From Stochastic Dominance to Black-Scholes: an Alternative Option Pricing Paradigm

By Ioan Mihai Oancea\(^1\) and Stylianos Perrakis\(^2\)

Abstract

This paper examines the relationship between option pricing models that use stochastic dominance concepts in discrete time, and the traditional arbitrage-based continuous time models. It derives multiperiod discrete time index option bounds based on stochastic dominance considerations for a risk-averse investor holding only the underlying asset, the riskless asset and (possibly) the option for any type of underlying asset distribution in which the market index is the single state variable. It then considers the limit behavior of these bounds as trading becomes progressively more frequent and the underlying asset tends to continuous time diffusion. It is shown that these bounds tend to the unique Black-Scholes-Merton option price. This result is extended to equity options by assuming a linear CAPM-type relationship between index and equity returns.

**Key words:** Stochastic dominance, option pricing, option bounds, incomplete markets, diffusion processes.

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\(^1\) University of Connecticut School of Business, One University Place, Stamford, CT 06901, (203) 987-6687, michael.oancea@business.uconn.edu.

\(^2\) RBC Distinguished Professor of Financial Derivatives, John Molson School of Business, Concordia University, 1455 de Maisonneuve Blvd. W, Montreal, QC H3G 1M8, Canada, Phone: (+1-514) 848-2424 ext. 2963, Fax: (+1-514) 848-4500, perrakis@jmsb.concordia.ca. Perrakis acknowledges the support of SSHRC, the Social Sciences and Humanities Research Council of Canada, and the RBC Distinguished Professorship of Financial Derivatives.
1. **Introduction**

The stochastic dominance (SD) approach to option pricing was introduced about 30 years ago in a series of articles that provided an alternative avenue of research to the then dominant arbitrage pricing of derivative assets.\(^3\) Although it did not attract much attention during the following years, it has more recently emerged as a counterpart to arbitrage in handling problems in an area where the latter paradigm is unable to derive useful results. Recall that arbitrage relies on two fundamental assumptions that cannot be relaxed easily in most applications. These are known as dynamic market completeness and frictionless trading. While there are extensions of the basic arbitrage methodology that can take care of several forms of market incompleteness, there is no satisfactory arbitrage-based approach to option pricing in the presence of transaction costs.\(^4\) By contrast SD also produces bounds in the presence of transaction costs, bounds that are independent of the number of trading periods, and have had important empirical applications that identify mispriced options whenever the bounds are violated.\(^5\)

The SD approach was developed piecemeal from the outset in a discrete time context, initially in frictionless markets and eventually extended to incorporate transaction costs. For this reason its relationship with the traditional option pricing methods that rely on arbitrage and replication in continuous time is unclear. For SD to emerge as an alternative paradigm in option pricing it must be shown that it produces the same results in the cases where arbitrage produces a single option price, the cases of complete frictionless markets. This demonstration forms the topic of this paper.

We formulate the SD approach in its most general form, deriving the discrete time SD bounds in the case of underlying asset returns that are Markovian but not necessarily independent and identically distributed (iid). We assume that the investor holds a portfolio consisting of the underlying asset and a riskless bond, as in the case where the underlying asset is a major market index. We then consider a general formulation of the continuous time limits of the asset return distribution that tends to state-dependent diffusion as the time partition tends to zero. We show that in this limiting process both SD bounds tend to the same limit, the Black-Scholes-Merton (1973) option price. We then extend the SD bounds to stock options, by relaxing the assumption that the investor holds only the index and assuming that the stock return consists of a term proportional to the index return and a random noise component. We find the same unique option price for this case as well. Thus, SD has all the advantages of arbitrage, while simultaneously being able to produce results in the presence of frictions, where the arbitrage approach fails.

In what follows we review the various extensions of the basic arbitrage option pricing model as they have been developed in the last thirty years to account for the cases of market incompleteness and frictions; since these are well known, only the most important studies will be mentioned in this brief review. At the end of this paper we indicate under what assumptions it is possible to extend the SD approach to these cases as well.

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\(^3\) This class of models was introduced by Perrakis and Ryan (1984) and extended by Levy (1985) and Ritchken (1985) in a single period context. The multiperiod extension was done in Perrakis (1986, 1988) and Ritchken and Kuo (1988).

\(^4\) See Merton (1989), and Soner et al (1996).

If there is absence of arbitrage then it can be shown that the price of any contingent claim on the underlying asset is equal to the discounted expected payoff of the claim under a risk neutral transformation of the underlying asset’s return distribution, generally denoted as the $Q$-distribution. Unfortunately this distribution is unique only in complete markets, which correspond to the binomial model and to a univariate diffusion process in discrete and continuous time respectively. The main sources of incompleteness in the financial literature are associated with bivariate diffusion models in which the volatility is a state variable together with the asset return, with jumps in the asset return and/or the volatility, or with a combination of these factors.

