A Structural Approach for Predicting Default Correlation

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Abstract

In this paper we investigate default correlation using structural frameworks based on the observable equity data. We first derive the Merton default correlation model based on the Merton credit risk model (1974) and next extend the Leland-Toft term structure model (1996) into a two-firm environment and derive the LT default correlation model. The empirical results suggest that the Merton default correlation model does a fine job of predicting default correlations between higher rated bonds over shorter time horizons. However, the LT default correlation model predicts default correlations more accurately for lower rated bonds over longer time horizons. In order to properly predict default correlations between different credit ratings over different time horizons, we recommend combining the two models in practice.

Keywords: Default correlation; defaultable bonds; structural model
JEL classification: G1, G2
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1. Introduction

Default correlation measures the extent to which two firms become insolvent simultaneously within a given time frame. When there are multiple risky bonds in a portfolio, a single default correlation (between two firms or two assets) can be extended to a default correlation matrix, which could cause massive defaults. The financial crisis of 2008 demonstrated how default correlation among underlying assets for asset backed securities led to massive defaults, resulting in billions of losses among financial institutions and investors. The severe consequence of the financial crisis called attention to better understand how to model and estimate default correlation. In this paper, we develop a consistent structural approach to directly predict default correlation based on observed equity correlation. To our knowledge, our work is one of the earliest attempts in the literature to apply the nature of the structural framework that allows default probability, debt and equity values to be simultaneously determined in a unified framework to predict default correlation from equity correlation in a cohesive structural model.

In the literature, the cumulative default probability of a single bond is commonly modeled by the structural framework. This type of models are pioneered by Merton (1974) and extended by others (see, for example, Black and Cox 1976; Ingersoll, 1977; Leland, 1994; Longstaff and Schwartz, 1995). Merton assumes that the evolution of firm asset value follows a geometric Brownian motion, the default boundary is the face value of debt, and default can only happen at debt maturity. To allow for default before debt maturity, Black and Cox (1976) introduce the first-passage-time model that specifies
default as an event of the first time that the firm’s asset value hits the default boundary. Merton’s assumption that the default boundary equals the face value of debt is also relaxed by Black and Cox (1976) and others. The boundary can be exogenously specified as a covenant to protect bondholders’ interests (see Black and Cox, 1976; Longstaff and Schwartz, 1995), or can be determined endogenously as a threshold at which stockholders maximize the equity value at default (see Leland, 1994; Leland and Toft, 1996).

Because the major components in the structural model are asset, debt, equity, and default boundary, any dependence of one component on another can induce default correlation. One can therefore differentiate default correlation models by their correlation channels. Earlier studies focus on the correlation between firms’ assets (Zhou, 2001; Frey et al. 2001). Asset values are treated as a function of common factors and firm-specific factors (see Finger, 1999; Frey et al., 2001) where common factors dictate asset return correlation between firms. More recent studies introduce the correlation between firms’ default boundaries in addition to that between firms’ assets to account for contagious credit risk effects. Giesecke (2003, 2006) assumes that each time a firm defaults, the true level of its default boundary is revealed and investors use this new information to update their beliefs about the default boundaries of other firms. To model the dependence of firms’ default boundaries, Giesecke employs copulas to link probability distributions of individual firm default boundaries to a joint distribution function of default boundaries.

However, a difficulty of implementing existing structural default correlation models is that asset value process or default boundary is unobservable. Zhou (2001) gets around this difficulty by assuming equity correlation equal to asset correlation. As
pointed out by de Servigny and Renault (2004) and Liu et al. (2013), this assumption is not appropriate. de Servigny and Renault (2004) empirically show that default correlations estimated based on equity correlations that proxy for asset correlations is weakly related to historical default correlations. Liu et al. (2013) demonstrate that as a firm’s leverage ratio goes up, equity correlation and model implied asset correlation deviate substantially. To tackle the problem with Zhou’s model, they propose a hybrid approach to apply the Leland and Toft (1996) model (hereafter, the LT model) to estimate asset correlation based on equity correlation and then input the model implied asset correlation to Zhou’s (2001) default correlation. However, their hybrid model is still limited by the assumption of constant leverage ratio in Zhou’s model and potential inconsistency between the LT model and Zhou’s model.

In this paper, we advance structural default correlation modeling by constructing an entirely self-consistent framework based on the Merton (1974) and the LT (1996) models for single firms, respectively. The advantages of our Merton default correlation model include easy implementation due to the closed-form solution and good prediction of default correlations between higher quality bonds. However, this default correlation model cannot capture the time trend of default correlation or fit default correlation between lower quality bonds well. In contrast, our LT default correlation model can capture the time trend of default correlation and predict default correlation related to lower quality bonds well. More importantly, the LT default correlation model is more realistic in recognizing that firms dynamically optimize their leverage ratio instead of sticking to a constant number given exogenously as in the Merton default correlation model. However, the implementation of this model requires cumbersome computation.
After comparing the estimated default correlations by our Merton and LT default correlation models with that by Zhou’s (2001) model and empirical default correlations, we find that our two default correlation models outperform Zhou’s model in more accurately predicting default correlations compared with the benchmark - empirical default correlations. Based on the cost and benefit analysis, we recommend to use the Merton default correlation model for higher quality bonds over shorter time horizon and the LT default correlation model for lower quality bonds and longer time horizon.

The remainder of the paper is organized as follows. Section 2 develops the Merton default correlation model and discusses model predicted default correlations. Section 3 develops the LT default correlation model and also discusses model predicted default correlations. Finally, Section 4 concludes the paper.

