2004), pure credit contracts (Carr and
First, our paper has a similar view on default as the Carr-Wu American cancellable call, Cornell et al.’s (1996) cash-infusions and distressed real-state model, or Hellwig and Lorenzoni’s (2009) equilibrium model, where positive levels of debt are sustainable because the interest rate is sufficiently low to provide repayment incentives, the meaning of a low $b$.

Second, we link our results to subprime consumer credit (Einav et al., 2012), where large down-payments limit defaults by screening out high-risky borrowers and lowering loan size—but, at the same time, increase equity value $E$—and to the fragility of repo markets (Martin et al., 2014), where more leveraged borrowers/less profitable ones are more fragile, analogous to a large $b$. Equation (1) also extends to a model of cash constraints (Anderson and Caverhill, 2012), where $b$ depends on the cost of cash savings as well, allowing us to integrate insolvency (or endogenous default) and illiquidity (where cash is exhausted).

Finally, our work relates to He and Milbradt (2016, HM), who also focus on the default point to analyze dynamic debt maturity. Whereas we obtain the equity Gamma at the default point, HM get this default boundary. Likewise, in HM’s setting, time to default is given near default by a ratio of equity prices to jump-adjusted net cash flows, which follows from a similar first-order Taylor expansion (HM’s equation (18)). Put simply, both of us, and like Garlappi and Yan (2011), learn credit risk by focusing first on the vicinity of default.

The rest of the paper is organized as follows: Section 2 studies endogenous default risk in a one-factor setting. Section 3 considers a multi-factor model. Section 4 uses Leland and Toft’s (1996) model to provide more intuition. Section 5 provides further implications of our results to default risk. Section 6 specializes in mortgage default. Section 7 concludes. Appendix A provides some extensions. Appendix B contains omitted proofs.

2 Default Risk in Endogenous Credit-Risk Models

We introduce the endogenous credit-risk model. $E$ is the equity value, $V$ is the underlying state variable (e.g., asset values or earnings), and $r$ is the risk-free rate.

We consider a free-boundary problem (Duffie, 2001, chapter 11), where equity can have finite maturity. Under the $Q$–risk-neutral measure, the dynamics of $V$ is given by

$$dV_t = \mu (V_t) \, dt + \sigma (V_t) \, dZ_t,$$

Wu, 2011), and CDS spreads (Bai and Wu, 2013, and Doshi et al., 2013). Subrahmanyam et al. (2014) use five similar covariates to those in Campbell et al. (2008) to control the credit risk of reference firms upon the inception of CDS trading. See Atkesson et al. (2014), who link Leland’s DD to the inverse of equity volatility.
where \(dZ\) is a Wiener process. Ito’s Lemma implies equity, \(E = E(t, V_t)\), solves the following PDE in the continuation (i.e., nondefaulting) region:

\[
rE = E_t + \mu (V_t) E_V + \frac{1}{2} \sigma^2 (V_t) E_{VV} + NC_t(V_t),
\]

where \(NC_t(V_t) \times dt\) is the instantaneous net cash flow to equity holders.

In endogenous credit-risk models, \(NC_t\) is the payout rate minus the net debt service (after tax-payments). The payout rate is the dividend yield (which is nonnegative), if \(V\) is the asset value instead of earnings. In Leland (1994), \(-NC_t\) is constant, the continuous coupon of a consol bond, whereas in Leland and Toft (1996) (or He and Xiong, 2012), \(-NC_t(V_t)\) includes the coupon and rollover cost of short-term debt minus dividends.

\(NC_t\) is a key variable to understand default. \(NC_t\) can be the result of an optimization process (e.g., the dividend policy). We assume \(-NC_t(V_t)\) depends on negative earnings and debt service minus dividends, but do not have to specify any of them. Likewise, the functional forms of \(\mu (V_t)\) and \(\sigma (V_t)\), upon which equity value \(E\) depends, are also left unspecified. See section 5 for the Leland-Toft model, where \(V\) is lognormal distributed.\(^{12}\)

Let \(e(V)\) denote the intrinsic value of equity, and let \(V_B(t)\) be the optimal default boundary. If \(V_t = V_B(t)\), \(E\) satisfies the following two boundary conditions:

\[
E(t, V_B(t)) = e(V_B(t)) \quad \text{and} \quad E_V(t, V_B(t)) = e_V(V_B(t)),
\]

the value-matching and smooth-pasting, respectively. If \(e \neq 0\), \(e\) is the equity holder’s “residual value” in default. It also holds that \(E_t(t, V_B(t)) = 0\) (because \(E_t + E_V V_B' = e_V V_B'\)).\(^{13}\)

### 2.1 Equity value in the vicinity of default

We explain the value of equity in the neighborhood of the default point. From equations (2) and (3), the option Gamma at the boundary (for \(V = V_B\)) is given by

\[
\frac{1}{2} E_{VV}(V_B) = \frac{-NC_t(V_B) + \left( r - \mu (V_B) \frac{e_V(V_B)}{e(V_B)} \right) e(V_B)}{\sigma^2(V_B)}
\]

\[= \frac{-NC_t(V_B)}{\sigma^2(V_B)}, \quad \text{if} \ e = 0. \]

\(^{12}\)The negative net cash-flows are paid by equity holders (e.g., by stock dilution, Duffie, 2001, p.263); otherwise, default will be optimally postponed if losses are financed by selling the firm’s assets.

\(^{13}\)A similar PDE holds in real option models (see Dixit and Pindyck, 1994), for example, the option to abandon a mine.
Consider the nondefaulting region of equity value,

\[ \{(t, V) : E(t, V) \geq e(V_B(t))\} . \]

From a second-order Taylor expansion in \( V_t \), with third-order error \( O(V_t - V_B)^3 \),

\[
E(t, V_t) \approx e(V_B) + e_V(V_B) \times (V_t - V_B) + \frac{1}{2} E_{VV}(V_B) \times (V_t - V_B)^2
\]
\[ = \frac{1}{2} E_{VV}(V_B) \times (V_t - V_B)^2, \quad \text{if} \ e = 0. \]

As a simple application, it is easy to derive the equity beta,

\[
\frac{E_V}{E} = \frac{e_V(V_B) + E_{VV}(V_B) \times (V_t - V_B)}{e(V_B) + e_V(V_B) \times (V_t - V_B) + \frac{1}{2} E_{VV}(V_B) \times (V_t - V_B)^2}.
\]

If we assume \( E_{VV}(V_B) > 0 \), in the limit (\( \lim V_t \downarrow V_B \)),

\[
\frac{E_V}{E} \approx \frac{1}{V_t - V_B} \to \infty \quad \text{if} \ e(V_B) = 0, \quad \text{but}
\]
\[
\frac{E_V}{E} = \frac{e_V(V_B)}{e(V_B)} < \infty \quad \text{if} \ e(V_B) > 0,
\]

in particular, \( \frac{E_V}{E} = 0 \) if \( e(V_B) = c \neq 0 \). Thus, this radically different behavior of equity risk near default, which depends on shareholders’ recovery value \( e(V_B) \), is also derived in Garlappi and Yan (2011, p.799) Corollary 1.3 (for a parametric model and by using other means) and is employed to link financial distress and equity returns in a novel way.

Instead of equity prices, we consider the “time value” of equity, that is, equity minus the intrinsic (or equityholder’s residual) value \( e \),

\[
E^{TV}(t, V) = E(t, V_t) - \left( e(V_B) + e_V(V_B) \times (V_t - V_B) + \frac{1}{2} e_{VV}(V_B) \times (V_t - V_B)^2 \right).
\]

We include (here) the case in which the intrinsic value is nonlinear or quadratic at the boundary, \( e_{VV}(V_B) \neq 0 \) (Gryglewicz, 2011). Note \( E^{TV} > 0 \) because, by definition, the value of equity is larger than the shareholders’ residual value in the nondefaulting region.

**Lemma 1** A second-order Taylor approximation of the time-value of equity is given by

\[
E^{TV}(t, V) \approx \left( -NCt(V_B) + \left( r - \mu(V_B) \frac{e_V(V_B)}{e(V_B)} \right) e(V_B) - \frac{1}{2} e_{VV}(V_B) \sigma^2(V_B) \right) \times \frac{V_t - V_B}{\sigma(V_B)}^2.
\]

Proof. It follows from equations (4) and (5).
In Ibáñez and Paraskevopoulos (2010), the left-hand side of equation (6) provides the cost of suboptimal exercise of an American option, which is approximated by the product of two terms: (i) the sensitivity to suboptimal exercise and (ii) the bias of the exercise policy (i.e., $V_B$ is the optimal exercise boundary), respectively. Ibáñez and Velasco (2016) extend this result to a multi-factor setting. We extend this result to credit risk.

2.2 Distance to default in the vicinity of default

We write distance to default (DD) in terms of equity prices and negative net cash-flows. We rename equation (6) as follows (assuming $e_{VV} = 0$ to simplify the exposition),

$$ E^{TV}(t, V_t) \approx \left( -NC_t(V_B) + \left( r - \mu(V_B) \frac{e_V}{e} \right) e \right) \times \left( \frac{V_t - V_B}{\sigma(V_B)} \right)^2 = b_t(V_B) \times a_t^2, \quad (7) $$

$b_t$ is the marginal value of default; $a_t = \frac{V_t - V_B}{\sigma(V_B)} > 0$ is the volatility-adjusted distance to the default boundary (DD). $b_t$ and $a_t$ represent the financial needs and the economic health of the firm, respectively.

The term $b_t$ depends on negative net cash-flows, $-NC_t$. If equityholders can negotiate some value at default, $e(V_B) > 0$, $b_t$ is adjusted by the dividend yield $r - \mu$ ($\mu$ is the drift under $Q$). $E^{TV} > 0$ implies that $b_t > 0$, and $b_t = b_t(V_B)$ is evaluated at the default point.

DD, the term $a_t$, is valid in a general setting but let us provide two examples: For a lognormal (an arithmetic) process, which are commonly used when $V_t$ is the asset value (operating revenues) of the firm, $\sigma(V_B) = \sigma \times V_B$ (and $\sigma(V_B) = \sigma$), which, intuitively, imply the distance to $V_B$ in relative (absolute) terms.

Because $V_B$ is not observable in practice, we use current negative net cash-flows $b_t(V_t)$ (where $b_t(V_t) \approx b_t(V_B) + O(V_t - V_B)$). We then invert equation (7),

$$ a_t^2 \approx \frac{E^{TV}(t, V_t)}{b_t(V_t)}, \quad (8) $$

which implies that, near the default point, DD is approximated by a low ratio of the equity price (i.e., equity time-value) to the firm’s negative net cash-flow. The error of equation’s (8) approximation is also 3rd-order, $O(V_t - V_B)^3$.

2.2.1 The approximate default event and its associated default boundary

From equation (8), which relates DD to the ratio $E^{TV}/b$, we define an approximate default event, and its associated default boundary $\tilde{V}_B$, in terms of this ratio. We then show that $\tilde{V}_B$ provides a good approximation to the optimal default boundary, $V_B$. 

7
For \( \alpha \geq 0 \), the following two events are equivalent,

\[
\left\{ \frac{V_t - V_B}{\sigma(V_B)} \leq \sqrt{\alpha} \right\} \equiv \left\{ V_t \leq V_B \times \left(1 + \frac{\sigma(V_B)}{V_B} \sqrt{\alpha} \right) \right\},
\]

the left-hand-side (lhs) event depends on DD; in the rhs event, the inequality rhs defines an explicit but slightly biased default boundary.

Because \( V_t \) is not observable, we approximate the DD’s lhs event as follows, for small \( \alpha \),

\[
\{ E^{TV}(t, V_t)/b_t(V_t) \leq \alpha \}, \tag{9}
\]

and, therefore, define an implicit but approximate default boundary, \( \bar{V}_B \), from

\[
V_t = \bar{V}_B : \ E^{TV}(t, \bar{V}_B)/b_t(\bar{V}_B) = \alpha, \tag{10}
\]

where \( \bar{V}_B = V_B \) if \( \alpha = 0 \).

First, if \( dE^{TV}/dV > 0 \) and \( db_t/dV \leq 0 \) (e.g., in Leland \( \frac{db_t(V_t)}{dV_t} = 0 \), in Leland-Toft \( \frac{db_t(V_t)}{dV_t} < 0 \)),

\[
\frac{d(E^{TV}_t/b_t)}{dV} > 0.
\]

Then, (i) there exists a unique point \( \bar{V}_B \geq V_B \) such that \( E^{TV}(t, \bar{V}_B)/b_t(\bar{V}_B) = \alpha \). And (ii), let \( \bar{V}^{(1)}_B \) and \( \bar{V}^{(2)}_B \) be solutions associated to \( \alpha_1 \) and \( \alpha_2 \), respectively. If \( 0 \leq \alpha_1 < \alpha_2, \bar{V}^{(1)}_B < \bar{V}^{(2)}_B \).