In the case of jump-diffusion asset return distributions arbitrage considerations are supplemented by market equilibrium conditions, in which a representative investor’s optimal portfolio choices define the $Q$-distribution. Unfortunately this distribution can only be defined for a specific type of investor utility function, the constant relative risk aversion class, and is a function of the risk aversion coefficient of the investor. This is an unobservable parameter that enters into the option price and whose value differs widely among various estimation exercises. On the other hand, the stochastic volatility models handle market incompleteness by specifying, somewhat arbitrarily, a “price” of the volatility risk in order to derive a risk neutral process, a form of the $Q$-distribution that can be used to price contingent claims.

The presence of frictions in option pricing is generally represented under the form of proportional transaction costs in trading the underlying asset. It has been known at least since Merton (1989) that in the presence of even a “small” but finite transaction cost rate the option replication strategy that underlies arbitrage in complete market models is only able to derive bounds on option prices that depend on the number of trading periods till option expiration. The bounds tend to the well-known Merton (1973) arbitrage bounds as the time periods increase, bounds that are too wide for any practical considerations. The impossibility of deriving useful results in a market with frictions even under simple diffusion asset dynamics was also confirmed theoretically.

Stochastic dominance is a concept almost as old as arbitrage, since it was initially developed almost simultaneously in two 1969 articles by Hadar and Russell, and Hanoch and Levy, that derived conditions for choices among investment prospects with uncertain outcomes. These conditions referred to pairwise comparisons of the terminal distributions of the risky prospects’ outcomes, in which one of the prospects was unanimously selected by all risk averse investors. It was introduced into option pricing by Perrakis and Ryan (1984) who did not use formally the SD concept but derived conditions on minimal and maximal admissible option prices for an investor who held the underlying asset and a riskless bond with and without the option. The theoretical framework was provided by incorporating SD concepts into an extension of the equilibrium option pricing models of Rubinstein (1976) and Brennan (1979); unlike these models, the SD

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8 Hull and White (1987) assume that the stochastic volatility risk is diversifiable (the price of volatility risk is zero); while Heston (1993) and Bates (1996) adopt particular forms of that price of risk. A more rigorous definition is in a recent article by Christoffersen, Heston and Jacobs (2012).
9 The discrete time approach within the binomial model or with periodic portfolio revisions was developed by Leland (1985), Boyle and Vorst (1992), and Bensaid et al (1992). For a discussion of the problems of this approach see Perrakis and Lefoll (2000, 2004). Soner, Shreve and Cvitanic (1996) demonstrated the impossibility in continuous time.
10 We limit ourselves to option pricing arising out of second order stochastic dominance considerations. Ritchken (1985) showed that first order dominance conditions derived essentially the same results as the arbitrage bounds of Merton (1973).
bounds are valid for any underlying asset distribution. Ritchken (1985) derived essentially the same results by a linear programming (LP) approach, as did Levy (1985), who used the traditional SD pairwise comparisons extending earlier results on an unrelated problem by Levy and Kroll (1978). The LP and the SD-equilibrium approaches were extended into a multiperiod discrete time context by Perrakis (1986, 1988) and Ritchken and Kuo (1988). It is the limiting form of these results as the number of partitions of the time to expiration of the option tends to infinity that forms the topic of this paper. More recently, stochastic dominance was extended to discrete time distributions that tend to jump-diffusion processes in continuous time, and to discrete time distributions that incorporate proportional transaction costs. All continuous time extensions of the SD approach rely on the proofs provided in this paper, that forms thus an indispensable stage in extending SD to becoming an alternative option pricing paradigm.

In the next section we summarize the existing SD results in the form of multiperiod recursive option pricing bounds. Section 3 examines the continuous time convergence in the case of index options, while Section 4 extends the results to equity options. Section 5 concludes.

II. The Option Pricing Model in Discrete Time

We consider a market with an underlying asset (initially identified as an index) with current price \( S_t \) and a riskless asset with return per period equal to \( R \). There is also a European call option with strike price \( K \) expiring at some future time \( T \). Time is initially assumed discrete \( t = 0, 1, \ldots, T \), with intervals of length \( \Delta t \), implying that \( R = e^{r\Delta t} - 1 = r\Delta t + o(\Delta t) \), where \( r \) denotes the interest rate in continuous time. In each interval the underlying asset has returns \( \frac{S_{t+\Delta t} - S_t}{S_t} \equiv z_{t+\Delta t} \), whose distribution may depend on \( S_t \).