2. The Merton Default Correlation Model

2.1. The model derivation

Following Merton (1974), we assume that equity represents a European call option on the firm’s asset with the strike price equal to the debt face value. Denote $V_1$ and $V_2$ as total assets of firms 1 and 2, respectively, and the dynamics of $V_1$ and $V_2$ are given by the following stochastic process:

$$d \ln V = \mu dt + \Sigma dW_t$$  \hspace{1cm} (1)

where $\ln V = (\ln V_1, \ln V_2)'$, $\mu = (\mu_1, \mu_2)'$, and $W_t = (W_{1,t}, W_{2,t})'$ are column vectors, $\mu_1$ and $\mu_2$ are instantaneous expected rates of return of firms per unit of time, and $W_{1,t}$ and $W_{2,t}$ are independent standard Wiener processes with volatilities $\sigma_1$ and $\sigma_2$, respectively.
Equity values $S_1$ and $S_2$ are call options on firm assets with maturity $T$ and strike price $B_i$. That is,

$$S_i(V_i,B,t) = e^{-r(T-t)}E(V_i(T) - B_i)^+ = V_i(t)N(d_{i,1}) - e^{-r(T-t)}B_iN(d_{i,2})$$ \hspace{1cm} (2)

for $i = 1, 2$, where $B_i$ is the face value of firm $i$’s debt and

$$d_{i,1} = \frac{\ln \frac{V_i(t)}{B_i} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$ \hspace{1cm} (3a)

$$d_{i,2} = d_{i,1} - \sigma \sqrt{T-t} \hspace{1cm} (3b)$$

If the firm’s asset value $(V_{i,T})$ at maturity is greater than the face value of debt $(B_i)$, the firm does not default and shareholders receive $V_{i,T} - B_i$. However, if $V_{i,T} < B_i$, the firm defaults on its debts, and debtholders take control of the firm.

We can derive equity return correlation (hereafter equity correlation) given the correlation between two firms’ asset returns. The correlation coefficient of equity returns $ho_S$ (see Appendix B) is defined as

$$\rho_S = \frac{\text{cov}(R_i, R_j)}{\sqrt{\text{var}(R_i) \text{var}(R_j)}} \hspace{1cm} (4)$$

The average return of equity over time period $T-t$ is defined as\(^1\)

$$R_i = \frac{1}{T-t} \ln \frac{S_i(V,B,t)}{S_i(V,B,T)} = \frac{1}{T-t}[\ln S_i(V,B,T) - \ln S_i(V,B,t)]$$ \hspace{1cm} (5)

and the correlation coefficient between two equities is

$$\rho_S = \frac{\text{cov}(\ln S_i(V,B,T), \ln S_j(V,B,T))}{\sqrt{\text{var}(\ln S_i(V,B,T)) \text{var}(\ln S_j(V,B,T))}} \hspace{1cm} (6)$$

\(^1\) The instantaneous return cannot be used here. The instantaneous return does not reflect the potential default of debt.
The computations of variance and covariance of equity returns are given in Appendix A.

Default correlation can be expressed as

\[
\rho_D = \frac{E(I_{[V_{i_2}>B_1, V_{j_2}>B_2]} ) - E(I_{[V_{i_2}>B_1]} )E(I_{[V_{j_2}>B_2]} )}{\sqrt{\text{Var}(I_{[V_{i_2}>B_1]} ) \text{Var}(I_{[V_{j_2}>B_2]} )}}
\]  

(7)

where \( I_{[u]} \) is an indicator function with a value equal to one if \( u \) is true, and zero otherwise. The option pricing model shows that

\[ E(I_{[V_{i,x}>B_1]} ) = N(d_{x,2}) \]

The variance \( \text{Var}(I_{[V_{i,y}>B_1]} ) \) of zero-one distribution is given by

\[ \text{Var}(I_{[V_{i,y}>B_1]} ) = E(I_{[V_{i,y}>B_1]} )[1 - E(I_{[V_{i,y}>B_1]} )] \]

(8)

Finally, the expectation, \( E(I_{[V_{i_2,j_2}>B_1]} ) \), can be expressed in terms of the bivariate normal distribution function, \( \Psi \), (see Appendix B).

\[
E(I_{[V_{i_2,j_2}>B_1]} ) = \Psi(\zeta_1 + \rho \zeta_2 + \frac{m_1 - \ln B_1}{\zeta_1}, \rho \zeta_1 + \zeta_2 + \frac{m_2 - \ln B_2}{\zeta_2})
\]

Equations (4) and (7) indicate that both equity correlation (\( \rho_S \)) and default correlation (\( \rho_D \)) are functions of asset correlation. Given equity correlation \( \rho_S \), we can numerically solve for asset correlation \( \rho \), and then substitute \( \rho \) into (7) to determine default correlation \( \rho_D \). Therefore, we establish the relation between equity correlation and default correlation directly without the information for asset correlation. That is, default correlation can be directly computed from equity correlation without \textit{a priori} knowledge of asset correlation. In the following, we implement this procedure and present the empirical results.
2.2. Predictions of the Merton default correlation model

To examine the performance of the Merton default correlation model to predict default correlations, we use equity data to infer default correlation over the sample periods of 1970-1993 and 1990-2010, respectively. Equity correlation and volatility are estimated using monthly stock return data over the two sample periods.\(^2\)

Table 1 lays out the empirical default correlations for time horizons of five and ten years over the sample periods of 1970 to 1993 and 1990 to 2010, respectively. The empirical default correlations are used as the benchmark to examine the performance of our proposed Merton and LT default correlation models versus Zhou’s (2001) default correlation model. The empirical default correlations for five and ten years over the sample period of 1970 to 1993 are obtained from Lucas (1995) and that for ten years over the sample period of 1990 to 2010 are obtained from Liu et al. (2013). We use Moody’s Corporate Default Risk Service database by following Lucas (1995) approach to estimate the empirical default correlations for five years over the sample period of 1990 to 2010 due to unavailability of the data in any existing work.