Hence, \( \bar{V}_B \) in equation (10) behaves properly. (Note a finite \( \alpha \geq 0 \) implies \( b_t(\bar{V}_B) > 0 \)).

Second, by using equation (8) approximation (with error \( O(V_t - V_B)^3 \sim O(\alpha^3) \)),

\[
E^{TV}(t, V_t)/b_t(V_t) \approx a^2_t + O(\alpha^3).
\]

To stay in the nondefaulting region (i.e., \( V_t > V_B \)), we take the positive root (of the latter equation) which implies that \( \bar{V}_B \geq V_B \). That is, plotting the latter equation in equation (10), for \( V_t = \bar{V}_B \),

\[
\bar{V}_B - V_B \approx V_B \sqrt{\alpha - O(a^2)} \times \frac{\sigma(V_B)}{V_B}.
\]

Because \( \alpha \rightarrow O(a^2), a \rightarrow O(\sqrt{\alpha}) \), and it follows that

\[
\bar{V}_B - V_B \approx V_B \sqrt{\alpha - O(\alpha^{3/2})} \times \frac{\sigma(V_B)}{V_B}. \tag{11}
\]

From equation (8), equations (10) and (11) are equivalent in terms of \( \bar{V}_B \) (for \( \bar{V}_B \geq V_B \)).
Proposition 1  Let the implicit (approximate) default boundary \( \tilde{V}_B \) solve equation (10) and \( \tilde{V}_B \geq V_B \). Assume that \( dE_t^{TV}/dV > 0 \) and \( db_t/dV \leq 0 \). Then, the following two (approximate) default events are equivalent for small \( \alpha \geq 0 \),

\[
\{ E^{TV}(t, V_t)/b_t (V_t) \leq \alpha \} \equiv \{ V_t \leq V_B \times \left( 1 + \frac{\alpha - O(\alpha^{3/2})}{\tilde{V}_B} \right) \},
\]

where we write \( \sigma(V_B) = \sigma \times V_B \). In particular; if \( \alpha \to 0 \), \( \tilde{V}_B = V_B \).

Proof. It follows from equations (8) to (11).

The purpose of Proposition 1 is to show, in a crystal-clear way, how the default event defined by a small ratio \( E/b \) (i.e., a small \( \alpha \)) is in terms of assets value and its endogenous default boundary, the lhs and the rhs of the latter equation, respectively. If the assets value \( V_t \) is not observable, \( \tilde{V}_B \) (and \( V_B \)) are neither.

Two counterexamples to equation (9)  (i) Let approximate the default event by \( \left\{ \frac{V_t - V_B}{V_B} \leq \sqrt{\alpha} \right\} \). The error of the boundary approximation is given by \( \tilde{V}_B - V_B = \sqrt{\alpha} \times V_B \).

This lacks to adjust DD by volatility,

\[
\left\{ \frac{V_t - V_B}{V_B} \leq \sqrt{\alpha} \right\} \equiv \left\{ \frac{V_t - V_B}{\sigma V_B} \leq \frac{\sqrt{\alpha}}{\sigma} \right\},
\]

implying to default (nondefault) if volatility \( \sigma \) becomes small (large), for a fixed DD.

(ii) The default event depends only on equity prices, \( \{ E_t^{TV} \leq \alpha \} \). This misses that no firm defaults if \( b_t \leq 0 \). From equation (8), this event can be approximated (for a small ratio \( E_t^{TV}/b_t \)) as follows,

\[
\{ E_t^{TV} \leq \alpha \} \equiv \left\{ \frac{E_t^{TV}}{b_t} \leq \frac{\alpha}{b_t} \right\} \approx \left\{ \frac{V_t - V_B}{\sigma (V_B)} \leq \frac{\sqrt{\alpha}}{b_t} \right\},
\]

which implies to default if \( b_t \) is small rather than large, which is counter intuitive.\(^{14}\)

2.3 The marginal value of default

We explain the value of the option to default. For simplicity, assume that \( E = E^{TV} (e = 0) \).

From equation (8), the marginal rate of substitution between \( a^2 \) and \( b \) is negative; i.e.,

\[
\frac{da^2}{db} \approx \frac{-E}{b^2} < 0.
\]

\(^{14}\)We take \( \alpha \geq 0 \) to work in the no defaulting region as in Lemma 1 (where \( V_t > V_B \) and \( E > 0 \)), where \( \alpha > 0 \) implies to accelerate default. Similar conclusions follow if \( \alpha < 0 \), which implies to delay default. But since we move to the defaulting region where \( E = 0 \), it is a bit different to show our results (\( \alpha < 0 \) implies that the equity price \( E \) becomes negative).
Hence, the largest the firm’s financial needs, \( b \) the lowest the economic health \( a \). The value of (the option to) default depends explicitly on the negative net cash-flows \( b \) associated to the leveraged security. The larger this cost, the larger the value of (and incentives to) default.\(^{15}\)

Let us illustrate the value of default and DD in the cross-section and in the time-series, respectively. Consider a sample of distress firms with the same value of equity (after scaling by size), \( E \). Equation (12) implies firms with large financial needs, large \( b \), are closer to default. Consider now a distress firm which has not defaulted in the last six months and the minimum value of \( E_t/b_t \) has happened in the past months (i.e., \( \min\{E_{t}/b_{t}\}_{t=1}^{6} \leq E_0/b_0 \)). Then, the firm is more far away from default today than in the past. See Duffie and Lando (2001) and Giesecke (2004) for related problems from unobservable state variables.

The variables \( E \) and \( b \) are market-based and accounting-based, respectively, which are readily available. By contrast, \( V - V_B \) is model dependent; \( V, V_B, \sigma \) are not observable. The ratio \( E/b \) is low-biased or conservative for less distressed firms, if equity grows less than quadratic in equation’s (7) approximation (i.e., \( E^{TV}(t,V_t) < b_t (V_B) \times a_t^2 \)). \( E/b \) becomes even negative (when net cash-flows are positive), if based on equation (8). In both cases, the ratio \( E/b \) is not a sufficient statistic to predict default; it is necessary to properly normalize this ratio, which produces a new DD measure.\(^{16}\) This is shown next.

### 2.4 Credit risk; the probability of default

To link our work to the credit risk empirical literature, which is based on discrete hazard models, we map our continuous-time model to a discrete-time setting by assuming no cash outflows between 0 and time \( t \) (e.g., \( t \) is next coupon/payment day); it is never optimal to default if net cash-flows are nonnegative.

Credit risk is the probability the ratio \( E/b \) becomes small, conditional on not defaulting

\(^{15}\) A case in point is BlackBerry’s CFO Thorsten Heins, after a near-$1bn fiscal 2nd-quarter loss, “We’re very disappointed with our operational and financial results this quarter (and have announced a series of major changes..) but we remain a financially strong company with $2.6bn in cash and no debt.” At the same time that it issued its profit warning last Friday, the company also announced the planned lay-off of 4,500 of its remaining 11,700 employees by the end of February. “BlackBerry eyes turnaround as results disappoint” by Paul Taylor (Financial Times, September 27, 2013). Favilukis et al. (2017), indeed, show labor obligations are a large component of operational leverage—and credit risk.

\(^{16}\) Yet, the variables \( a^2 \) and a positive ratio \( E/b \) are highly correlated; both depend on \( V \); see Section 4.3 for Leland-Toft model. The \( E/b \) ratio, which is dimensionless, is indeed close to the “price–earnings” ratio—a simple measure of how expensive a company is which fluctuates around sixteen for the S&P 500 index. Both ratios differ only in the sign if a company has little debt.
today. Credit risk depends on the event \( \{ E_t/b_t \leq \alpha \} \), if we assume that \( b_s \leq 0 \) for \( s \in (0, t) \).

Let \( E_0^P \) denote the conditional expectation under a \( P \)-measure (e.g., the \( \mathcal{P} \)-objective or \( Q \)-risk-neutral measures). The quantile

\[
p(t) = E_0^P \left[ 1_{\{E_t \leq \alpha \times b_t \}} \right]
\]

yields the probability that a firm is less than \( \sqrt{\alpha} \) DD units (by \( t > 0 \)), which approximates the corporate default event.

To compute the default probability \( p(t) \), we just need to model equity prices and net cash-flows. Let denote by \( \mu_E \) and \( \sigma_E^2 \) the first two moments of equity returns,

\[
\mu_E - \frac{\sigma_E^2}{2} = \frac{1}{t} \left[ \ln \frac{E_t}{E_0} \right] \quad \text{and} \quad \sigma_E^2 = \frac{1}{t} \left[ \ln \frac{E_t}{E_0} + \left( \ln \frac{E_t}{E_0} \right)^2 \right].
\]

These two events are equivalent; i.e.,

\[
\{ E_t \leq \alpha \times b_t \} \equiv \left\{ \frac{\ln \frac{E_t}{E_0} - \left( \mu_E - \frac{\sigma_E^2}{2} \right) t}{\sigma_E \sqrt{t}} \leq -\frac{\ln \frac{E_0}{b_t} + \left( \mu_E - \frac{\sigma_E^2}{2} \right) t}{\sigma_E \sqrt{t}} + \ln \alpha \right\}, \tag{13}
\]

where \( Z(0, 1) \) is a (zero-mean and unit-variance) normalized random variable, and \( D^E \) is a new DD measure based on endogenous models (conditional on \( b_t > 0 \)).\(^{17}\)

This tail event depends on expected (in practice, past) equity returns \( \mu_E \) and equity volatility \( \sigma_E \), debt service and losses (negative profitability) \( b_t \), and the price per share \( E_0 \); similar to Campbell et al. (2008, Table III and page 2,912) five most important covariates.

Because \( p \) is a cumulative probability (from \(-\infty \) until \(-D^E_{0,t} \))

\[
\frac{dp}{dD^E_{0,t}} = -p' < 0,
\]

and because equity returns are normalized, the larger \( D^E_{0,t} \) the lower default-risk (a lower \( p \)).

\(^{17}\)For completeness, using standard Black-Scholes notation (\( V_t \) is asset value, \( K \) is strike price, \( \sigma \) is assets’s volatility, \( \mu \) is assets’s drift, and \( \tau \) is maturity), Merton’s DD is given by

\[
D^M = \frac{\ln \left( \frac{V_t}{K} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.
\]

Merton’s DD is implicit in the price of a European call and depends on financial leverage. \( K \) is approximated by short- and long-term debt mix. Merton’s default probability is given by the normal function, \( \Phi(-D^M) \). See Leland (2004) for a comparative of default probabilities in Merton (1974) and Leland and Toft (1996) models; and see Duffie and Singleton (2003) for a review of credit-risk.
The same way, the term-structure of default probabilities depends on the compound event

$$\left\{ \bigcup_{t>0} \{ E_t \leq \alpha \times b_t \} \right\} \equiv \left\{ \bigcap_{t>0} \{ E_t > \alpha \times b_t \} \right\},$$

which is equivalent to no surviving. Computing the expectation of this multiperiod event (a first-passage problem), which requires to specify $E$ and $b$ dynamics, is beyond the scope of this paper. Moreover, equity prices $E$, the equity market, pave the way for a large correlation or contagion between two firms in a downturn market, even if cash-flows $b$ do not. See also Duffie et al. (2007) who address both problems, dynamic and correlated default, in a reduced-form default-intensity setting.