Except for the trivial case where \( z_{t+\Delta t} \) takes only two values the market for the index is incomplete in a discrete time context. The valuation of an option in such a market cannot yield a unique price. Our market equilibrium is derived under the following set of assumptions that are sufficient for our results:

- There exists at least one utility-maximizing risk averse investor (the trader) in the economy who holds only the index and the riskless asset
- This particular investor is marginal in the option market
- The riskless rate is non-random

Each trader holds a portfolio of \( x_t \) in the riskless asset and \( y_t \) in the stock by maximizing recursively the expected utility of final wealth over the periods \( t = 0, 1, \ldots, T' \) of length \( \Delta t \), where \( T' \geq T \). The current value

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12 Although this assumption may not be justified in practice, its effect on option values is generally recognized as minor in short- and medium-lived options. It has been adopted without any exception in all equilibrium based jump-diffusion option valuation models that have appeared in the literature. See the comments in Bates (1991, p. 1039, note 30) and Amin and Ng (1993, p. 891). In order to evaluate the various features of option pricing models, Bakshi, Cao and Chen (1997) applied without deriving it a risk-neutral model featuring stochastic interest rate, stochastic volatility and jumps. They found that stochastic interest rates offer no goodness of fit improvement.
13 The results are unchanged if the traders are assumed to maximize the expected utility of the consumption stream.
function is \( \Omega(x_t + y_t \mid S_t) = \max_{y_t} E[\Omega((x_t - v_t)(1 + R) + (y_t + v_t)(1 + z_{t+\Delta t}) \mid S_t] \), where \( v_t \) denotes the optimal portfolio revision or stock purchase from the riskless account. The first order conditions of this maximization yield the pricing kernel \( Y(z_{t+\Delta t}) \), the state-contingent discount factor or normalized marginal rate of substitution of the trader evaluated at her optimal portfolio choice. Assuming no transaction costs and no taxes, the following relations characterize market equilibrium in any single trading period \((t, t+\Delta t)\),

\[
E[Y(z_{t+\Delta t}) \mid S_t] = (1 + R)^{-1}, \quad E[(1 + z_{t+\Delta t}) Y(z_{t+\Delta t}) \mid S_t] = 1.
\]  

(2.1)

Because of the assumed risk aversion and portfolio composition of our traders it can be easily seen that the pricing kernel \( Y(z_{t+\Delta t}) \) would be monotone (either non-increasing or non-decreasing) in the stock return \( z_{t+\Delta t} \) for every \( t = 0, 1, \ldots, T \).

This property is sufficient for the derivation of tight option bounds for all stock return distributions. It can be easily seen from the second relation in (2.1) that under such an assumption \( Y(z_{t+\Delta t}) \) must be non-increasing if the optioned stock is a “positive beta” one, with expected return exceeding the riskless rate, since this implies that the trader will always hold a positive amount of the stock. Since this is the case for the overwhelming majority of stocks, this is the assumption that will be adopted here. These assumptions may be restrictive for options on individual stocks, but their validity in the case of index options cannot be doubted, given the fact that numerous surveys have shown that a large number of US investors follow indexing strategies in their investments. These market equilibrium assumptions are quite general, insofar as they allow the existence of other investors with different portfolio holdings than the trader. They do not assume the existence of a representative investor, let alone one with a specific type of utility function. The results presented in this section are derived for unspecified discrete time asset dynamics, and are applied to the specific case of diffusion in the next section.

The derivation of option pricing bounds under a non-increasing pricing kernel \( Y(z_{t+\Delta t}) \) is done by an extension of the expected utility comparisons under a zero-net-cost option strategy introduced by Perrakis and Ryan (1984) and extended by Constantinides and Perrakis (2002, 2007) to incorporate transaction costs. An alternative approach is the LP formulation of minimum and maximum values of the option price. That approach requires a discrete state distribution of the random return \( z_{t+\Delta t} \) and solves the following problem:

\[
\max (\min)_{Y(z_{t+\Delta t})} E[C(S_t(1 + z_{t+\Delta t}))Y(z_{t+\Delta t})], \text{ subject to (2.1) and } Y(z_{t+\Delta t}) \text{ non-increasing.}
\]  

(2.2)

The LP approach yields identical results with the one adopted here.

In the expected utility comparisons approach an option upper bound is found by having the trader open a short position in an option with price \( C \), with the amounts \( \alpha C \) and \( (1 - \alpha)C \) added respectively to the riskless asset and the stock account. For the option lower bound a long position is financed by shorting an amount \( \beta S_t \), \( \beta < 1 \) of stock, with the remainder invested in the riskless asset. Both bounds are found as limits on the call price \( C \) such that the value function of the investor with the open option position would exceed that of the trader who does not trade in the option if the write (purchase) price of the call lies above (below) the upper (lower) limit on \( C \).