Table 3 reports the predictions of default correlations by the Merton default correlation model using stock return data. We find that the Merton model generates a pattern of default correlation similar to that shown by the empirical default correlations. First, default correlations become stronger as ratings of both bonds decline. For example, the predicted default correlation between two Aa-rated bonds for five years over the two

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\(^2\) Equity correlations for two sample periods are obtained from Liu et al. (2013). The reason to choose these two sample periods is that the empirical default correlations estimated by Lucas (2005) and Liu et al. (2013), the benchmark for our model performance, span these two sample periods. To estimate equity volatility, we first calculate equal-weighted stock returns in a particular rating group for each month over the two sample periods, respectively. Next, we calculate the standard deviations of the equal-weighted stock returns over the two sample periods, respectively.
sample periods is 0, while that between two B-rated bonds increases to 14.7% and 11.7%, respectively. Second, given the rating for one bond, default correlation generally increases as the rating for another bond declines. For example, the predicted default correlation between two A-rated bonds for five years over both sample periods is 0.5% but increases to 2.2% and 1.5% between A- and B-rated bonds over the two sample periods, respectively.

However, the Merton model tends to underestimate default correlations for lower grade bonds. For example, the predicted default correlation between two B-rated bonds for ten years over the sample periods of 1970 to 1993 and 1990 to 2010 is 16.4% and 13.2%, respectively. However, the empirical default correlations for the two sample periods are 38% and 32%, respectively. Moreover, the Merton default correlation model cannot capture the effect of time horizon on default correlation well. For example, when the time horizon is five years, the Merton model predicts that default correlation between B-rated bonds is 14.7% and 11.7% over the two sample periods. As the time horizon lengthens to ten years, the predicted B-rated default correlation only moderately increases to 16.4% and 13.2%, respectively. In contrast, the empirical default correlations show significant increases as time horizon gets longer. The B-rated empirical default correlations for five years are 29% and 19.1% over the two sample periods, but that increase to 38% and 32% for ten years.

As we compare the performance of our proposed Merton default correlation model with that of Zhou’s (2001) default correlation model that is reported in Table 2, we find that both models can do a fine job of predicting default correlations between higher rated bonds over shorter time horizons. In addition, both models have closed-form
solutions and can be implemented easily. For some specific cases, our Merton model outperforms Zhou’s model. For example, the default correlations between Aa-rated bonds over the sample period of 1970 to 1993 for five and ten years are 0 and 1.4% by our Merton model versus 0 and 1% by the empirical estimates. However, the estimates by Zhou’s model are 0.1% and 1.9%. As like the Merton model, Zhou’s model much underestimates default correlations between lower rated bonds. Also, Zhou’s model cannot well capture the effect of time horizon on default correlation either. Nevertheless, our Merton model has an advantage over Zhou’s model in that the Merton default correlation model coherently links equity correlation to asset correlation and then infers default correlation. Instead, Zhou’s model roughly assumes that asset correlation is equal to equity correlation.

We think that a possible reason for the underestimation of default correlation by the Merton default correlation model is that the model allows the asset process of a “bankrupt” firm to continue evolving and to become solvent again over time before debt maturity with finite chances. This setup could lead to an increasing divergence between predicted and realized default correlations over longer time horizon. To fix this problem, we consider the first-passage-time model that allows bankruptcy to occur once a firm’s assets hit its default boundary and prohibits the firm from resurrection.³ Theoretically speaking, the first-passage-time model can potentially improve predictions of default correlations than the Merton model. In the following section, we propose a LT default correlation model to estimate default correlation using the first-passage-time method to permit default before maturity.

³ In some cases, bankrupt firms do come back after bankruptcy reorganization. However, after the reorganization, the firm’s capital structure differs from that before bankruptcy and the asset process differs too.
3. The LT Default Correlation Model

3.1. The model derivation

We extend the LT (1996) model, originally for a single firm, to a two-firm environment to predict default correlation from the equity correlation. Compared with the Merton and Zhou’s default correlation models, the LT model not only considers corporate taxes and bankruptcy cost, but also endogenously optimizes firm leverage that is treated as a critical indicator of credit risk. In contrast, the Merton and Zhou’s models simply assume leverage as a constant.

The LT model assumes the firm’s asset process \( V \) to follow a drifting geometric diffusion process as follows:

\[
\frac{dV}{V} = \left[ \mu(V,t) - \delta \right] dt + \sigma dZ, \tag{9}
\]

where \( \mu(V,t) \) is the expected rate of return on the firm’s assets, \( \delta \) is the payout, which is the proportion of the firm value paid to all security holders, \( \sigma \) is the constant volatility of asset returns, and \( Z \) is a standard Wiener process. The asset value \( V \) includes the net cash flows generated by the firm’s activities.