**Example** Assume lognormal equity prices, $b_t$ is deterministic, and no cash outflows between $t-1$ and $t$; we focus on one random variable, equity prices. We obtain one and two periods (of length $\Delta t$) ahead conditional default probabilities, where

$$D_{E_{t-1},t}^{E_{0,t}} = \frac{\ln \frac{E_{t-1}}{\alpha \times b_t} + (\mu_E - \frac{\sigma_E^2}{2}) \Delta t}{(\sigma_E \sqrt{\Delta t})}.$$ 

Because $\ln \frac{E_t}{E_{t-1}} = N \left( (\mu_E - \frac{\sigma_E^2}{2}) \Delta t, \sigma_E \sqrt{\Delta t} \right)$ is Gaussian,

$$p(1) = \Phi \left( -D_{E_{0,1}}^{E_{0,1}} + \frac{\ln \alpha}{\sigma_E \sqrt{\Delta t}} \right) = 1 - \Phi \left( \frac{\ln \alpha}{\sigma_E \sqrt{\Delta t}} \right),$$

where, of course, $\alpha > 0$ because a lognormal process is strictly positive. For two periods, the default probability (i.e., one minus the surviving probability) is given by

$$p(2) = 1 - \int_{b_t \times \alpha}^{\infty} \frac{1}{E_1 \sqrt{2\pi \sigma_E \sqrt{\Delta t}}} e^{-\frac{\left( \frac{\ln \frac{E_1}{E_0} - (\mu_E - \frac{\sigma_E^2}{2}) \Delta t}{\sigma_E \Delta t} \right)^2}{\sigma_E \Delta t}} \times \Phi \left( D_{E_{1,2}}^{E_{0,2}} - \frac{\ln \alpha}{\sigma_E \sqrt{\Delta t}} \right) \times dE_1.$$ 

**2.5 Comparative static analysis of the new $D_{0,t}^{E}$**

The new DD’s measure has a clear comparative analysis. The larger (lower) $E_0$ and $\mu_E$ ($b_t$ and $\sigma_E$) the larger this distance, $D_{0,t}^{E}$. For $\sigma_E$ this holds only if the firm is far from distress (i.e., $\frac{E_0}{\alpha \times b_t} \geq 1$), see below. If $b$ is small, equity prices need to collapse so that the firm is close to default. $D_{0,t}^{E}$ is also valid in a discrete-time setting, where $\alpha = 1$ and $b_t$ is the negative net cash-flow at time $t$ (instead of the continuous flow $b_t \times dt$), see Section 2.6.

Three implications follow: (1) Computing partial derivatives (and abstracting from the
mean term, \((\mu_E - \frac{\sigma_E^2}{2}) \approx 0)\),

\[
\frac{dD_{0,t}^E}{dE_0} = \frac{\left( \frac{E_0}{b_t} \right)^{-1}}{\sigma_E \sqrt{t}} > 0,
\]

\[
\frac{dD_{0,t}^E}{d\sigma_E} = \frac{\ln \frac{E_0}{\alpha \times b_t} \times -1}{\sigma_E \sqrt{t}} < 0 \text{ (or } > 0) \text{ , if } \frac{E_0}{\alpha \times b_t} > 1 \text{ (or } < 1).\]

\(D_{0,t}^E\) is more sensitive to the ratio \(\frac{E_0}{b_t}\) than to equity volatility \(\sigma_E\) if

\[
\left( \frac{E_0}{b_t} \right)^{-1} > \left| \ln \frac{E_0}{\alpha \times b_t} \right| \times \frac{1}{\sigma_E} \iff \sigma_E > \frac{E_0}{b_t} \times \left| \ln \frac{E_0}{\alpha \times b_t} \right|,
\]

e.g., if the firm is near distress, \(\frac{E_0}{b_t} \approx 1\). The opposite holds if either \(\frac{E_0}{b_t} \to 0\) or \(\frac{E_0}{b_t} \to \infty\), a severe distress or a very healthy firm, respectively. Large equity volatility will help in both scenarios, to escape from and to bring a firm closer to default.\(^{18}\)

(2) Consider a firm that is near distress (i.e., \(\frac{E_0}{b_t} < 1\)),

\[
\text{if } \frac{E_0}{b_t} < 1, \quad \frac{dD_{0,t}^E}{dE_0} \sim E_0^{-1} > b_t^{-1} \sim -\frac{dD_{0,t}^E}{db_t}.
\]

(14)

For this distressed firm, it helps more to rise its future prospects than to reduce costs, \(E\) and \(b\), respectively. In the case of underwater mortgages, it calls for reducing the principal (which rises equity value, \(E\)) better than interest payments \(b\). In the case of indebted sovereigns, long-term structural funds better than lower interest rates.

And (3), consider a setting with only financial leverage, \(b_t = B_0 \times r_b\), \(B_0\) is the face value of debt, \(r_b\) is the debt service rate, and \(\ln (E_0/B_0)\) is financial leverage (and \(\frac{\sigma_E^2}{2} \approx 0\)). Then, if all debt must be repaid, \(r_b = 1 + \mu_b\), where \(\mu_b\) is the debt cost (and \(\mu_b \approx \ln (1 + \mu_b)\)),

\[
\frac{\ln \frac{E_0}{b_t} \times \mu_E}{\sigma_E} \approx \frac{\ln \frac{E_0}{B_0} \times \mu_E - \mu_b}{\sigma_E},
\]

which depends on leverage and a carry cost between equity and debt, \(\mu_E - \mu_b\). But if all debt is refinanced or rollover \(r_b = \mu_b > 0\),

\[
\frac{\ln \frac{E_0}{b_t} \times \mu_E}{\sigma_E} \approx \frac{\ln \frac{E_0}{B_0} \times \mu_E - \ln \mu_b}{\sigma_E},
\]

which is quite sensitive to rising the debt cost, \(\ln \mu_b\) (where \(\frac{d}{d\mu_b} [-\ln \mu_b] = \frac{-1}{\mu_b} < 0\)).

\(^{18}\)Equity volatility \(\sigma_E\) can be approximated from realized (implied) volatilities under the \(P\) (\(Q\)) measures.
2.6 A discrete-time setting

We consider a discrete-time setting (e.g., Merton, 1974, or Geske’s, 1978, coupons bond).\textsuperscript{19} The “default” constraint is simply given by

\[ E_n \geq b_n, \ n = \{1, 2, ..., T\}, \]  

(15)

\( T \) is the number of coupon payments. Predicting default one-period ahead (of length \( \Delta t \)), distance-to-default is given by

\[ D^E = \frac{\ln \left( E_0/b_0 \right) + \left( \mu_E - \frac{\sigma_E^2}{2} \right) \times \Delta t}{\sigma_E \times \sqrt{\Delta t}}, \]

because the default event is given by

\[ \left\{ Z(0, 1) \leq -D^E_0 + \frac{\ln \alpha}{\sigma_E \sqrt{\Delta t}} \right\}, \]

where, we assume, \( b_t \approx b_0 > 0 \) (and \( \alpha = 1 \)).

Let us show how \( D^E \) and the associated default event, which depends on \( \alpha \), include both a continuous-time and a discrete-time setting. If \( E_0/b_0 < 1 \) (i.e., \( \ln \left( E_0/b_0 \right) < 0 \)), \( D^E \) becomes large and negative if \( \Delta t \) is small, where a small \( \Delta t \) also approximates a continuous-time setting. This precisely is our definition of default risk in continuous-time; a small \( E_0/b_0 \). On the other hand, if either \( E_0/b_0 \) or \( \Delta t \) are not so small (the 2nd-order approximation is not so accurate or a discrete-time setting, respectively), \( E_0/b_0 \) is not a substitute of \( D^E \).

\( \alpha = 1 \) does not change these implications, while the equity price, \( E_0 \), must be really small if \( \alpha \to 0 \) (because of the term \( \ln \left( \alpha/E_0 \right) \)). In continuous-time, \( \alpha \) can be seen as the time than equity \( E_t \) can support losses at a rate \( b_t \times dt \) before defaulting, and \( b \) represents the sum of all net cash-flows per year (like the dividend-yield approximation from cash dividends).

2.7 Strategic models of default

Consider now a firm that can renegotiate some value for equityholders before going to liquidation. After all, bankruptcy is costly for bondholders. This is formulated if the equity value at default is not zero but\textsuperscript{20}

\[ e \left( V_B \right) = \phi V_B \ \text{and} \ e_{V} \left( V_B \right) = \phi, \ \phi > 0. \]

\textsuperscript{19}Structural credit-risk models assume a time homogenous debt for tractability. Firms, however, choose more flexible capital structures such as lumpy maturity structure and active maturity management. This is a very feature of debt markets, see Chen et al. (2012), Choi et al. (2012), and Mian and Santos (2012).

Equation (6) simplifies to

$$E^{TV}(t, V_t) \approx \left( -NC_t + \left( r - \frac{\mu (V_B)}{V_B} \right) \phi V_B \right) \times \frac{(V_t - V_B)^2}{a^2} \cdot$$

(16)

Three implications follow. First, strategic default (i.e., $\phi > 0$) rises the value of default (i.e., a larger $b$) if $r - \mu > 0$, which is like a dividend-yield.

Second, we show that $\frac{dV_B}{d\phi} > 0$ (see Appendix B), which implies default is accelerated (delayed) if $\phi > 0$ ($\phi < 0$). That is, from $\frac{da}{dV_B} < 0$,

$$\frac{da}{d\phi} = \frac{da}{dV_B} \times \frac{dV_B}{d\phi} < 0,$$

and DD lowers (rises) if $\phi > 0$ ($\phi < 0$). A negative recovery value $\phi < 0$, which can be seen as a penalty for equityholders—costly default, implies default is optimally delayed. A negative $\phi$ is similar to a collateral pledged to the loan, which is lost only at default. We will apply this result to mortgages and households.

And third, from $E^{TV}(t, V) = E(t, V_t) - \phi V_t$, if $\phi > 0$,

$$\{E_t \leq \alpha \times b_t\} \subset \{E_t \leq \alpha \times b_t + \phi V_t\} \equiv \{E_t^{TV} \leq \alpha \times b_t\}$$

and

$$E^P_0 \left[ \{E_t \leq \alpha \times b_t\} \right] < E^P_0 \left[ \{E_t^{TV} \leq \alpha \times b_t\} \right].$$

For a company with the same equity value $E_t$ (and $b_t$) but positive recovery value $\phi > 0$, default is accelerated too.

In the three cases: fixed equity time-value, fixed asset value, and fixed equity value, respectively; $\phi > 0$ speeds up default. Shareholders’ recovery value is an additional variable to forecast default; see Garlappi and Yan (2011), who show that recovery value explains equity risk and returns of firms in financial distress.

3 Default Risk in Endogenous Multi-factor Models

The same 2nd-order equity value approximation holds for a model with stochastic parameters. We focus on stochastic volatility (we consider cash savings next).\(^{21}\) The only difference now is that the asset volatility $\sigma (V_B)$ is replaced by the overall multifactor volatility. It changes DD but not $b$. We abstract from this overall volatility by focusing on the ratio $E/b$.

\(^{21}\)Stochastic volatility makes Leland model intractable (see McQuade (2013)), but it is of first order importance in option-pricing.
Consider that volatility is defined either, $\sigma (V_t) = \sigma_t \times V_t$ or $\sigma (V_t) = \sigma_t$, where
\[ d\sigma_t = \alpha (\sigma_t) \, dt + \beta (\sigma_t) \, dW_t, \]
and $\rho$ is the correlation between $dW_t$ and $dZ_t$. The associated PDE is given by
\[ rE = E_t + \mu (V_t) \, EV + \frac{1}{2} \sigma^2 (V_t) \, E_{VV} + NC_t + \alpha (\sigma_t) E_{\sigma} + \frac{1}{2} \beta^2 (v) E_{\sigma \sigma} + \rho \sigma (V_t) \beta (\sigma_t) \, E_{V \sigma}, \quad (17) \]
and the exercise boundary depends also on volatility, $V_B(t, \sigma_t)$. We assume a second smooth-pasting condition
\[ E_{\sigma} (t, V_B, \sigma_t) = 0. \]
For $V = V_B(t, \sigma_t)$ (see Ibáñez and Velasco (2016)),
\[ E_{\sigma \sigma} = E_{VV} \times \left( \frac{\partial V_B}{\partial \sigma} \right)^2 \quad \text{and} \quad E_{V \sigma} = -E_{VV} \times \frac{\partial V_B}{\partial \sigma}. \]
Denoting by $\Sigma (t, V_B(t, \sigma_t), \sigma_t)$ the overall volatility at the default boundary, i.e.,
\[ \Sigma (t, V_B(t, \sigma_t), \sigma_t) = \sqrt{\sigma^2 (V_B) + \beta^2 (\sigma_t) \left( \frac{\partial V_B}{\partial \sigma} \right)^2 - 2 \rho \sigma (V_B) \beta (\sigma_t) \frac{\partial V_B}{\partial \sigma}}, \]
which shows that the curved default boundary $V_B(t, \sigma_t)$ can be hit from lower $V_t$ or lower $\sigma_t$ too (if $\frac{\partial V_B}{\partial \sigma} < 0$). Then, the time-value of equity is given by
\[ E^{TV} (t, V_t, \sigma_t) \approx \left( -NC + \left( r - \mu (V_B) \frac{e_V}{e} \right) e \right) \times \frac{\left( V_t - V_B(t, \sigma_t) \right)}{\Sigma (t, V_B(t, \sigma_t), \sigma_t)} \left( \frac{\Sigma (t, V_B(t, \sigma_t), \sigma_t)}{a_t^2} \right). \quad (18) \]
In a multi-factor setting, the novel term is the boundary sensitivity to the additional state variables; i.e., the term $\partial V_B/\partial \sigma$ in $\Sigma (t, V_B(t, \sigma_t), \sigma_t)$. The volatility drift does not matter (near default) because $E_{\sigma} (t, V_B, \sigma_t) = 0$. Distance-to-default, $a_t^2 = \frac{V_t - V_B(t, \sigma_t)}{\Sigma (t, V_B(t, \sigma_t), \sigma_t)}$, is approximated by $E^{TV}/b_t$; and the default event is approximated as equation (9), $\left\{ E^{TV}/b_t \leq \alpha \right\}$.