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14 Bogle (2005) reports that in 2004 index funds accounted for about one third of equity fund cash inflows since 2000 and represented about one seventh of equity fund assets.
The distribution of the return $z_{t+\Delta t}$ is assumed continuous without loss of generality, with the discrete case available in the online appendix. We denote this distribution by $P(z_{t+\Delta t} | S_t)$, which may depend on $S_t$ in the most general case; for notational simplicity this dependence is suppressed in the expressions that follow. Let also $\tilde{C}_t(S_t)$ and $C_t(S_t)$ denote respectively the upper and lower bounds on admissible call option prices supported by the market equilibrium (3.1), the asset dynamics and the monotonicity of the pricing kernel assumption.

If the option price $C(S_{t+\Delta t}) = C(S_t(1 + z_{t+\Delta t}))$ is known then bounds on $C_t(S_t)$ are found as the reservation write and reservation purchase prices of the option under market equilibrium that excludes the presence of stochastically dominant strategies, namely strategies that augment the expected utility of all traders. Violations of the bounds imply that any such trader can improve her utility by introducing a corresponding short or long option in her portfolio.

Let also $\hat{\tilde{z}}_{t+\Delta t} = E[z_{t+\Delta t} | S_t] \equiv \tilde{z}$, and by assumption $\hat{\tilde{z}} \geq R$. Similarly, let $z_{\min,t+\Delta t}$ denote the lowest possible return, which will be initially assumed strictly greater than -1.

The following important result, proven in the appendix, is a direct consequence of the monotonicity of the pricing kernel $Y_t(z_{t+\Delta t})$.

**Lemma 1:** If the option price $C_t(S_t)$ is convex for any $t$ then it lies within the following bounds

$$
\frac{1}{1 + R} E^U_t [C_t(S_t(1 + z_{t+\Delta t}))] \leq C_t(S_t) \leq \frac{1}{1 + R} E^U_t C_t(S_t(1 + z_{t+\Delta t})),
$$

(2.3)

where $E^U_t$ and $E^L_t$ denote respectively expectations taken with respect to the distributions

$$
U_t(z_{t+\Delta t}) = \begin{cases} 
P(z_{t+\Delta t} | S_t) & \text{with probability } \frac{R - z_{\min,t+\Delta t}}{E(z_{t+\Delta t} - z_{\min,t+\Delta t})}, \\
1_{z_{\min,t+\Delta t}} & \text{with probability } \frac{E(z_{t+\Delta t}) - R}{E(z_{t+\Delta t}) - z_{\min,t+\Delta t}} \equiv Q 
\end{cases},
$$

(2.4)

$$
L_t(z_{t+\Delta t}) = P(z_{t+\Delta t} | S_t, z_{t+\Delta t} \leq z_t^*) , \text{ such that } E(z_{t+\Delta t} | S_t, z_{t+\Delta t} \leq z_t^*) = R
$$

With this result it can be shown that the bounds $\tilde{C}_t(S_t)$ and $C_t(S_t)$ may be derived recursively by a procedure described in Proposition 1. This procedure yields a closed form solution, which relies heavily on the

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15 Similar expressions as the ones presented in Lemma 1 and Proposition 1 also hold when we have a “negative beta” stock, with $Y(z_{t+\Delta t})$ is non-decreasing and $\hat{\tilde{z}}_{t+\Delta t} < R$. The limiting results of the next section also hold for this case as well, with minor modifications.
assumed convexity of the option price $C_t(S_t)$, itself a consequence of the convexity of the payoff. The convexity property clearly holds for the diffusion and jump-diffusion cases examined in this paper.\footnote{The convexity of the option with respect to the underlying stock price holds in all cases in which the return distribution had iid time increments, in all univariate state-dependent diffusion processes, and in bivariate (stochastic volatility) diffusions under most assumed conditions; see Merton (1973) and Bergman, Grundy and Wiener (1996).}

**Proposition 1:** Under the monotonicity of the pricing kernel assumption and for a discrete distribution of the stock return $z_t$ all admissible option prices lie between the upper and lower bounds $\overline{C}_t(S_t)$ and $\underline{C}_t(S_t)$, evaluated by the following recursive expressions

\[
\overline{C}_t(S_t) = C_t(S_t) = (S_T - K)^+
\]

\[
\overline{C}_t(S_t) = \frac{1}{1 + R} E^u_t \left[ \overline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \right] |S_t|
\]

\[
\underline{C}_t(S_t) = \frac{1}{1 + R} E^d_t \left[ \underline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \right] |S_t|
\]

(2.5)

where $E^u_t$ and $E^d_t$ denote expectations taken with respect to the distributions given in (2.4).