Suppose that there is an identical but levered firm issuing a risky debt \( d \) per unit time with \( t \) periods to maturity, a continuous constant coupon flow \( c(t) \) and a principal \( p(t) \). The firm remains solvent until the asset value \( V \) hits a default boundary \( V_B \). Upon bankruptcy, bondholders receive a fraction \( \chi = (1 - \beta) \) of the asset value \( V_B \), where \( \beta \) is the bankruptcy cost ratio and \( \beta V_B \) is loss due to bankruptcy. Further, we assume that \( r \) represents the continuous interest rate paid by a default-free asset and that investors follow a buy-and-hold investment strategy. Under the risk-neutral valuation, it can be shown that the value of the debt, \( d \), is given by
\[ d(V, V_B, t) = \frac{c(t)}{r} + e^{-\nu} \left[ p(t) - \frac{c(t)}{r} \right] (1 - F(t)) + \left[ \chi V_B - \frac{c(t)}{r} \right] G(t), \]  

(10)

where \( F(t) \) and \( G(t) \) are given in Leland and Toft (1996). The total outstanding debt \( D \) is the integration of the debt flow \( d(V, V_B, t) \) over \( T \), the maturity of newly issued debt:

\[ D(V, V_B, T) = \int_{t=0}^{T} d(V, V_B, t) \, dt. \]  

(11)

The integral can be carried out numerically. The tradeoff between the benefit of tax shields and bankruptcy costs suggests that there exists an endogenously determined bankruptcy threshold \( V_B \) that maximizes firm value. The equity value, as a function of default boundary \( V_B \) and asset value \( V \), is given by,

\[ S(V, V_B, T) = V + \tau_c \frac{C}{r} \left[ 1 - \left( \frac{V_B}{V} \right)^{a+z} \right] - \beta V_B \left( \frac{V_B}{V} \right)^{a+z} - D(V, V_B, T), \]  

(12)

where \( C \) is the annual coupon payment, and \( \tau_c \) is the corporate income tax rate. Parameters \( a \) and \( z \) are functions of asset volatility \( \sigma \) and interest rate \( r \).4

Equation (12) establishes a link between asset process \( V \) and equity process \( S \). If asset value \( V(t) \) follows a geometric Brownian motion, equity value \( S(V) \) as a function of \( V(t) \) will also exhibit a similar stochastic process.

To extend the LT model into a two-firm setting, we consider two firms with asset values \( V_1 \) and \( V_2 \). The dynamics of \( V_1, V_2, S(V_1), \) and \( S(V_2) \) are specified by Equations (9) to (12). Asset returns \( \Delta \ln V_1 \) and \( \Delta \ln V_2 \) are correlated with a coefficient \( \rho \), and have volatilities \( \sigma_1 \) and \( \sigma_2 \). We define equity correlation \( \rho_S \) as

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4 Detailed derivations of Equations (10) and (12) are given by Leland and Toft (1996).
\[ \rho_s = \frac{\text{cov}(\Delta \ln S(V_1), \Delta \ln S(V_2))}{\sqrt{\text{var}(\Delta \ln S(V_1)) \text{var}(\Delta \ln S(V_2))}}. \]  

(13)

A correlation between two asset processes, \( V_1 \) and \( V_2 \), will undoubtedly lead to a correlation between two corresponding equity processes, \( S(V_1) \) and \( S(V_2) \). However, these two correlations can diverge significantly because as time evolves, both leverage ratio and asset volatility change. In general, when a firm uses debt, equity and debt do not necessarily move in tandem. This can be seen from the option perspective where equity can be viewed as a call option on a firm’s underlying assets, and debt as a shorted put plus a constant. When asset volatility increases, the equity value tends to go up while the debt value can move in an opposite direction. When corporate taxes are considered, the debt-equity movements can be further distorted.\(^5\) This is why substituting equity correlation \( \rho_s \) for asset correlation \( \rho \) can be problematic, especially for lower ratings.

3.2. The model implementation

The first step to implement the extended LT model is model calibration. This is because a key variable in the model is asset volatility \( \sigma \) that is unobservable. The way to pin down its value is to try different possible values until the model-generated default probability matches the observed default rate. This procedure is well explained in papers such as Huang and Huang (2012). Table 4 lists the cumulative default rate for the sample periods of 1970 to 1993 and 1990 to 2010. To minimize estimation error, in the actual

\(^5\) We note that the extended LT model considers corporate taxes while the Merton and Zhou’s (2001) default correlation models do not. The effect of corporate taxes on debt and equity is not linear because of differential treatments for capital gains tax rate, ordinary income tax rate, and tax rebate due to default losses.
calibration, we choose an asset volatility $\sigma$ that minimizes the squared difference of log odds between the implied and observed default probabilities,

$$\sigma = \arg \min_{\sigma \in \mathbb{R}^+} \sum_{i=1}^{10} \left[ \ln P_i(\sigma) - \ln \overline{P}_i \right]^2,$$

(14)

where $P_i$ is the model-implied default probability by year $i$, and $\overline{P}_i$ is the corresponding observed default rate. Since we input equity premiums into the model, these probabilities are physical probabilities. The model-implied asset volatility $\sigma$ for the various bond ratings are reported in Panels A and B of Table 4. To value debt and equity with the model, we return to the risk-neutral measure by retaining asset return volatility $\sigma$ and forcing equity premium to zero as required by the equivalent martingale measure approach. We choose interest rate $r = 8\%$ and payout $\delta = 6\%$ for 1970-1993, which are in line with Huang and Huang (2012). For 1990-2010, we choose $r = 4\%$ and $\delta = 2\%$, which are the averages for the S&P 500 companies over the same time period of our data. Corporate tax rate $\tau_c$ is set at 35% and the bankruptcy ratio $\beta$ is set at 20% of the going concern value right before the default based on the estimates in Andrade and Kaplan (1995).