### 3.1 Default risk with liquidity constraints: insolvency and illiquidity

A factor that can produce credit risk is illiquidity and cash shortages. The theoretic models of Anderson and Carverhill (2012), Gryglewicz (2011), and Murto and Terviö (2014) consider constraints of this type. Firms need to hold cash and optimally decide the amount of cash and dividends. If revenues fall short of debt payments, they will incur in extra costs (e.g., new issues of equity) or simply default.

The need to save cash (which is denoted by $C$) has two impacts on default; (i) if cash earns a lower rate, and (ii) if cash is constrained to be above a lower bound ($C \geq 0$). We distinguish
between insolvency and illiquidity; i.e., default when $C > 0$ and $C = 0$, respectively. That is, $C$ can vanish and hit zero because dividends payments, negative earnings, debt service, or rollover costs of expiring debt.\footnote{In the liquidity models of Acharya et al. (2012), Bolton et al. (2011), and Decamps et al. (2011), earnings follow an arithmetic process, which implies constant business conditions, and default happens only when cash exhausts. Our results do not apply in their setting, default is only because illiquidity (no insolvency).}

If saving cash is costly, this cost explicitly appears in $b$. In Gryglewicz (2011), who solves the two models without and with cash constraints, we show the equity time-value is the same in both models (see Appendix A, Section 3). Yet the equityholders’ recovery value becomes nonlinear in the model with cash reserves (i.e., $e_{VV}(V_B) \neq 0$), and the reason (why the equity time-value is the same) is that cash reserves are not costly.

Because cash is the firm’s balance equation, its dynamics must be of the type

$$dC_t = (r_C \times C + NC_t) \, dt + \xi dW_t - dD_t,$$

where $r_C = r_C (C)$ is the interest earned on cash and $dD_t$ the instantaneous dividend ($dD_t \geq 0$). Now, $NC_t$ are net earnings and $dD_t$ is the true net cash-flow to equityholders. We add a Gaussian factor $\xi dW_t$ (where $E[dZdW] = \rho dt$), which allows for stochastic cash-flows and a general two-factor setting. $dD_t$ is the result of some optimization process taken as given.

The associated PDE (which holds only in a subset of the state space if $C$ is constrained) is given by

$$\tau E = \mu (V_t) \, E_V + \frac{1}{2} \sigma^2 (V_t) \, E_{VV} + dD_t + (r_C C + NC_t - dD_t) \, E_C + \frac{1}{2} \xi^2 E_{CC} + \rho \sigma (V_t) \xi E_{VC},$$

where $E = E(C, V)$. We denote by $V_B(C_t)$ the optimal default boundary, which depends on cash, and assume the following equity value at default

$$E(C, V_B(C)) = C + \phi V_B, \, \phi \geq 0,$$

and (assuming smooth-pasting)

$$E_C(C, V_B(C)) = 1 \, \text{ and } E_V(C, V_B(C)) = \phi.$$

The firm retains cash (i.e., $E_C = 1$, as it could spend all cash immediately before default) plus some strategic value ($E_V = \phi \geq 0$). The effect of the dividend policy ($dD_t$) depends on $E_C$, being irrelevant at default if $E_C = 1$.\footnote{This is because dividends are paid from cash. If they are financed from the assets of the company the net effect at default will be $1 - E_V$ (instead of $1 - E_C$), see Leland and Toft model below.}
Similar to the stochastic volatility model, for \( V = V_B(C_t) \),
\[
E_{CC} = E_{VV} \left( \frac{\partial V_B}{\partial C} \right)^2 \quad \text{and} \quad E_{VC} = -E_{VV} \frac{\partial V_B}{\partial C}.
\]
Denoting by \( \Sigma(t, V_B(t, C_t), C_t) \) the overall volatility at default, i.e.,
\[
\Sigma(V_B(C_t), C_t) = \sqrt{\sigma^2(V_B) + \xi^2 \left( \frac{\partial V_B}{\partial C} \right)^2 - 2\rho \sigma(V_t) \xi \frac{\partial V_B}{\partial C}},
\]
it follows that
\[
\frac{1}{2} E_{VV} = \left( -NC_t + (r - r_C) C + \left( r - \frac{\mu(V_B(C))}{V_B(C)} \right) \times \phi V_B(C) \right) \times \frac{1}{\Sigma^2(V_B(C_t), C_t)}. \tag{20}
\]
The value of default \( b \) depends also on the cost of cash, \( r_C \). We assume that cash is costly; i.e., \( r_C = r_C(C) \) and
\[
(r - r_C) \times C \geq 0.
\]
By defaulting, the firm stops paying a (negative) net cash-flow, costly cash, and a carry cost.

The time-value of equity is given by
\[
E^{TV} \approx \left( -NC_t + (r - r_C) C + \left( r - \frac{\mu(V_B(C))}{V_B(C)} \right) \times \phi V_B(C) \right) \times \frac{\left( V_t - V_B(C) \right)^2}{\Sigma(V_B(C), C)} \quad \text{near insolvency.} \tag{21}
\]

**Insolvency and illiquidity** Although a new borrowing/lending rate \( r_C \) applies to cash, we did not include any constraint on cash \( C \). Consider that borrowing is forbidden; otherwise, the firm will default because illiquidity. A constraint such as
\[
C_t \geq 0,
\]
does not change equations (19) and (20), it reduces only the state space. Default now is because either insolvency \( (V = V_B(C) \text{ and } C_t > 0) \) or illiquidity \( (C_t = 0) \). Clearly, the value of equity lowers and hence the default boundary rises, but the same formula applies to the time-value of equity near insolvency.\(^{24}\)

Assume that \( V_B(C) \) is decreasing in \( C \) and \( C > 0 \) (see Appendix B). Then, a lower \( C \) lowers the value of default from \( (r - r_C) C \); and if \( NC_t \) depends on \( V_B(C) \), a lower \( C \) implies (a larger \( V_B(C) \) and) a lower \( -NC_t \) and lower value of defaulting too. So, defaulting when
\[^{24}\text{This is easy to see in Boyle and Guthrie (2003), who add a liquidity constraint to the standard real option model and contains a call payoff. The call is exercised earlier (compared to the unconstrained case) and hence the value is lower.}
the value of default $b$ is low indicates costly cash and illiquidity concerns, which is consistent with better business conditions, i.e., larger $V_B(C)$ (where $\phi = 0$).

In the case of defaulting because illiquidity (i.e., $C = 0$), the formula does not apply (no smooth-pasting). If a firm defaults with zero or positive net cash-flows, $-b \geq 0$, it must be because illiquidity ($C = 0$). A loss-making firm by definition will be close to illiquidity, yet insolvency and illiquidity are closely linked (if $b > 0$ is small relative to the unconstrained case); low cash savings depend mainly on two factors, past negative earnings (and a low initial cash endowment).

Assume now that $C_t \geq -c$, where $c > 0$ is a borrowing limit. If $C_t < 0$, a more negative $C_t$ rises the value of default from $(r - r_C)C_t$. But this is intuitive too, risk-free borrowing is not forbidden in this market but is expensive.

In sum, in its more general form, the default event can be likewise described by

$$\left\{ \left\{ \frac{E_t - \phi_t}{E_0} \leq \alpha \times \frac{\text{debt service + losses - dividends + cost of cash \times cash}}{E_0} \right\} \cup \{\text{cash} \leq C\} \right\},$$

$\phi_t$ is shareholders’ recovery value and $C \leq 0$ a lower risk-free, if expensive, firm dependent borrowing limit. See Davydenko (2013) for an empirical study of insolvency and illiquidity of firms in financial distress; and see Campbell and Cocco (2015) in the case of mortgages, which are known as default dual triggers. Default probabilities also depend on cash holdings in Campbell et al. (2008).

### 3.2 Regime switching models

The same intuition and results carry over for regime switching models (Bhamra et al., 2010, or Chen, 2010). In this case, a system of PDE, value-matching, and smooth-pasting conditions hold at the default boundary. Consider two regimes; one bad, one good. In the bad (good) regime, the value of default is reduced (raised), because there is a chance to move to the other regime, where prospect looks better (worse). So, a new macroeconomic term $\Pi_t$ appears in $b_t$,

$$b_t = -NC_t - \Pi_t,$$

where $b_t$ depends on the regime and $\Pi_t$ is given by the expected gains/losses of moving between regimes, the new minus the old regime.

---
Bhamra et al. (2010) and Chen (2010) find that (asset values are procyclical but) earnings $NC_t$ at default are countercyclical in a macroeconomic regime switching model. $b$ depends on negative earnings and the negative gain of moving between regimes, hence $b$ (or the value of default) is “procyclical” in these models, because $\Pi$ is negative in the good regime (i.e., $-\Pi > 0$). A business cycle or macroeconomic factor may be relevant when predicting default.

### 3.3 Bankruptcy and liquidation

Some models that include a bankruptcy and a liquidation process, chapter 11 and chapter 7, respectively, have associated two exercise boundaries (Broadie et al., 2007). In the bankruptcy boundary, most models assume a new optimality condition so-called super-contact condition which requires continuity of the 2nd derivative (Dumas, 1988). In the liquidation boundary, the same, equation (7), 2nd-order approximation holds.

### 4 Default Risk in Leland and Toft (1996)

We consider Leland-Toft model to further illustrate our results and provide a numerical exercise. We use the same notation than Leland-Toft and He and Xiong (2012), but follow He-Xiong mathematical layout. We denote by $r$ the risk-free interest rate, $\delta$ the dividend-yield, and $\sigma$ the constant volatility of the unlevered assets value $V_t$, which follows a geometric Brownian motion under the $Q$-risk-neutral probability measure,

\[
\frac{dV_t}{V_t} = (r - \delta) dt + \sigma dZ_t,
\]

where $dZ_t$ is a standard Wiener process.

We denote by $E(V_t)$ the equity value and by $d(V_t, m)$ the debt value (with $m$-maturity). Equity value solves the following differential equation

\[
rE = (r - \delta)V_tE + \frac{1}{2}\sigma^2V_t^2E_{VV} + \delta V_t - (1 - \pi)C + d(V_t, m) - p,
\]

$NC_t \times dt$ is the “net cash-flow” to equity holders. $\delta V_t - (1 - \pi)C$ is the payout-rate less after-tax coupon payments ($\pi$ is the tax-rate). $p - d(V_t, m)$ are the refinancing costs of the expiring debt; $p$ is the notional of the expiring debt, and $d(V_t, m)$ is the price of the new raised debt. The capital letter $C$ indicates that the total coupon paid for the whole debt is $c \times m = C$, whereas only a bond with notional $p$ expires at $t$. 

20
The model is fully specified by providing the boundary conditions. Let \( V_B \) be the optimal default point, if \( V_t = V_B \),

\[
E(V_B) = 0 \quad \text{and} \quad E_V(V_B) = 0,
\]

which denote value-matching and smooth-pasting conditions, respectively. Equity is worthless if the firm defaults. The value of debt simplifies to

\[
d(V_B, m) = \alpha \frac{V_B}{m},
\]

where \( \alpha \) gives the firm’s recovery value in default, \( 0 \leq \alpha \leq 1 \).

4.1 Equity value near default

For \( V_t = V_B \), from equations (24) and (25), equation (23) simplifies to

\[
\frac{1}{2} \sigma^2 V_B^2 E_{VV}(V_B) = - \left( \delta V_B - (1 - \pi) C + \alpha \frac{V_B}{m} - p \right),
\]

which is the Gamma of the equity value at the default point.