**Proof:** We use induction to prove that (2.5) yields expressions that form upper and lower bounds on admissible option values. It is clear that (2.4) holds at $T$ and that $\overline{C}_T(S_T)$ and $\underline{C}_T(S_T)$ are both convex in $S_T$. Assume now that $\overline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta}))$ and $\underline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta}))$ are respectively upper and lower bounds on the convex function $C_{t+\Delta t}(S_t(1 + z_t))$, implying that

\[
\underline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \leq C_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \leq \overline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \quad (2.6)
\]

By Lemma 1 we also have

\[
\frac{1}{1 + R} E^d_t \left[ C_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \right] \leq C_t(S_t) \leq \frac{1}{1 + R} E^u_t \left[ C_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \right] \quad (2.7)
\]

(2.6) and (2.7), however, imply that

\[
\underline{C}(S_t) = \frac{1}{1 + R} E^u_t \left[ \underline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \right] \leq C_t(S_t) \leq \frac{1}{1 + R} E^d_t \left[ \overline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta})) \right] = \overline{C}(S_t), \quad (2.8)
\]
QED.

An important special case arises when \( z_{\min,t+\Delta t} = -1 \), implying that the stock can become worthless within a single elementary time period \((t, t + \Delta t)\). In such a case the lower bound given by expectations taken with (2.4) remains unchanged, but the upper bound takes the following form, with \( E^p \) denoting the expectation under the actual return distribution \( P(z_{t+\Delta t} | S_t) \):

\[
\overline{C}_T(S_T) = (S_T - K)^+, \quad \overline{C}_t(S_t) = \frac{E^p[\overline{C}_{t+\Delta t}(S_t(1 + z_{t+\Delta t})) | S_t]}{E[1 + z_{t+\Delta t} | S_t]}.
\] (2.9)

When the returns are iid then (2.9) corresponds to the expected payoff given \( S_t \) discounted by the risky asset’s return. The upper bound of (2.9) has been extended to allow for proportional transaction costs. The same is true for the lower bound given by (2.4).

It can be easily seen from (2.4) that the distributions are risk neutral, with \( E^{U_t}(z_{t+\Delta t}) = E^{L_t}(z_{t+\Delta t}) = R \). These distributions are independent of option characteristics such as the strike price or time to expiration. Note also that the pricing kernel \( Y(z_{t+\Delta t}) \) corresponding to the upper bound has a “spike” at \( z_{\min,t+\Delta t} \) and is constant thereafter, while the kernel of the lower bound is constant and positive till a value \( z_t^* \) such that \( E[z_{t+\Delta t} | z_{t+\Delta t} \leq z_t^*] = R \), and becomes zero for \( z_{t+\Delta t} > z_t^* \). These pricing kernels are boundary marginal utilities that do not correspond to a CPRA utility function or, indeed, to any class of utility functions with continuously decreasing marginal utilities.

The distributions \( U_t \) and \( L_t \) are the incomplete market counterparts of the risk neutral probabilities of the binomial model, the only discrete time complete market model. If, in addition to payoff convexity, the underlying asset returns are iid then \( U_t \) and \( L_t \) are time-independent and independent of the stock price \( S_t \). In all cases, however, the distributions \( U_t \) and \( L_t \) depend on the entire actual distribution of the underlying asset, and not only on its volatility parameter, as in the binomial and the BSM models. In particular, they depend on the mean \( \hat{z} \) of the distribution. If \( \hat{z} = R \) then (2.4) imply that the two distributions \( U_t \) and \( L_t \) coincide. As \( \hat{z} \) increases above \( R \) the bounds widen, reflecting the incompleteness of the market. The dependence of \( U_t \) and \( L_t \) on convexity and on the entire return distribution may appear restrictive, but in fact the approach is quite general. The stochastic dominance assumptions may still be used to find the tightest bounds that can be supported by the market equilibrium monotonicity condition when convexity does not hold, with the bounds now depending in general on option characteristics. Recall that arbitrage and equilibrium models are able to provide expressions for option prices only under specific assumptions about asset dynamics. By contrast the stochastic dominance approach can accommodate any type of asset dynamics, including time- and state-varying distributions, provided a suitable discrete time representation can be found. As shown in the next section, the dependence on many parameters of the distribution, including \( \hat{z} \) in the diffusion case, disappears at the continuous time limit.

An additional advantage of the SD approach is that the derived bounds also point out to profitable strategies that generate superior returns for all risk averse investors when there are observed option prices that violate the bounds. The strategies arise out of the proofs for the derivation of the bounds in Lemma 1, presented in the appendix. Such strategies, modified for the presence of frictions, were applied successfully in Constantinides et al (2011).

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17 See Propositions 1 and 5 of Constantinides and Perrakis (2002).
III. Index Option Pricing for Diffusion Processes

We model the return \( z_{t+\Delta} \) in the following general form\(^\text{18}\) that guarantees convergence to diffusion as \( \Delta t \to 0 \)

\[
\begin{align*}
z_{t+\Delta} &= \mu(S_t, t) \Delta t + \sigma(S_t, t) \varepsilon \sqrt{\Delta t}.
\end{align*}
\]  

(3.1)

In this expression \( \varepsilon \) has a bounded distribution of mean zero and variance one, \( \varepsilon \sim D(0,1) \) and \( 0 < \varepsilon_{\text{min}} \leq \varepsilon \leq \varepsilon_{\text{max}} \), but otherwise unrestricted.