In Monte Carlo simulations of our model, for each iteration we generate a time-series sample path according to Equation (9) with the starting asset value $V_i(0)$ normalized to 100, where $i = 1, 2$ denoting the two firms. For each random movement in $V_i(0)$ at time $t$, we apply Equations (10) to (12) to obtain debt value $D_i(t)$ and equity value $S_i(t)$. For the next random movement in $V_i(t + \Delta t)$ at time $t + \Delta t$, we again apply the model to obtain $D_i(t + \Delta t)$ and $S_i(t + \Delta t)$ while keeping the coupon, principal and default boundary unchanged. This is to recognize the fact that the stationary capital structure of
the LT model rules out any debt restructuring after the optimization is done. The procedure is repeated until we reach the horizon at $t = T$. This allows us to map out one sample path. For a second iteration, the same procedure is repeated to generate another sample path of $V_i(t)$ (and thereby $D_i(t)$ and $S_i(t)$ as well) for each firm.

To permit correlation $\rho$ between the returns of two asset processes, we employ the following return dynamics:

$$
\begin{align*}
\Delta V_1 &= V_1 \left[ \frac{\mu_i}{n} + \frac{\sigma_i}{\sqrt{n}} \times \Delta Z_1 \right] \\
\Delta V_2 &= V_2 \left[ \frac{\mu_i}{n} + \frac{\rho \sigma_i}{\sqrt{n}} \times \Delta Z_1 + \frac{\sqrt{1 - \rho^2} \sigma_i}{\sqrt{n}} \times \Delta Z_2 \right],
\end{align*}
$$

(15)

where $n$ denotes the number of time intervals partitioned for each year and $\mu_i$ is the net drift rate. The random variables $\Delta Z_1$ and $\Delta Z_2$ follow two independent standard normal distributions. Volatility for each period is $\sigma_i / \sqrt{n}$ where $\sigma_i$ is the annualized asset return volatility for firm $i$. For example, when the time interval is in months, $n$ is set to 12 and $\sigma_i / \sqrt{12}$ is the monthly asset return volatility. In each simulation, $t$ is represented by the number of time steps within the period $[0, t]$. The convergence of Monte Carlo simulations can be achieved by a large number of iterations. For each rating pair (e.g., Baa and Ba firms), we generate 30,000 sample paths. Correlations of assets, equities and debts are calculated for each sample path and then we take their averages over all the sample paths, respectively.
3.3. Predictions of the LT default correlation model

Table 5 reports the predicted default correlations based on our LT default correlation model over the two sample periods. Obviously, the LT default correlation model can capture the pattern of the empirical default correlations well. That is, as the time horizon becomes longer and the rating of bonds becomes lower, default correlation gets stronger. Compared with the Merton and Zhou’s default correlation models, the LT default correlation model has great improvement in two ways. First, the LT model can capture the time effect on default correlation much better. If we use the previous example of the default correlation between B-rated bonds for five years versus ten years, the LT model much outperforms the Merton and Zhou’s models. The LT model predicts the five-year default correlation of 22% and 24.8% over the sample periods of 1970 to 1993 and 1990 to 2010, respectively. As the time horizon extends to ten years, the predicted default correlations increase to 39.3% and 35.5%, respectively. The model predicted results are pretty close to the B-rated empirical default correlations for five years of 29% and 19.1% over the two sample periods and 38% and 32% for ten years. Second, the LT model generally predicts default correlation between any rated bond and lower rated bonds much better. For example, the empirical 5-year default correlation is 3.6% for AA-B pair over the sample period of 1990 to 2010. The corresponding prediction by the Merton and Zhou’s models is 0.4% and 0.25%, while that by the LT model is exactly 3.6%. For B-B pair, the ten-year empirical benchmark is 38% and 32% over the two sample periods; The Merton model’s prediction is 16.4% and 13.2%, the Zhou’s model’s prediction is 13.7% and 11%, but the LT model predicts 39.3% and 35.5%. The evidence shows that the LT
default correlation model does a better job of predicting default correlations related to lower rated bonds.

However, the LT model could overestimate default correlation related to higher rated bonds. For example, the empirical default correlation between Aa- and A-rated bonds for five years over the sample period of 1970 to 1993 is 1%, but the predicted default correlation by the LT model is 4%.

To summarize, we think that the Merton default correlation model should be used to estimate default correlation between higher rated bonds over shorter horizon, and instead, the LT default correlation model should be used to estimate default correlation related to lower rated bonds over longer horizon. Obviously, a single Zhou’s (2001) model cannot perform the predication of default correlations between different rating bonds and over different time horizons well.

4. Conclusions

Default correlation is a critical component for risk management in areas such as fixed income portfolios, bank management, insurance industry, and working capital management and for the valuation of the credit derivatives underlying on a credit portfolio. Different types of structural default correlation models have been developed in the literature in which default correlation is modeled by correlated asset processes or correlated default boundaries. However, a big problem with implementing the models is that neither asset process nor default boundary is observable in the market. In this study, we propose a consistent structural framework for the estimation of default correlation that is built on the Merton credit risk model and the extended LT credit risk model. In our
proposed frameworks, we inherently link equity correlation to asset correlation and asset correlation to default correlation. As such, the observable equity correlation, instead of unobservable asset correlation or default boundary correlation, becomes the key input variable of our default correlation models.