In the continuation region, \( V_t \geq V_B \), we approximate the equity value by a 2nd-order Taylor expansion at the default point \( V_B \); i.e.,

\[
E(V_t) \approx E(V_B) + E_V(V_B) \times (V_t - V_B) + \frac{1}{2} E_{VV}(V_B) \times (V_t - V_B)^2 + O(V_t - V_B)^3
\]

\[
= \frac{r(1-\pi)C + p - (\delta + \frac{\alpha}{m}) V_B}{\sigma^2 V_B^2} \times (V_t - V_B)^2,
\]

where the 2nd equality uses equation (24). This approximation resembles the time-value of a perpetual American put option; \( K = \frac{(1-\pi)C + p}{r} \) is the strike price and \( \gamma = \delta + \frac{\alpha}{m} \geq 0 \) is the dividend-yield (for a given boundary \( V_B \)). That is, if \( V_t \geq V_B \),

\[
E(V_t) \approx \left[ \frac{V_t}{\text{“assets”}} - \frac{K}{\text{“strike”}} \right] + \left[ K - V_t + \frac{rK - \gamma V_B}{\sigma^2 V_B^2} \times (V_t - V_B)^2 \right],
\]

equity is given by the firm’s asset value \( V_t \), minus the present value of debt \( K \), plus an American put providing protection against equity becoming negative.

Remarks. Equations (22) to (25) are the same than in He and Xiong (2012); e.g., our equation (23) is exactly their PDE (11). He and Xiong (2012) model, which also includes a rollover risk component, differs from Leland and Toft (1996) model in the value of debt, \( d(V_t, m) \), and implies a different value of equity. In our case, we do no have to make explicit the value of debt, \( d(V_t, m) \). The net cash-flow term, \( NC_t \), also appears in equation (12) in Leland and Toft (1996). If \( p = 0 \) and \( d = 0 \), equity also corresponds to Leland’s (1994) model, where \( C \) is the consol bond coupon. Another rollover risk model is Chen et al. (2012), see their equity pricing equations (11) to (13), but which is given by a two-state continuos-time Markov chain.
We write the firm’s value relative to the endogenous boundary, $\frac{V_t - V_B}{\sigma V_B} \geq 0$,

$$E(V_t) \approx (1 - \pi) C + p - \left( \delta + \frac{\alpha}{m} \right) V_B \times \left( \frac{V_t - V_B}{\sigma V_B} \right)^2 .$$  \hspace{1cm} (29)$$

In Leland (1994),

$$E(V_t) \approx ((1 - \pi) c - \delta V_B) \times \left( \frac{V_t - V_B}{\sigma V_B} \right)^2$$

$$= (1 - \pi) c \times \left( \frac{V_t - V_B}{\sigma V_B} \right)^2 \text{ if } \delta = 0,$$

where $c$ is the fixed coupon of a consol bond.

### 4.2 DD in terms of equity prices and negative net cash-flows

The value of equity near default is approximated by the product of two components,

$$E(V_t) \approx (1 - \pi) C + p - \left( \delta + \frac{\alpha}{m} \right) V_B \times \left( \frac{V_t - V_B}{\sigma V_B} \right)^2 = b \times a_t^2 .$$  \hspace{1cm} (31)$$

The negative net cash-flows $b$ is given by

$$b = -NC_t = (1 - \pi) C - \delta V_B + p - \frac{\alpha}{m} V_B ,$$

where $d(V_B, m) = \frac{\alpha}{m} V_B$ is the debt value at default. $a_t = \frac{V_t - V_B}{\sigma V_B}$ is DD for a lognormal process. Equation (9), the corporate default event, is given by

$$\left\{ \frac{E(V_t)}{-(\delta V_t - (1 - \pi) C + d(V_t, m) - p)} \leq \alpha \right\}.  \hspace{1cm} (32)$$

### 4.3 Numerical exercise

We compute the value of equity based on the exact Leland–Toft formula and on the 2nd-order approximation (in equation (31)). We use the same parameters than He and Xiong (2012) calibration exercise for a typical US firm (during the 90’s), which are based on Huang and Huang (2012) empirical work. That is, $r = 0.08$, $\pi = 0.27$, $\sigma = 0.23$, $\alpha = 1 - 0.6$, $\delta = 0.02$, $T = 1$, $V_0 = 100$, and $C = 6.39$ and $P = 61.68$. The optimal default boundary is $V_B \approx 65.05$.

The large $b = b(V_B) = 39.02$ implies that the marginal value of default is high. The closest the asset value to the default boundary, the lowest the equity value and distance-to-default errors (Table 1). For example, if $V_0 = 70$, then $E = 2.31$ and $\frac{V_0 - V_B}{\sigma V_B} = 0.33$, and the 2nd-order approximation implies 4.27 and 0.24, respectively.

The quadratic approximation $b \times \left( \frac{V_0 - V_B}{\sigma V_B} \right)^2$ is upper-biased compared to $E$, and the ratio $E/b$ is conservative ($\sqrt{E \over b} < \frac{V_0 - V_B}{\sigma V_B}$ for $V_0 > V_B$). However, because $E/b$ is unbiased at the
boundary \( \sqrt{E/b} = \frac{V-V_B}{\sigma \sqrt{V_B}} = 0 \), if \( V \rightarrow V_B \), it follows that \( E/b \) classifies or disentangles perfectly between defaulting and nondefaulting firms. Further, \( \frac{V-V_B}{\sigma \sqrt{V_B}} \) and \( \sqrt{E/b} \) are highly correlated; both rise with \( V \). For example, this correlation is above 99% for the ten examples in Table 1, which are between 0 and 1 standard deviations. Because \( b(V_B) > b(V_t) \), \( E(V_t) < \frac{E(V)}{b(V_B)} \), and \( \frac{E(V)}{b(V)} \) may provide a better approximation to DD but worse to \( V_B \).

Finally, if we define the ratio \( \frac{E(V)}{b(V)} = \alpha \), we recover the approximate default boundary \( \tilde{V}_B \) from \( \tilde{V}_B = V_0 \). For example, from the third row (where \( V_0 = 66 \)), \( \sqrt{\alpha} = 0.051 \) implies that \( \tilde{V}_B = 66 \) (instead of the true boundary \( V_B = 65.05 \)).

<table>
<thead>
<tr>
<th>Table 1: Equity values and standardized distance to default, Leland and Toft (1996)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 0.08, \pi = 0.27, \sigma = 0.23, \alpha = 0.4, \delta = 0.02, T = 1, ) and ( C = 6.39 ) and ( P = 61.68 ).</td>
</tr>
<tr>
<td>Exercise boundary, ( V_B = 65.05 ). ( E \approx b \times \left( \frac{V_0-V_B}{\sigma \sqrt{V_B}} \right)^2 ) and ( b = 39.02 ).</td>
</tr>
<tr>
<td>\begin{tabular}{</td>
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</tbody>
</table>

5  Further Implications on Default Risk

The term \( b \) depends on many other features which alter the default decision; e.g., time-varying borrowing rates. If borrowing rates are floating rates (e.g., adjusted-rates mortgages), the leveraged security will swing between being closer (far away) to default when variable rates rise (lower) precisely to avoid “larger” debt service payments. Time-varying rates are also linked to short-term borrowing or roll-over debt (e.g., Leland and Toft, 1996), and a lower credit quality (hence larger refinancing costs) will lead to a large \( b \)—and earlier default. The
timing of default will coincide with payment days (e.g., maturity or coupon-days), which resembles the optimal exercise of American equity options around cash dividend days.\footnote{See the extreme case of the developer Reyal Urbis, a €4.3 billion—the 2nd largest bankruptcy in Spain, which filed for bankruptcy one day before a €73m tax debt expiry (\textit{Cinco Días} (Madrid), April 1, 2013).}

That a large $b$ is associated with default fits well with the empirical evidence on default triggers across asset classes. Campbell and Cocco (2015) report that Adjusted-(Fixed-)Rate Mortgages defaults tend to occur when interest rates are high (low) for moderate declines in house prices; and the clustering of balloon/interest-only loan defaults at maturity. Davydenko (2013) finds that corporate default is associated with economic distress (i.e., low assets value) but also negative profitability or high financial costs; and, analogously, the clustering of firm defaults around coupon days.\footnote{See Davydenko’s (2013) Figure 4, entitled on the timing of default relative to scheduled debt payments for a sample of defaulted firms, which is fully consistent with this.} Gopalan et al. (2013) show that default (and lower credit quality) are related to a larger proportion of maturing debt. Performance-sensitive debt also leads to earlier default (Manso et al., 2010).\footnote{Glober (2013) finds that firms with low recovery at default tend to default later, since, endogenously, are less leveraged. Harford et al. (2013) report that US firms have reduced the maturity of debt but simultaneously rising cash holdings, and hence, reducing (the default risk which is associated to) roll-over risk.}

In reduced-form econometric models that predict default, most studies focus on Merton’s DD, the value of assets at default (Davydenko, 2012), negative equity (Foote et al., 2008), the gain at default which is a put payoff (Fay et al., 2002, for household bankruptcy), besides a battery of control variables. We show endogenous default depends on low equity prices and large negative net cash-flows, with the debt payments calendar present, and the path of these two variables helps to explain many robust empirical findings on financial distress.\footnote{There is a literature based on one-period default models, Bulow and Shoven (1978) or Foote et al. (2008).}

An estimator of $\frac{V-V_B}{\sigma V_B}$ will become redundant if we control by $E$ and $b$; which explains the little significance of Merton’s DD in other studies. Depressed stock prices are closely related to financial distress if we control by $b$; in Garlappi and Yan (2011) stocks with per-share price less than $5$ are associated with high levels of Moody’s KMV expected default frequency.\footnote{The asset-pricing literature relates anomalous returns with financial distress, but these low price stocks are precisely those exclude in most empirical studies to avoid microstructure issues (Garlappi and Yan (2011)).} If past returns predict (low) equity prices, they will predict DD too; Duffie et al. (2009, p.2102) wonder why one-year trailing stock returns have a high impact on default intensities.

Martin et al. (2014) derive two reciprocal “liquidity” and “collateral” constraints in a discrete-time equilibrium model of repo runs and study (the fragility of) different repo markets
from the static comparative of these two constraints; finding “...more leveraged borrowers, or less profitable ones, are more fragile (p.967),” which is precisely the meaning of $b$. Indeed, the static comparative of the three illiquidity, collateral, and default constraints in their model is the same; hence, more fragile borrowers are also riskier (and illiquidity always precedes insolvency, which is ruled out in their model). See Appendix B.

Olney (1999) notes that “consumer spending collapsed in 1930, turning a minor recession into the Great Depression.. Households were shouldering an unprecedent burden of installment debt. Down payments were large.. Missed installment payments triggered repossession, household lost all acquired equity. Cutting consumption was the only viable strategy in 1930 for avoiding default..” However, “Institutional changes lowered the cost of default by 1938. When recession began again, indebted households chose to default rather than reduce consumption.” Our results can be used to understand this. If $-\phi < 0$ is household recovery value at default, a penalty, $\frac{da}{d\phi} < 0$ implies that $\frac{da}{d(-\phi)} > 0$, which rises DD (and delays default).

6 Mortgage Default

Because of its importance, we apply our results to mortgage default, which is closely related to corporate default if dividends are renting income and assuming no cash ($C = 0$). Let $V_t$ be the value of the unlevered house.

Near default, the value of the mortgaged house (i.e., equity or equation (16)) is given by

$$E^{TV}(t, V_t) \approx \left(-NC_t + \left(r - \frac{\mu (V_B)}{V_B}\right) \phi V_B \right) \times \left(\frac{V_t - V_B}{\sigma (V_B)}\right)^2 = b_t \times a_t^2,$$

$$NC_t = \delta V_t - c_t \times (1 - \tau),$$

where $\delta V_t$ is the renting income rate minus a maintenance cost, $c_t$ the debt service (which may include interest and principal), and $\tau \in [0, 1]$ a tax relief. $r - \mu \geq 0$ is the depreciation rate of the house and $\phi V_B$ a residual value for the owner (if she walks away). Here, $V_t$ no $E_t$ is observable (and traded) in practice.

Specifically, the difference between a Leland’s firm and a mortgage or leveraged house is twofold: In Leland’s model, (i) dividends, $\delta$, are paid from the assets of the company, which implies that $r - \mu = \delta$; and (ii) dividends are in cash, while the mortgage provides a housing service, which is either consumed or rented. If consumed, it cannot be used to cover interest rates payments, which has direct (nonpositive) liquidity implications.

We are interested in the following sensitivity sign $\frac{da}{d\phi}$, where (from equation (61) in
Appendix B) for \( x \neq \sigma \),
\[
\text{sign} \left( \frac{da}{dx} \right) = -\text{sign} \left( \frac{\partial V_B}{\partial x} \right) \approx -\text{sign} \left( \frac{b_x}{b} - \frac{E_x}{E} \right).
\]

We focus on cash-flow parameters, which then implies (if \( \text{sign}(b_x) = -\text{sign}(E_x) \))
\[
\text{sign} \left( \frac{da}{dx} \right) \sim -\text{sign} \left( \frac{\partial b}{\partial x} \right),
\]
i.e., the larger the value of default, \( b \), the lower the DD, \( a \).