The discretization (3.1) can be easily shown to converge to diffusion.\(^\text{19}\) The main result of this section, however, is the convergence of the transformed return distributions that underlie the two option bounds. We use the weak convergence criterion for the two return processes. For any number \( m \) of time periods to expiration we define a sequence of stock prices \( S^m \) and an associated probability measure \( P_m \). The weak convergence property for such processes\(^\text{20}\) stipulates that for any continuous bounded function \( f \) we must have

\[
\lim_{\Delta \to 0} E^{P_m} \left[ f(S^m_T) \right] \to E^{P} \left[ f(S_T) \right],
\]

where the measure \( P \) corresponds to diffusion limit of the process, to be defined shortly. \( P_m \) is then said to converge weakly to \( P \) and \( S^m_T \) is said to converge in distribution to \( S_T \). A necessary and sufficient condition for the convergence to a diffusion is the Lindeberg condition, which was used by Merton (1982, 1992) to develop criteria for the convergence of multinomial processes. In a general form, if \( \phi \) denotes a discrete stochastic process in \( d \)-dimensional space the Lindeberg condition states that a necessary and sufficient condition that \( \phi \) converges weakly to a diffusion, is that for any fixed \( \delta > 0 \) we must have

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\|\phi - \phi^0\| < \delta} Q_{\Delta} (\phi, d\phi) = 0
\]

(3.2)

where \( Q_{\Delta} (\phi, d\phi) \) is the transition probability from \( \phi = \phi^0 \) to \( \phi_{t+\Delta} = \phi \) during the time interval \( \Delta t \). Intuitively, it requires that \( \phi^0 \) does not change very much when the time interval \( \Delta t \) goes to zero.

When the Lindeberg condition is satisfied, the following limits exist

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\|\phi - \phi^0\| < \delta} (\phi - \phi^0)Q_{\Delta} (\phi, d\phi) = \mu(\phi)
\]

(3.3)

\(^\text{18}\) For simplicity dividends are ignored throughout this paper. All results can be easily extended to the case where the stock has a known and constant dividend yield, as in index options. In the latter case the instantaneous mean in (2.1) and (2.2) is net of the dividend yield.

\(^\text{19}\) The discretization (3.1) is sometimes referred to as the Euler discretization. The proof of its convergence is available from the authors on request.

\(^\text{20}\) For more on weak convergence for Markov processes see Ethier and Kurz (1986), or Strook and Varadhan (1979).
In our case the state variable vector is one-dimensional and \( \phi_i = S_i \), while the limits in (3.3) and (3.4) are 

\[
\mu(S_i, t) \equiv \mu_i \quad \text{and} \quad \sigma(S_i, t) \equiv \sigma_i
\]

respectively. At the limit we have

\[
\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dW
\]

(3.5)

The recursive procedure described in (2.4) and (2.5) can be applied directly to the stock returns \( z_{r \Delta t} \) given by (3.1) in order to generate the upper and lower bounds at time zero. Of particular interest, however, is the existence of a limit to these bounds as \( \Delta t \to 0 \) and (3.1) tends to (3.5). These limits are expressed by the following proposition whose proof is in the appendix.

**Proposition 2:** When the underlying asset follows a continuous time process described by the diffusion (3.5) then both upper and lower bounds (2.3)-(2.4) of a European call option evaluated on the basis of the discretization of the returns given by (3.1) converge to the same value, equal to the expectation of the terminal payoff of an option on an asset whose dynamics are described by the process

\[
\frac{dS_i}{S_i} = r dt + \sigma(S_i, t) dW,
\]

(3.6)

dispersed by the riskless rate.

This result establishes the formal equivalence of the bounds approach to the prevailing arbitrage methodology for plain vanilla index option prices whenever the underlying asset dynamics are generated by a diffusion or Ito process, no matter how complex. Note that the univariate Ito process is the only type of asset dynamics, corresponding to dynamically complete markets, for which options can be priced by arbitrage considerations alone. The two bounds (2.4)-(2.5), therefore, by defining the admissible set of option prices for any discrete time distribution corresponding to such a dynamic completeness, generalize the binomial model to any type of discrete time distribution.

(Figure 3.1 about here)

Figure 3.1 illustrates the convergence of the two bounds to the BSM value for an at-the-money call option with \( K = 100 \) and \( T = 0.25 \) years for the following instantaneous annual parameters:

\[ r = 3\%, \mu = 5\% \text{ to } 9\%, \sigma = 10\% \]

The diffusion process was approximated by a 300-period trinomial tree constructed according to the algorithm of Kamrad and Ritchken (1991). The two option bounds were evaluated as discounted expectations of the payoffs under the risk neutral probabilities obtained by applying the expressions (2.4)-(2.5) to subtrees of the 300-period trinomial tree.