We perform the empirical studies to examine the performance of our Merton and LT default correlation models versus Zhou’s (2001) default correlation model that assumes the equality of asset correlation with equity correlation. We first estimate default correlations based on the three models for five and ten years over the sample periods of 1970 to 1993 and 1990 to 2010, respectively. We next compare the model predicted default correlations with the corresponding empirical default correlations for the same time horizons over the same sample periods. The comparison shows that our Merton default correlation model and Zhou’s default correlation model can predict default correlations between higher quality bonds over shorter time periods well. However, both models much underestimate default correlations related to lower quality bonds. Also, both models cannot capture time trend of default correlations. In contrast, our LT default correlation model can do a good job of predicting default correlations related to lower quality bonds. Also, it can capture time trend of default correlations well. Nevertheless, our LT model generally overestimates default correlations between higher quality bonds.

Our results suggest that first, with only one basic correlated stochastic factor, i.e. the stochastic asset process, it is possible to work within the structural framework to properly model the correlated default risk and yield reasonable predictions compared to the empirical default correlation. This is in contrast with some other existing models that rely on more complex factors in order to boost the prediction. Second, a single default
correlation model, such as Zhou’s (2001) model, may not be enough to predict default correlations between different rating bonds over different time horizons. Finally, we recommend two models be used to estimate default correlations properly. Our Merton default correlation model that can be easily implemented is proper to estimate default correlations between higher quality bonds over shorter time periods. Our LT default correlation model that requires cumbersome computations is proper to estimate default correlations related to lower quality bonds over longer time horizon.
APPENDIX A1  Derivation of Asset Correlation from Equity Correlation for the Merton Default Correlation Model

In this appendix, we present the results needed to calculate asset correlation from equity correlation based on the Merton-type model. Merton (1974) models equity $S_i$ of firm $i$ as a call options of the firm’s assets $V_i$ on the face value of the firm debt $B_i$. The dynamic process of $V_i$ is defined in Equation (1) and the equity value is given by the Black-Scholes option pricing model:

$$ S_i(V,B,t) = e^{-(T-t)} E[(V_i(T) - B_i)^+] $$

$$ = e^{-(T-t)} E\left\{ V_i(t) \exp\left[ \left( r - \frac{\sigma^2}{2} \right)(T-t) + Z_i(T) - Z_i(t) \right] - B_i \right\}^+ $$

$$ = V_i(t) N(d_{i,1}) - e^{-(T-t)} B_i N(d_{i,2}), $$

where $Z_i(T) - Z_i(t)$ is normally distributed, $r$ is risk free interest rate, $T$ is maturity, $E$ is the expectation under the risk-neutral probability, and

$$ d_{i,1} = \frac{\ln \frac{V_i(t)}{B_i} + \left( r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} $$

$$ d_{i,2} = d_{i,1} - \sigma \sqrt{T-t} $$

The correlation coefficient of equity returns $R_i$ is

$$ \rho_S = \frac{\text{cov}(R_i, R_j)}{\sigma_{S,i} \sigma_{S,j}} $$

The continuous compounding return of equity is defined as

$$ R_i = \frac{1}{T-t} \ln \frac{S_i(V,B,T)}{S_i(V,B,t)} = \frac{1}{T-t} \left[ \ln S_i(V,B,T) - \ln S_i(V,B,t) \right] $$

under the condition $V_i(T) > B$.

The equity return correlation coefficient is defined as
\[ \rho_S = \frac{\text{cov}(R_i, R_j)}{\sqrt{\text{var}(R_i) \cdot \text{var}(R_j)}} \]

\[ = \frac{\text{cov}(\ln S_j(V, B, T), \ln S_j(V, B, T))}{\sqrt{\text{var}(\ln S_j(V, B, T)) \cdot \text{var}(\ln S_j(V, B, T))}} \]

where

\[ \text{var}(\ln S_j(V, V_B, T)) = E[(\ln S_j(V, V_B, T))^2] - [E(\ln S_j(V, V_B, T))]^2 \]

and

\[ \text{cov}(\ln S_j(V, V_B, T), \ln S_j(V, V_B, T)) = E[(\ln S_j(V, V_B, T) \times \ln S_j(V, V_B, T))] - E(\ln S_j(V, V_B, T))E(\ln S_j(V, V_B, T)) \]

The expectation and second moment can be computed as

\[ E(\ln S_j(V, V_B, T)) \]

\[ = E\left\{ \ln V_j(t) \exp\left[ r - \frac{\sigma^2}{2} \right] (T-t) + Z_j(T) - Z_j(t) - B_j \bigg| V_j(T) > B \text{ and } V_j(T) > B_j \right\} \]

\[ E[(\ln S_j(V, V_B, T))^2] \]

\[ = E\left\{ \ln V_j(t) \exp\left[ r - \frac{\sigma^2}{2} \right] (T-t) + Z_j(T) - Z_j(t) - B_j \bigg|^2 \bigg| V_j(T) > B \text{ and } V_j(T) > B_j \right\} \]

and

\[ E(\ln S_j(V, V_B, T) \times \ln S_j(V, V_B, T)) \]

\[ = E\left\{ \ln V_j(t) \exp\left[ r - \frac{\sigma^2}{2} \right] (T-t) + Z_j(T) - Z_j(t) - B_j \bigg| V_j(T) > B \text{ and } V_j(T) > B_j \right\} \]

\[ \ln V_j(t) \exp\left[ r - \frac{\sigma^2}{2} \right] (T-t) + Z_j(T) - Z_j(t) - B_j \bigg| V_j(T) > B \text{ and } V_j(T) > B_j \right\} \]
The condition $V_i(T) > B_i$, i.e. 