Then, it is straightforward the following static comparative analysis in Table 2. For instance, the signs of Loan-to-Value and Loan-to-Income on mortgage default-risk, \(+\) and \(0\), respectively (where \(0\) can be explained by a large loan but a better home), agree with those of the rich household model in Chen et al. (2013).

<table>
<thead>
<tr>
<th>parameter, ( x )</th>
<th>( da/dx ) or ( \partial b/\partial x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan or Loan-to-Value, ( c ) or ( c - \delta ):</td>
<td>( \partial b/\partial x &gt; 0 )</td>
</tr>
<tr>
<td>Floating- and Fixed-Rate Mortgage, ( c ):</td>
<td>( \partial b/\partial c &gt; 0 )</td>
</tr>
<tr>
<td>Loan-to-Income, ( c + \delta ):</td>
<td>( \partial b/\partial x \approx 0 ) (or undetermined)</td>
</tr>
<tr>
<td>Renting cost, ( \delta ):</td>
<td>( \partial b/\partial \delta &lt; 0 )</td>
</tr>
<tr>
<td>Taxes (on interest payments), ( \tau ):</td>
<td>( \partial b/\partial \tau &lt; 0 )</td>
</tr>
<tr>
<td>Reduced interest payment, (-c):</td>
<td>( -\partial b/\partial c &lt; 0 ) (temporary, delays default)</td>
</tr>
<tr>
<td>Constant payment (amortizable debt), ( c ):</td>
<td>( \partial b/\partial c &gt; 0 ); but after amortization ( dE &gt; 0 ), and hence, ( \frac{da^2}{dE} = \frac{1}{b} &gt; 0 )</td>
</tr>
<tr>
<td>Only-Interest-Payment, (-c):</td>
<td>( -\partial b/\partial c &lt; 0 ) before ( T ) (but ( &gt; 0 ) at maturity)</td>
</tr>
<tr>
<td>Reduced principal:</td>
<td>( da/dx &gt; 0 ) (since ( dE &gt; 0 ) and ( \frac{da^2}{dE} = \frac{1}{b} &gt; 0 ))</td>
</tr>
<tr>
<td>Value, ( V ):</td>
<td>( da/dV &gt; 0 )</td>
</tr>
<tr>
<td>Recovery value at default, ( \phi ):</td>
<td>( da/d\phi &lt; 0 )</td>
</tr>
<tr>
<td>Negative externality at default, (-\phi):</td>
<td>( -da/d\phi &gt; 0 ) (# children, # years in the neighborhood,..)</td>
</tr>
<tr>
<td>Liquidity/cash problems:</td>
<td>cash constraint model</td>
</tr>
</tbody>
</table>

In terms of loan modification, a policy debate in the US is about either reducing the principal or temporary reductions in payments (Elul et al. (2010), Foote et al (2008)). Based on the bankruptcy cost for lenders in foreclosure, Goodman (2010) proposes to lower the
principal of some no-very-deep under-water mortgage holders, as the only way of solving the housing market overhang with millions of properties going into default.\footnote{Goodman (2010) reports that 50\% of the US mortgages are either voluntarily prepaid or go into default after four (98.5\% after fifteen) years.}

Table 2 is clear in this debate, the temporary reduction mostly delays or postpones default until the household has to face debt payments again. It helps in the case of liquidity problems such as a job loss. Reducing the principal rises equity value (i.e., $dE > 0$), and hence, lowers the risk of default and foreclosure; i.e., $da^2 \approx \frac{4E}{b} > 0$. Indeed, this is better than lowering interest-rates for a severe underwater house ($E/b < 1$, see equation (14)). Given this restructuring (i.e., reducing) of the principal, it is “in the best interest” of the debtor to stay in her house. The lender has rights and covenants (also, information on the borrower financial situation) to induce this process.

The parameter $\phi$ can accommodate heterogeneity among homeowners, where a positive (negative) recovery value induces earlier (later) default; see Section 2.7. In the case of mortgages, default also depends on the attachment/number of years living in the house (Campbell and Cocco (2015)), which can enter negatively in this residual value, $\phi < 0$.

\section{Concluding Remarks}

As noted by Duffie et al. (2007) and (2009), interpreting default risk is not easy. Consider the following three covariates. The firm’s trailing one-year stock return offers significant incremental explanatory power of default risk, yet we lack a structural interpretation. Merton’s DD is a key covariate to predict default risk, yet it is based on the simplest zero coupon debt model. The trailing average default rate may proxy for unobserved covariates, but it violates an independence assumption in the likelihood function.

This paper offers an original description of default risk based on endogenous credit-risk models. We show that endogenous default (which is defined by a sufficiently low asset value) can likewise be described in terms of depressed equity prices $E$ and large negative net cash flows $b$, a low ratio $E/b$. By defaulting, shareholders stop incurring a negative net cash-flow, which is given by debt service and negative earnings minus dividends. The larger this cost, the larger the value of (and incentives to) default.

The determinants of the event (and probability) of default are negatively related to price per share, profitability, expected (or past) equity returns, and cash holdings, and are positively
related to equity volatility and the debt-service calendar (and to, a lesser extent, the cost of cash, shareholders’ recovery, and the business cycle). From all these variables, if we want to hide or understate default-risk, earnings are easier to manipulate (or inflate). The financial missreporting, from Enron to, recently, Gowex, is a clear example of this.

Our results extend to a multi-factor setting and imply all leveraged securities face similar tradeoffs. It is a simple and theoretically grounded measure, which allows us to understand credit risk. As a clear application, for a distressed firm (or underwater house), namely, \( E/b < 1 \), lowering the principal is more helpful to equityholders (homeowners), which in turn increases equity value \( E \), than temporarily reducing the interest (mortgage) payments \( b \).

We consider a model with a second state variable cash reserves and with liquidity constraints (e.g., borrowing is forbidden). This approach allows us to integrate insolvency and illiquidity. Now the value of default depends also on the cost of cash and the level of cash reserves, which are inversely related to earnings at default. A lower value of default, \( b \), is associated with lower cash-reserves (and less negative earnings), indicating default is more a result of illiquidity than insolvency. Yet insolvency and illiquidity are closely linked; low cash savings depend mainly on two factors, past negative earnings (and a low initial cash endowment).

Finally, equity prices \( E \) (i.e., the equity market) pave the way for a large correlation or contagion between two firms in a downturn market—even if cash-flows \( b \) do not. See Duffie et al. (2007) who address both problems, dynamic and correlated default, in a reduced-form default-intensity setting but using Merton-based covariates.

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University of Toronto.


Huang J. and M. Huang (2012), “How Much of the Corporate-Treasury Yield Spread is


8 Appendix A: Firm Default, Extensions and Robustness

We add several features to Leland–Toft model. We work on a case by case basis. For easy of comparison with the original papers, we use their same original notation but we will strictly limit to the needed results (see the original papers for full details).

8.1 Strategic default

Shareholders can seize some value before liquidation. The equity value at default is not zero but $\phi V_B$, $\phi \geq 0$. The new boundary conditions are given by

$$ E(V_B) = \phi V_B \text{ and } E_V(V_B) = \phi. $$

For $V_t = V_B$, then \( \left( r - (r - \delta)V_t \frac{E_V}{E} \right) E = \delta \phi V_B \), and the option Gamma is given by

$$ \frac{1}{2} E_{VV}(V_B) = \frac{-NC + \delta \phi V_B}{\sigma^2 V_B^2} = - \frac{\left( \delta (1 - \phi) V_B - (1 - \pi) C + \alpha \frac{V_B}{m} - p \right)}{\sigma^2 V_B^2}. \quad (33) $$

In the non-default region, $V_t \geq V_B$, from a 2nd-order Taylor expansion at the point $V_B$,

$$ E(V_t) \approx \phi V_t + (-NC + \delta \phi V_B) \times \left( \frac{V_t - V_B}{\sigma V_B} \right)^2, \quad (34) $$

and the time-value by

$$ E^{TV}(t, V) = E(t, V_t) - \phi V_t \approx (-NC + \delta \phi V_B) \times \left( \frac{V_t - V_B}{\sigma V_B} \right)^2. \quad (35) $$

The effect of the dividend-yield on default, which is negative, is reduced to $-\delta (1 - \phi)$.

8.2 Stochastic volatility

From equations (18) and (26), $E^{TV}(t, V_t, \sigma_t) = E(t, V_t, \sigma_t)$, and

$$ E(t, V_t, \sigma_t) \approx \left( (1 - \pi) C + p - \left( \delta + \frac{\alpha}{m} \right) V_B(\sigma_t) \right) \times \left( \frac{V_t - V_B(\sigma_t)}{\sqrt{\sigma_t^2 V_B^2 + b^2(\sigma_t) \left( \frac{\partial}{\partial V_B} V_B \right)^2 - \rho \sigma_t V_B b(\sigma_t) \frac{\partial}{\partial V_B} V_B}} \right)^2. $$

The distance-to-default, $a_t$, can be inverted from equity value in the same way; i.e.,

$$ a_t = \frac{V_t - V_B(\sigma_t)}{\sqrt{\sigma_t^2 V_B^2 + b^2(\sigma_t) \left( \frac{\partial}{\partial V_B} V_B \right)^2 - \rho \sigma_t V_B b(\sigma_t) \frac{\partial}{\partial V_B} V_B}} \approx \sqrt{\frac{E(t, V, \sigma_t)}{b_t}}. \quad (36) $$
8.3 Liquidity: Gryglewicz (2011)

Gryglewicz (2011) model is interesting to us; the shareholders’ recovery value is not lineal, providing a robust test for our results. Gryglewicz provides close-form solutions in a model with cash reserves. The firm generates a stochastic flow of earnings before interest and taxes (EBIT),

\[ dX_t = \mu dt + \sigma dZ_t, \]  

(37)

where \( \mu \) is unknown but takes either of two values, \( \mu_L \) or \( \mu_H \), \( \mu_L < \mu_H \). All parties share a common prior expectation \( \mu_0 \) about \( \mu \), with \( \mu_0 \in (\mu_L, \mu_H) \). From optimal filtering theory,

\[ dX_t = \mu_t dt + \sigma dZ_t \quad \text{and} \quad d\mu_t = \frac{(\mu_t - \mu_L)(\mu_H - \mu_t)}{\sigma} dZ_t. \]  

(38)

We denote volatility of average earnings by \( \Sigma(\mu_t) = \frac{(\mu_t - \mu_L)(\mu_H - \mu_t)}{\sigma} \).

The dynamic of cash reserves is given by

\[ dC_t = rC_t dt + (1 - \tau) [dX_t - k dt] - dDiv_t, \]  

(39)

where \( k \) is the coupon rate of the consol bond and \( dDiv_t \) is the dividend. It is required that

\[ C_t \geq 0 \quad \text{and} \quad dDiv_t \geq 0, \]  

(40)

borrowing and negative dividends are forbidden. If \( C_t = 0 \), illiquidity produces default.