As the figure shows, the two bounds converge to their common limit uniformly from below and above respectively. The speed of convergence varies inversely with the size of the risk premium, but convergence is essentially complete after 300 periods even for the largest premium of 6%. This speed of convergence may also be helpful for the cases where no closed-form expression for the option price exists, as in complex cases of state-dependent univariate diffusions, like the Constant Elasticity of Variance (CEV) model. In such cases valuation of the option by Monte Carlo simulation of the bounds is certainly an alternative to an option value computed as a discounted payoff of paths generated by the Monte Carlo simulation of (3.1). While there may not be any computational advantages in going through the bounds route to option valuation, the fact that both upper and

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lower bound tend to the same limit from above and from below may provide a benchmark for the accuracy of the valuation, in contrast to the direct simulation of (3.1).

So far we have established the equivalence of SD and arbitrage in the pricing of options on a well-diversified index, corresponding to the portfolio policies of a group of investors. Since the BSM model can also be derived by arbitrage, for equity options, this needs also to be demonstrated for the SD approach. This is done in the next section.

IV. The SD Approach for Stock Options

In this section we extend the set of assumptions about market equilibrium, by defining the risky asset held by the trader as a market index portfolio and allowing the trader to adopt additional marginal positions in a single stock, as well as in options on that stock. Let now \( I_t \) denote the current value of the index and \( S_t \) the value of the stock, with the returns \( \frac{I_{t+\Delta t} - I_t}{I_t} \equiv z_{t+\Delta t} \) and \( \frac{S_{t+\Delta t} - S_t}{S_t} \equiv v_{t+\Delta t} \). The market equilibrium conditions (2.1) are now as follows

\[
E[Y(z_{t+\Delta t})|I_t, S_t] = (1 + R)^{-1},
\]

\[
E[(1 + z_{t+\Delta t})Y(z_{t+\Delta t})|I_t, S_t] = E[(1 + v_{t+\Delta t})Y(z_{t+\Delta t})|I_t, S_t] = 1,
\]

(4.1)

\[
C_t(S_t, I_t) = E[C_{t+\Delta t}(S_t(1 + v_{t+\Delta t}), I_t, 1 + z_{t+\Delta t})] = 1.
\]

(4.2)

Assume now a joint discrete distribution of the two returns, and set

\[
E[v_{t+\Delta t} | z_{t+\Delta t} = z_{j,t+\Delta t}, I_t, S_t] = \tilde{v}_{j,t+\Delta t}. \]

The equilibrium relations (4.1) imply certain restrictions on the parameters of the joint distribution. These are expressed by the following Lemma. It covers the case of diffusion and can be extended to cover jump-diffusion for that joint return distribution.

**Lemma 2:** If the function \( \tilde{n}_j(z_{j,t+\Delta t}) \) is linear, \( \tilde{n}_j(z_{j,t+\Delta t}) = q + z z_{j,t+\Delta t} \), then the following relation must hold:

\[
R(1 - z) = q.
\]

Further, if in addition \( E[C_{t+\Delta t}(S_t(1 + v_{t+\Delta t}), I_t, 1 + z_{t+\Delta t})|z_{t+\Delta t}] \) can be written as a function

\[
\hat{C}_{t+\Delta t}(S_t(1 + \tilde{v}(z_{t+\Delta t})), I_t(1 + z_{t+\Delta t})) \]

then \( C_t(S_t, I_t) \) takes the form \( C_t(S_t) \), independent of the index level \( I_t \).

**Proof:** We write the last relation in (4.1) by replacing \( \tilde{n}(z_{t+\Delta t}) \). For the second part of the Lemma, we use induction. Since the
Lemma obviously holds at T-1, we assume that it holds at \( t + \Delta t \) and apply the last relation of (4.1) to \( \hat{C}_{t+\Delta t}(S_t(1+\nu(z_{t+\Delta t})) \), QED.

Assume now that Lemma 4 holds and define

\[
\begin{align*}
\overline{C}_T(S_T) &= \hat{C}_T(S_T) = (S_T - K)^+ \\
\overline{C}_{t+\Delta t}(S_t, z_{t+\Delta t}) &= E^T[\hat{C}_{t+\Delta t}(S_t(1+\nu_{t+\Delta t}))|z_{t+\Delta t}], \ t \leq T-1 \\
\underline{C}_{t+\Delta t}(S_t, z_{t+\Delta t}) &= E^T[\hat{C}_{t+\Delta t}(S_t(1+\nu_{t+\Delta t}))|z_{t+\Delta t}], \ t \leq T-1. \\
\overline{C}_t(S_t) &= \frac{1}{1+R} E^{L_t}[\overline{C}_{t+\Delta t}(S_t, z_{t+\Delta t})|S_t] \\
\hat{C}_t(S_t) &= \frac{1}{1+R} E^{L_t}[\underline{C}_{t+\Delta t}(S_t, z_{t+\Delta t})|S_t] \\
\end{align*}
\]

(4.3)

The distributions \( U_t, L_t \) are those given by (2.4) with \( P \) denoting the original distribution. The following result extends Lemma 1 to stock options.