$$V_i(t) \exp \left( r - \frac{\sigma^2}{2} \right)(T - t) + Z_i(T) - Z_i(t) > B_i$$

equivalently

$$Z_i(T) - Z_i(t) > -\frac{\ln V_i(t)}{B_i} + \left( r - \frac{\sigma^2}{2} \right)(T - t)$$

where $Z_i(T) - Z_i(t)$ is normally distributed with standard deviation $\sigma \sqrt{T - t}$. Introducing standard normal distribution with zero mean and unit standard deviation

$$W = \frac{Z(T) - Z(t)}{\sigma \sqrt{T - t}}.$$

We can rewrite the expectations as

$$E(\ln S_j(V, V_B, T))$$

$$= \frac{1}{(T - t)^2} \int \int \ln \left( V_i(t) \exp \left( r - \frac{\sigma^2}{2} \right)(T - t) + \sigma \sqrt{(T - t)W_i} \right) - B_i \right)dW_i$$

$$E(\ln S_j(V, V_B, T)^2)$$

$$= \int \int \ln \left( V_i(t) \exp \left( r - \frac{\sigma^2}{2} \right)(T - t) + \sigma \sqrt{(T - t)W_i} \right) - B_i \right) \right)^2dW_i$$

and

$$E(\ln S_j(V, V_B, T) \times \ln S_j(V, V_B, T))$$

$$= \frac{1}{(T - t)^2} \int \int \ln \left( V_i(t) \exp \left( r - \frac{\sigma^2}{2} \right)(T - t) + \sigma \sqrt{(T - t)W_i} \right) - B_i \right) \right)$$

$$\ln \left( V_j(t) \exp \left( r - \frac{\sigma^2}{2} \right)(T - t) + \sigma \sqrt{(T - t)W_j} \right) - B_j \right) dW_i dW_j$$

The last calculation involves double integrals over $W_i$ and $W_j$ and $W_i$ and $W_j$ are correlated with coefficient $\rho$. Applying the change of variable theorem to the integral we
can have an integral over two uncorrelated variables and see the next appendix for the
details of changing variables.
APPENDIX B1 Derivation of Default Correlation from Asset Correlation for the Merton Default Correlation Model

Given the results in Appendix A, we can derive asset correlation from equity correlation for the Merton default correlation model. The equity value of firm $i$ is the call value on the underlying firm asset process $V_{i,t}$. At time $T$ the firm value is given by Equation (5).

Default correlation is defined as

$$
\rho_D = \frac{E\left( I_{\{V_{i,t} > B_i, V_{j,t} > B_j\}} \right) - E\left( I_{\{V_{i,t} > B_i, V_{j,t} = B_j\}} \right) E\left( I_{\{V_{i,t} > B_i\}} \right)}{\sqrt{\text{Var}\left( I_{\{V_{i,t} > B_i, V_{j,t} > B_j\}} \right) \text{Var}\left( I_{\{V_{i,t} > B_i\}} \right)}}
$$

The option pricing model shows that

$$
E\left( I_{\{V_{i,t} > B_i\}} \right) = N(d_{i,2})
$$

We need to compute the joint default probability. Since $\ln V$ is normally distributed (see Appendix A), we integrate the probability density function $\psi$ over $\{\ln(V_{i,t}) > \ln(B_i), \ln(V_{j,t}) > \ln(B_j)\}$

$$
E\left( I_{\{V_{i,t} > B_i, V_{j,t} > B_j\}} \right) = \int_{\ln(B_i)}^{\infty} \int_{\ln(B_j)}^{\infty} \psi(\ln(V_{i,t}) \ln(V_{i,t})) d\ln(V_{i,t}) d\ln(V_{j,t})
$$

For simplicity we introduce $\xi_i^2 = \sigma_i^2 T$ and $m_i (i = 1, 2)$, which are the variance and the expected value of $\ln V_i$ for firm $i$. Suppose that $\sigma_i \sigma_j \neq 0$ and $|\rho| < 1$, then we can write

$$
\psi(\ln V) = \frac{1}{2\pi \sqrt{\det \Sigma_T}} \exp \left[-\frac{1}{2} \left( (\ln V - m)^T \Sigma_T^{-1} (\ln V - m) \right) \right]
$$

(A13)

where

$$
m = (m_1, m_2)
$$
\[ \Sigma_T \cdot \Sigma_T' = \begin{bmatrix} \frac{\zeta^2}{\zeta_1} & \rho \zeta_1 \zeta_2 \\ \rho \zeta_1 \zeta_2 & \frac{\zeta^2}{\zeta_2} \end{bmatrix}. \]

By an affine transformation

\[ U = \ln V \cdot F + \nu \quad \text{(A14)} \]

where

\[ U = (U_1, U_2) \]
\[ F = \begin{bmatrix} 1 \\ \frac{\rho}{\zeta_1 \sqrt{1 - \rho^2}} \\ \frac{1}{\zeta_2 \sqrt{1 - \rho^2}} \end{bmatrix} \]
\[ \nu = (\nu_1, \nu_2) = -mF. \]

\[ U_1 \text{ and } U_2 \text{ are independent and identically distributed standard normal random variables and} \]

\[ \ln V = U \cdot F^{-1} + m. \]