Gryglewicz shows the minimum level of cash reserves which is consistent with (40) is given by

\[ \overline{C}(\mu_t) = (1 - \tau) \times \left[ \frac{\sigma^2}{\mu_H - \mu_L} \ln \left( \frac{\mu_t - \mu_L}{\mu_H - \mu_t} \frac{\mu_H - \mu_t}{\mu_L - \mu_t} \frac{\mu_L - \mu^*}{\mu^* - \mu_L} \right) + \frac{1}{\tau} \left( k - \frac{\mu_H + \mu_L}{2} \right) \right]. \]  

(41)

For \( C_t = \overline{C}(\mu_t) \), under the optimal dividend policy, equity value depends only on the mean, \( \mu_t \), and solves

\[ rE(\mu_t) = \frac{1}{2} \sigma^2 (\mu_t) E''(\mu_t) + \frac{1}{dt} dDiv_t^*, \]  

(42)

\[ \frac{1}{dt} dDiv_t^* = \frac{(1 - \tau) \tau \sigma^2}{\mu_H - \mu_L} \ln \left( \frac{\mu_t - \mu_L}{\mu_H - \mu_t} \frac{\mu_H - \mu_t}{\mu_L - \mu_t} \frac{\mu_L - \mu^*}{\mu^* - \mu_L} \right) + (1 - \tau) \left[ \frac{\mu_H + \mu_L}{2} - k \right]^{+}, \]

where \( E'' \) denotes a 2nd derivative, \( \mu^* \) is the optimal default boundary, and \( dDiv_t^* \) is the associated optimal dividend. The value-matching and smooth-pasting boundary conditions depend on cash reserves and are given by

\[ E(\mu^*) = \overline{C}(\mu^*) \quad \text{and} \quad E'(\mu^*) = \overline{C}'(\mu^*), \]  

(43)

respectively. Note that \( \overline{C}(\mu_t) > 0 \) and \( dDiv_t^* > 0 \), if \( \mu_t > \mu^* \) (since \( \frac{\mu_t - \mu_L}{\mu_H - \mu_t} \frac{\mu_H - \mu_t}{\mu_L - \mu_t} \frac{\mu_L - \mu^*}{\mu^* - \mu_L} > 1 \)).
8.3.1 The value of default in Gryglewicz

Now, we explain the value of default, which is not in Gryglewicz. At the optimal default boundary, $\mu_t = \mu^*$,

$$C(\mu^*) = \frac{1}{r} (1 - \tau) \left\{ k - \frac{\mu_H + \mu_L}{2} \right\}^+ ,$$  \hspace{1cm} (44)

implying that

$$(1 - \tau) \left\{ k - \frac{\mu_H + \mu_L}{2} \right\}^+ = \frac{1}{2} \Sigma^2 (\mu^*) E''(\mu_t) + (1 - \tau) \left\{ \frac{\mu_H + \mu_L}{2} - k \right\}^+ \hspace{1cm} (45)$$

and

$$\frac{1}{2} E''(\mu^*) = \frac{(1 - \tau) \times \left( k - \frac{\mu_H + \mu_L}{2} \right)}{\Sigma^2 (\mu^*)}. \hspace{1cm} (46)$$

Because $C$ is not lineal, we have to compute its 2nd derivative, i.e.,

$$C'(\mu^*) = \frac{(1 - \tau) \sigma^2}{\mu_H - \mu_L} \times \left( \frac{1}{\mu_H - \mu^*} + \frac{1}{\mu^* - \mu_L} \right) \hspace{1cm} \text{and} \hspace{1cm} (47)$$

$$C''(\mu^*) = \frac{(1 - \tau)}{\mu_H - \mu_L} \left( \frac{(\mu^* - \mu_L)^2 - (\mu_H - \mu^*)^2}{(\mu_H - \mu^*)(\mu^* - \mu_L) / \sigma} \right)^2 = (1 - \tau) \frac{2\mu^* - (\mu_L + \mu_H)}{\Sigma^2 (\mu^*)}, \hspace{1cm} (48)$$

Hence, in the subset $\{ (\mu, C) : \mu \geq \mu^* \text{ and } C = C(\mu) \}$, the time-value of equity is given by

$$E^{TV}(\mu_t) \approx \frac{1}{2} \left( E''(\mu^*) - C''(\mu^*) \right) \times (\mu_t - \mu^*)^2 = (1 - \tau) \times (k - \mu^*) \times \left( \frac{\mu_t - \mu^*}{\Sigma (\mu^*)} \right)^2 \hspace{1cm} (49)$$

and

$$\frac{\mu_t - \mu^*}{\Sigma (\mu^*)} \approx \sqrt{\frac{E^{TV}(\mu_t)}{(1 - \tau) \times (k - \mu^*)}}. \hspace{1cm} (50)$$

The value of default depends on minus the net cash-flows; i.e., the coupon rate minus average profits, $k - \mu^*$, which must be positive. Moreover, in Gryglewicz (2011) model without cash constraint, the 2nd-order approximation of the option value is also equal to the rhs of (49).\textsuperscript{33} It turns out that both $\mu^*$’s, equations (8) and (29) in his paper are the same, only the optimal coupon differs between both models. So, cash constraints do not affect the decision to default if cash is invested at the same rate, $r$.

\textbf{Remark} For completeness, we show the conditions under which Gamma is the same in equations (49) and (19), where (19) depends on $V$ and $C$, and is given by

$$rE = \mu (V_t) E_V + \frac{1}{2} \sigma^2 (V_t) E_{VV} + dD_t + (rC + NC_t - dD_t) E_C + \frac{1}{2} \sigma^2 E_{CC} + \rho \sigma (V_t) \xi E_{VC},$$

\textsuperscript{33}The same PDE (42) holds with payout rate $dDiv_t = (1 - \tau) (\mu_t - k) dt$, which is the net cash-flow, and boundary conditions $E(\mu^*) = 0$ and $E'(\mu^*) = 0$. See equations (5) and (23) in Gryglewicz (2011).
with boundary condition at default

\[ E(V, C) = C, \quad E_V = 0 \quad \text{and} \quad E_C = 1. \]

We have that \( r_C = r, \, \mu(V_t) = 0 \) (from optimal filtering), and \( NC_t = (1 - \tau) \times (\mu^* - k) \) and \( E = C(\mu^*) \) at default. Then, for \( V_t = \mu_t \) and \( \sigma(V_t) = \Sigma(\mu_t) \), equation (19) simplifies to

\[
\frac{1}{2} \left( \sigma^2 (V_B) + \xi^2 \left( \frac{\partial V_B}{\partial C} \right)^2 - 2 \rho \sigma (V_t) \xi \frac{\partial V_B}{\partial C} \right) E_{VV} = -NC_t = (1 - \tau) \times (k - \mu^*),
\]

which implies the same Gamma as in equation (50) if \( \xi = 0 \).

### 8.4 Liquidity: Anderson and Carverhill (2012)

Anderson and Carverhill (2012) consider a two-factor problem, which is solved by numerical methods. The rate of operating revenues of the firm, \( dS_t \), is given by

\[
dS_t = \rho_t dt + \sigma dw_t^\sigma,
\]

\[
d\rho_t = \kappa (\bar{\rho} - \rho_t) dt + \eta \rho_t dw_t^\rho,
\]

where \( dw_t^\sigma \) and \( dw_t^\rho \) are two orthogonal Wiener processes. The profit of the firm is \( dS_t - f dt \), where \( f \) are operating costs. The firm has a “cash reserve” \( C \) with dynamics

\[
dC_t = (1 - \tau) [dS_t - ((f + q) - r_C C_t) dt] - dD_t,
\]

where \( \tau \) is the tax rate, \( r_C \) a different cash borrowing/lending rate \( ((r - r_C) \times C \geq 0) \), and \( dD_t \) the dividend payment. \( q \geq 0 \) is the continuous coupon of a perpetual bond, and the firm is leveraged if \( q > 0 \).

Equity value \( J^q(\rho, C) \), for a given coupon \( q \), is a function of the expected revenues and cash reserves (\( \rho_t \) and \( C_t \), respectively) defined in the state space \( \{ (\rho, C) : C \geq C(\rho) \} \), where \( C(\rho) \) is an exogenous borrowing limit given by

\[
C(\rho) = - \max \left\{ 0, (1 - \alpha) J^0(\rho, C) - \frac{q}{\tau} \right\},
\]

\( \alpha, \quad 0 \leq \alpha \leq 1 \), is the loss given default, \( J^0 \) is the value of the unlevered firm, and \( q/\tau \) is the value of the perpetual risk-free debt. (Intuitively, \( -C(\rho) \geq 0 \) is the firm liquidation value after repaying all debt). At every instant, the firm decides “optimally” (i) the dividend policy \( dD_t \geq 0 \), (ii) whether to issue new stock, or (iii) whether to default. Within the so-called saving region, \( S \), it is optimal to no pay dividends \( \frac{dD_t}{dt} = 0 \), and the firm will issue new equity if the value of the firm is not negative and if it is cheaper than borrowing money.
Hence, we focus on the saving region, \((\rho, C) \in S\), where \(\frac{dD_t}{dt} = 0\). The firm satisfies the following PDE,

\[
\frac{\partial}{\partial t} J_t^q + \kappa (\bar{\rho} - \rho_t) \frac{\partial}{\partial \rho_t} J_t^q + \frac{\sigma^2}{2} \rho_t \frac{\partial^2}{\partial \rho_t^2} J_t^q + (1 - \tau) [\rho_t - (f + q) - rC_tC_t] \frac{\partial}{\partial C_t} J_t^q + \frac{1}{2} \frac{\partial^2}{\partial C_t^2} J_t^q = rJ_t^q,
\]

subject to

\[
1 \leq \frac{\partial}{\partial C} J_t^q (\rho, C) \leq \frac{1}{\theta},
\]

where \(C \geq C (\rho)\) and equity has limited liability \(J_t^q (\rho, C) \geq 0\). \(\theta\) is the cost of issuing new shares (costly, if \(\theta < 1\)). In this region, positive cash-flows are saved in cash (better than paid in dividends, \(1 \leq \frac{\partial}{\partial C} J_t^q (\rho, C)\)) and negative cash-flows are withdraw from the cash reserve (better than issuing new stock, \(\frac{\partial}{\partial C} J_t^q (\rho, C) \leq \frac{1}{\theta}\)).

Anderson and Carverhill do not specify the smooth-pasting boundary condition for default, because the problem is solved numerically by maximizing among paying dividends, issue new shares, or default (see their Appendix A numerical techniques). They assume that shareholders get (where \(T\) is the default stopping-time),

\[
J_T^q (\rho, C) = \max \left\{ 0, (1 - \alpha) J_0^q (\rho, 0) + C - \frac{q}{r} \right\}.
\]

### 8.4.1 The value of default in Anderson and Carverhill

Now, we explain the value of default, which is not in Anderson-Carverhill. The default boundary, \(\{B (\rho) : \rho \leq \tilde{\rho} < \infty\}\), is part of the region \(S\). Default happens only because insolvency; the firm can always fund losses (by issuing new stock) at a cost which is given by \(\frac{1}{\theta}\). For \(\rho \to \infty\), it does not make sense to default at all, the firm will rise cash to any cost (which is bounded by \(\frac{1}{\theta}\)). Hence, \(\tilde{\rho} < \infty\).

For \(\{(\rho, B (\rho)) : \rho \leq \tilde{\rho} \text{ and } B (\rho) > C (\rho)\}\), and assuming that \(\frac{\partial}{\partial \bar{\rho}} J_t^q = 0\), the PDE simplifies to

\[
\frac{1}{2} \left( \eta^2 \rho_t \frac{\partial^2}{\partial \rho_t^2} + \sigma^2 \frac{\partial^2}{\partial C_t^2} \right) J_t^q = -(1 - \tau) [\rho_t - (f + q)] \times \frac{\partial}{\partial C_t} J_t^q + \left( r - rC_t \times \frac{\partial}{\partial C_t} J_t^q \right) B (\rho) - \kappa (\bar{\rho} - \rho_t) \times \frac{\partial}{\partial \rho_t} J_t^q
\]

\[
+ r \times (J_t^q - B (\rho)),
\]

where the first three terms in the rhs are similar to those in equation (20).
If shareholders keep cash and save some strategic value at default \((\phi \geq 0)\), i.e.,
\[
J^0_T (\rho, B (\rho)) = B (\rho) + \phi (\rho - \bar{\rho}),
\]
the equity gamma in the previous PDE is obtained from
\[
\frac{\partial}{\partial \rho} J^0_T (\rho, B (\rho)) = \phi \quad \text{and} \quad \frac{\partial}{\partial C} J^0_T (\rho, B (\rho)) = 1,
\]
\(B\) is the default boundary. Let \(V_B\) denote the optimal default boundary associate to \(\phi\), where \(V_B = V^{(0)}_B\) in the case of zero recovery. Let \(E^{(\phi)}(V_t, B)\) denote equity value for recovery \(\phi\) and a given default policy \(B\), where \(V_t \geq B\) and \(E^{(\phi)}(V_t, B)|_{V_t=B} = \phi B\).

9 Appendix B

9.1 Proof that \(\frac{dV^{(\phi)}_B}{d\phi} > 0\), where \(\phi V\) is shareholders’ recovery value at default

For simplicity, we assume a time homogeneous problem and a constant default boundary. Let \(V^{(\phi)}_B\) denote the optimal default boundary associate to \(\phi\), where \(V_B = V^{(0)}_B\) in the case of zero recovery. Let \(E^{(\phi)}(V_t, B)\) denote equity value for recovery \(\phi\) and a given default policy \(B\), where \(V_t \geq B\) and \(E^{(\phi)}(V_t, B)|_{V_t=B} = \phi B\).