The Jacobian of the transformation is

\[ J(U) = \left| \frac{\partial \ln V}{\partial U} \right| = \frac{1}{\zeta_1 \zeta_2 \sqrt{1 - \rho^2}}. \]

The region under the integration in (A13) is bounded in \((\ln V_1, \ln V_2)\) plane by

\[ V_1 = B_1, V_2 = B_2. \]

The image under the transformation in \((U_1, U_2)\) plane is bounded by

\[ U_1 = b_1 \]
\[ U_2 = -a_2 U_1 + b_2 \]

\[ b_1 = \frac{\ln B_1 - m_1}{\zeta_1}, \quad a_2 = \frac{\rho}{\sqrt{1 - \rho^2}}, \quad b_2 = \frac{\ln B_2 - m_2}{\zeta_2 \sqrt{1 - \rho^2}}. \]

The change of variable theorem gives
\[ E(I_{V_1 > b_{1,Y}, V_2 > b_{2,Y}}) = \int_{a_{U_1}}^{m} \int_{a_{U_2}}^{m} \phi(U_1) \phi(U_2) J(U_1) dU_1 dU_2 \]

where \( \phi \) is the standard normal probability density function. Alternatively, we can rewrite

\[ E(I_{V_1 > b_{1,Y}, V_2 > b_{2,Y}}) = \int_{a_{U_1}}^{m} \int_{a_{U_2}}^{m} \phi(U_1 - (\xi_1 + \rho \xi_2)) \phi(U_2 - \xi_2 \sqrt{1 - \rho^2}) dU_1 dU_2 \]

Integrating with respective to \( U_2 \) yields

\[ E(I_{V_1 > b_{1,Y}, V_2 > b_{2,Y}}) = \int_{a_{U_1}}^{m} \phi(U_1 - (\xi_1 + \rho \xi_2)) \phi(a_{U_1} - b_2 - \xi_2 \sqrt{1 - \rho^2}) dU_1 dU_2 \]

The integral can be expressed in terms of the bivariate normal distribution function, \( \Psi \), (Chuang, 1996)

\[ E(I_{V_1 > b_{1,Y}, V_2 > b_{2,Y}}) = \Psi \left( \xi_1 + \rho \xi_2 + \frac{m_1 - \ln B_1}{\xi_1}, \rho \xi_1 + \xi_2 + \frac{m_2 - \ln B_2}{\xi_2} \right) \]
References


Table 1

This table reports the empirical default correlation estimates for different credit ratings over time horizons of five and ten years. All estimates are in percentages. The empirical default correlations over the sample period of 1970 to 1993 are obtained from Lucas (1995) and that for ten years over the sample period of 1990 to 2010 are obtained from Liu et al. (2013). The five-year default correlations over the sample period of 1990 to 2010 are estimated based on Moody’s Corporate Default Risk Service database following the approach in Lucas (1995).

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<td>1 1 0</td>
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Panel B: Sample period 1990-2010

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<td>3.4 6.2 9.1 17.3</td>
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<td>4.1 10.8 11.4 20.2 32.0</td>
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Table 2
Default correlations predicted by Zhou’s (2001) default correlation model

This table reports the predicted five-year and ten-year default correlations for different credit ratings based on Zhou’s (2001) default correlation model over the sample periods of 1970 to 1993 and 1990 to 2010, respectively. All numbers are in percentages.

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<tr>
<td>B</td>
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Table 3
Default correlations predicted by the Merton default correlation model

This table reports the predicted five- and ten-year default correlations based on the Merton default correlation model over the sample periods of 1970 to 1993 and 1990 to 2010, respectively. The model inputs of equity correlation are obtained from Liu et al. (2013). All numbers are in percentages.

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<td>(T = 10 \text{ years})</td>
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<td>8.9  13.2</td>
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Table 4  
Calibration parameters for the LT default correlation model

This table reports the calibration parameters for the LT default correlation model. The historical cumulative default rates for the sample periods of 1970 to 1993 and 1990 to 2010 are from Fons (1994) and Liu et al. (2013), respectively. Equity premium for the sample period of 1970 to 1993 is from Bhandari (1988) and other parameters are from Liu et al. (2013). All numbers are in percentages.

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<td>3.7</td>
<td>10.7</td>
<td>28.4</td>
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Table 5
Default correlations predicted by the LT default correlation Model

This table reports the predicted five- and ten-year default correlations based on the LT default correlation model over the sample periods of 1970 to 1993 and 1990 to 2010, respectively. The original LT Model (1996) is extended to a two-firm environment to generate time series of asset and equity processes. The model is calibrated such that the model-implied default probability matches the observed historical cumulative default rates for various ratings, and the simulated equity process has the same correlation as the observed equity correlation (1970-1993 and 1990-2010). The model is iterated 30,000 times to ensure a desirable convergence of the mean value, i.e., a standard deviation $\sigma_{\rho_D} < 1\%$. With 30,000 sample paths of two firms, we are able to compute asset correlation $\rho$, default correlation $\rho_D$, and their means. This table shows the mean default correlation $\rho_D$ for various rating pairs when the iteration number is 30,000 in the Monte Carlo simulation. All numbers are in percentages.

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