\(V_B\) is the optimal policy associated to \(\phi = 0\), and hence,
\[
\frac{dE^{(0)}(V_t, B)}{dB} \bigg|_{B=V_B} = 0.
\]

The key insight is that
\[
E^{(\phi)}(V_t, B) = E^{(0)}(V_t, B) + E^{Q}_t [e^{-r\tau} \phi B|V_t],
\]
where \(\tau\) is the stopping-time associated to \(V_t = B\) (i.e., \(\tau = t\) if \(V_t = B\)). Then,
\[
\frac{dE^{(\phi)}(V_t, B)}{dB} \bigg|_{B=V_B} = 0 + \phi \times \left( E^{Q}_t [e^{-r\tau}] + V_B \frac{dE^{Q}_t[e^{-r\tau}|V_t]}{dB} \right)
\]
\[
> 0 \quad \text{if} \quad \phi > 0 \quad \text{(and} \quad < 0 \quad \text{if} \quad \phi < 0),
\]
where \(\frac{dE^{Q}_t[e^{-r\tau}|V_t]}{dB} \geq 0\) because \(V_t \geq B\). Consequently, the optimal default policy holds that
\[
V^{(\phi)}_B > V_B \quad \text{if} \quad \phi > 0; \quad \text{and} \quad V^{(\phi)}_B < V_B \quad \text{if} \quad \phi < 0.
\]
Next,
\[
E^{(\phi + \Delta \phi)} (V_t, V_B^{(\phi)}) = E^{(0)} (V_t, V_B^{(\phi)}) + E_t^Q \left[ e^{-r_T} (\phi + \Delta \phi) V_B^{(\phi)} | V_t \right],
\]
\[
dE^{(\phi + \Delta \phi)} (V_t, V_B^{(\phi)}) \frac{dV_B^{(\phi)}}{dV_B^{(\phi)}} = d \left( E^{(0)} (V_t, V_B^{(\phi)}) + E_t^Q \left[ e^{-r_T} \phi V_B^{(\phi)} | V_t \right] \right) + dE_t^Q \left[ e^{-r_T} \Delta \phi V_B^{(\phi)} | V_t \right]
\]
\[
= 0 + \Delta \phi \times \left( E_t^Q \left[ e^{-r_T} \right] + V_B^{(\phi)} dE_t^Q \left[ e^{-r_T} | V_t \right] \right) > 0 , \text{ if } \Delta \phi > 0.
\]
It follows that \( V_B^{(\phi + \Delta \phi)} > V_B^{(\phi)} \), and in the limit, \( \frac{dV_B^{(\phi)}}{d\phi} = \lim_{\Delta \phi \to 0} \frac{V_B^{(\phi + \Delta \phi)} - V_B^{(\phi)}}{\Delta \phi} > 0 \).

\[\blacksquare\]

### 9.2 Proof that \( \frac{dV_B^{(C)}}{dC} > 0 \), where \( C \) is cash-holdings

Let \((c_2, V_B(c_2))\) be a point of the default boundary. If it is optimal to default at a point \((c_2, V_1)\) for \( V_1 < V_B(c_2) \), we assume the strict inequality
\[
E(c_2, V_1) < c_2 + \phi V_1.
\]

Let \( c_1 \) be such that \( V_1 = V_B(c_1) \). If \( c_1 \leq c_2 \), we can pay \((c_2 - c_1)\) in cash and move to the boundary, where
\[
E(c_1, V_B(c_1)) + (c_2 - c_1) = c_1 + \phi V_B(c_1) + (c_2 - c_1) = c_2 + \phi V_B(c_1) = c_2 + \phi V_1,
\]
and consequently it is not optimal to default at the point \((c_2, V_1)\). Hence, \( c_1 > c_2 \). And from \( V_1 = V_B(c_1) < V_B(c_2) \), the default boundary is decreasing. This also holds in Figure 1 in Murto and Terviö (2013) and Anderson and Carverhill (2012), where \( V \) are earnings, and in Davydenko’s (2013) data plot Figure 1.

\[\blacksquare\]

### 9.3 Martin et al. (2014) repo runs model

In a equilibrium model of repo runs Martin et al. (2014) derive a liquidity (and collateral) constraints to show that “...more leveraged borrowers, or less profitable ones, are more fragile.” The liquidity constraint is given by (see their equation (13))
\[
\beta^2 R\bar{T} \geq (1 - \alpha + \beta) b,
\]
where \( R\bar{T} \) and \( b \) are earnings and debt, respectively; \( \beta < 1 \) is a discount factor and \( (1 - \alpha + \beta) \) is the amount of debt to be repaid.
One can derive a similar default constraint, if profits are nonnegative (i.e., $\pi \geq 0$ and $\pi$ is given by their unenumerated equation in page 967),

$$\beta^2 \frac{R - 1}{1 - R} \geq (1 - \alpha + \beta) b.$$ 

First, the liquidity and default constraints have the same static comparative analysis to profitability $R$, size $I$ and debt $b/I$. Second, the default constraint is strictly weaker than the liquidity constraint. If the liquidity constraint is binding (i.e., $\beta^2 R I = (1 - \alpha + \beta) b$), then

$$\beta^2 \frac{R - 1}{1 - R} \geq (1 - \alpha + \beta) b < \beta (1 - \alpha + \beta) b,$$

due to $eta < 1$ and $\beta^2 R > 1$ imply $\beta R > \beta^2 R > 1$. Next

$$\beta^2 (R - 1) I - (1 - \beta) (1 - \alpha + \beta) b = -\beta^2 I + \beta (1 - \alpha + \beta) b > 0.$$ 

Hence, illiquidity precedes default—a bankruptcy bank is also illiquid.

### 9.4 DD’s comparative static to structural parameters

We take into account that $V_B$ is an endogenous variable that depends on the structural parameters. From equation’s (7) approximation, where $b = b(V_B)$,

$$d (a^2) \approx d \left( \frac{E}{b} \right),$$ 

and $d ()$ denotes differential. We can derive the DD’s static comparative from either the left- or the right-hand-side. We assume that $\sigma(V_B) = \sigma \times V_B$.

**The default boundary, $V_B$**  The default point is a key input but, like DD, is not observable and endogenous. It follows that

$$d \left( \frac{V - V_B}{\sigma V_B} \right) = \frac{-V}{\sigma V_B^2} dV_B < 0,$$

and if the exercise boundary goes up (i.e., $dV_B > 0$), DD lowers and default-risk rises.

If the exercise boundary $V_B(x)$ depends on a structural parameter $x$ ($x \neq \sigma$), $dV_B(x) = \frac{\partial V}{\partial x} dx$ and $\frac{\partial V}{\partial x}$ determines default-risk. That is,

$$d \left( \frac{V - V_B}{\sigma V_B} \right) = \frac{-V}{\sigma V_B^2} \frac{\partial V_B}{\partial x} \Rightarrow \text{sign} \left( \frac{da}{dx} \right) = -\text{sign} \left( \frac{\partial V_B}{\partial x} \right).$$
On the other hand, where \( b = b(x, V_B(x)) \),

\[
\frac{d (E/b)}{dx} = \frac{bE_x - E \frac{db}{dx}}{b} = \frac{bE_x - E \left( b_x + \frac{\partial b}{\partial V_B} \frac{\partial V_B}{\partial x} \right)}{b},
\]

From the approximation \( d \left( \frac{V - V_B}{\sigma V_B} \right)^2 /dx \approx d \left( \frac{E}{b} \right) /dx),

\[
-2a \frac{V}{\sigma V_B^2} \frac{\partial V_B}{\partial x} \approx \frac{bE_x - E \left( b_x + \frac{\partial b}{\partial V_B} \frac{\partial V_B}{\partial x} \right)}{b^2}, \text{ and}
\]

\[
\frac{\partial V_B}{\partial x} \approx \left( \frac{-\frac{\partial b}{\partial V_B} E}{b^2} + 2a \frac{V}{\sigma V_B^2} \right)^{-1} \times \frac{E}{b} \times \left( \frac{b_x}{b} - \frac{E_x}{E} \right).
\]

**Corollary** Let \( \frac{\partial b}{\partial V_B} \leq 0 \), which is reasonable because \( b \) is given by negative net cash-flows and \( V \) are the firm’s assets or earnings (e.g., Leland and Leland–Toft models). In particular, if we consider \( b(x, V_I) \) (instead of \( b(x, V_B(x)) \)), \( \frac{\partial b}{\partial V_B} = 0 \). Then, if \( x \neq \sigma \),

\[
\text{sign} \left( \frac{da}{dx} \right) = -\text{sign} \left( \frac{\partial V_B}{\partial x} \right) \approx -\text{sign} \left( \frac{b_x}{b} - \frac{E_x}{E} \right).
\]

The tradeoff between the elasticities of the negative net cash-flows and equity prices \((b \text{ and } E, \text{ respectively})\) determines the sensitivity of the optimal default boundary. If this tradeoff is positive (negative), the exercise boundary rises (lowers)—and the DD lowers (rises). This result is convenient because \( V_B \) is endogenous.

Two cases. (1) if \( b_x = 0 \) (e.g., \( x \) is interest rates),

\[
\text{sign} \left( \frac{da}{dx} \right) = -\text{sign} \left( \frac{\partial V_B}{\partial x} \right) \approx \text{sign} \left( E_x \right),
\]

and the equity price determines DD by itself.

And (2), if \( \text{sign}(b_x) = -\text{sign}(E_x), \text{ e.g., in the case of cash-flow parameters,} \)

\[
\text{sign} \left( \frac{da}{dx} \right) = -\text{sign} \left( \frac{\partial V_B}{\partial x} \right) \approx -\text{sign} \left( b_x \right),
\]

which is also the intuition of the negative marginal rate of substitution between \( a \) and \( b \).

With regard to volatility, \( x = \sigma \),

\[
\frac{d \left( \frac{V - V_B}{\sigma V_B} \right)}{d \sigma} = -VV_B \frac{1 + \frac{\partial \ln V_B}{\partial \ln \sigma}}{(\sigma V_B)^2} + \frac{1}{\sigma^2} = -VV_B \left( 1 + \frac{\partial \ln V_B}{\partial \ln \sigma} - \frac{V_B}{V} \right).
\]

Default-risk depends on the boundary–volatility elasticity, which is negative in Leland-type models \( \frac{\partial V_B}{\partial \sigma} < 0 \). Two clear cases, close to and far away from default, respectively,

if \( V \to V_B \), \( \text{sign} \left( \frac{da}{d \sigma} \right) = -\text{sign} \left( \frac{\partial \ln V_B}{\partial \ln \sigma} \right) > 0 \), and

if \( V \to \infty \), \( \text{sign} \left( \frac{da}{d \sigma} \right) = -\text{sign} \left( 1 + \frac{\partial \ln V_B}{\partial \ln \sigma} \right) \).
where for $\sigma$ sufficiently large, $-1 < \frac{\partial \ln V_B}{\partial \ln \sigma} < 0$ and $d \left( \frac{V - V_B}{\sigma V_B} \right) / d\sigma < 0$. Note how we address both cases $\sigma_E$ and $\sigma_V$ separately, equity volatility and asset volatility, respectively.\(^{34}\)

**Two companies with the same value of equity, $E$** We apply equation (12), $da^2/db < 0$. From $db = \frac{db}{dx} dx$, and $\frac{db}{dx}$ determines default-risk for two firms with the same equity price. For any parameter $x$ (including $x = \sigma$), we have that

$$\frac{db}{dx} = b_x + \frac{\partial b}{\partial V_B} \frac{\partial V_B}{\partial x},$$

but sign($\frac{db}{dx}$) is not easy to pin down.

Let $\frac{\partial b}{\partial V_B} < 0$ as Leland-Toft model. If $b_x = 0$ (e.g., interest-rates or volatility),

$$\text{sign} \left( \frac{db}{dx} \right) = -\text{sign} \left( \frac{\partial V_B}{\partial x} \right).$$

In the case of assets volatility, if $\frac{\partial V_B}{\partial \sigma} < 0$, then $\frac{db}{d\sigma} > 0$; i.e., the firm with larger volatility is closer to default for the same equity prices. This result is relevant as we could not determine if $\frac{\partial \ln V_B}{\partial \ln \sigma} < -1$.

In Leland model, where $\frac{\partial b}{\partial V_B} = 0$ (if the dividend-yield is zero), it follows that

$$\frac{db}{d\sigma} = b_\sigma = 0.$$

Consequently, two zero-dividends Leland firms, which have the same equity price, have a similar DD although they have different volatility. This is another example where equity prices determine default-risk. In Leland-Toft this does not necessarily hold; the cost of roll-over debt depends on $V$ (and ultimately on $V_B$).

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\(^{34}\)For other parameters or variables, $V$ and $E$, the static comparative is as follows, respectively,

$$d \left( \frac{V - V_B}{\sigma V_B} \right) = \frac{1}{\sigma V_B} > 0 \quad \text{and} \quad \frac{d \left( \frac{V - V_B}{\sigma V_B} \right)^2}{dE} \approx \frac{d(E/b)}{dE} = \frac{1}{b} > 0.$$