**Lemma 3:** Under the conditions of Lemma 2, if the option price \( C_t(S_t) \) is convex in \( S_t \) then the relations (4.3) define bounds such that \( \overline{C}_t(S_t) \leq C_t(S_t) \leq \hat{C}_t(S_t) \).

**Proof:** We show that Lemma 1 holds, using again the convexity of \( C_t(S_t) \). This can be demonstrated using again the expected utility comparisons of Section 2. However, a much easier proof is by using the LP (2.2) applied to the objective function \( E[C_{t+\Delta t}(S_t(1+\nu_{t+\Delta t}))Y(z_{t+\Delta t})] = E[\hat{C}_{t+\Delta t}(S_t(1+\nu(z_{t+\Delta t}))Y(z_{t+\Delta t})] \), subject to the same constraints as (2.2) plus the additional one that \( E[(1+\nu(z_{t+\Delta t}))Y(z_{t+\Delta t})] = 1 \). If, however, the linearity condition of Lemma 2 holds then this last constraint is redundant and any feasible solution of the LP (2.2) with the modified objective function satisfies also this additional constraint. Since the linearity condition also implies that \( \hat{C}_{t+\Delta t}(S_t(1+\nu(z_{t+\Delta t})) \) is convex in \( z_{t+\Delta t} \), the bounds of \( C_t(S_t) \) are found by taking expectations of \( C_{t+\Delta t}(S_t(1+\nu_{t+\Delta t})) \) with respect to the distributions \( U_t, L_t \) given by (2.4)-(2.5), QED. The proof of Lemma 3 then follows directly by using induction, as in the proof of Proposition 1, QED.

Proposition 3 shows that both bounds converge to the BSM option price in the case of diffusion, as with index options. Define

\[\text{---}\]

\[22\] In evaluating \( U_t, L_t \) the terms \( n(\hat{a}_{t+\Delta t}) \) and \( n(z_{\min, \Delta t}) \) should replace \( \hat{a}_{t+\Delta t} \) and \( z_{\min, \Delta t} \).
with $\varepsilon \sim D_{\nu}(0,1)$ and $\eta \sim D_{\nu}(0,1)$. It is clear that the Lindeberg condition holds and both index and stock converge to the following bivariate diffusion

$$
\begin{align*}
\frac{dL_t}{L_t} &= \mu_t dt + \sigma_t dW_t, \\
\frac{dS_t}{S_t} &= m_t dt + \sigma^*_t dW_2,
\end{align*}
$$

(4.5)

where $\mu_t, m_t$ are the instantaneous means, $\sigma_t, \sigma^*_t$ the corresponding volatilities, and $E[dW_1 dW_2] = \rho dt$.

Further, the conditions of Lemma 2 hold and

$$
\sum_{j=1}^n (z_{jt+Dt}) = (m_t - m_t \frac{rS^n_t}{S_t})D_t + \frac{rS^n_t}{S_t} z_{jt+Dt} = m_t D_t + rs^n_t \varepsilon \sqrt{D_t}.
$$

(4.6)

We then have the following result, proven in the appendix.

**Proposition 3:** When both the index and the underlying asset follow continuous time processes described by the bivariate diffusion (4.5) then both upper and lower bounds (2.4)-(2.5) of a European call option evaluated on the basis of the discretization of the returns given by (4.4) converge to the same value, equal to the expectation of the terminal payoff of an option on an asset whose dynamics are described by the process

$$
\frac{dS_t}{S_t} = r dt + \sigma^*_t dW.
$$

(4.7)

This expectation is, however, the BSM price of an equity option, which emerges again as a result of the SD approach.

V. **Conclusions**

We have shown that SD yields identical results as arbitrage in the diffusion cases for both index and equity options. Since these are the only cases where arbitrage is by itself sufficient to generate option prices without any additional assumptions, SD emerges as an alternative option pricing paradigm that can also work in cases where arbitrage does not yield any useful results. Such cases are market incompleteness arising out of jump-diffusion and proportional transaction costs.
In the case of jump diffusion SD is also able to generate useful results without invoking elements that lie outside the model or are unobservable, like the representative investor. Since the asset return is the only random variable driving the underlying asset dynamics, the results of Proposition 1 still apply. It is sufficient to apply a discretization that converges to jump diffusion in continuous time. In contrast to simple diffusion, the market remains incomplete in continuous time, with the two bounds converging to different limits. This result is also particularly useful in the case of catastrophe-linked derivative instruments, where the underlying risk is by definition a rare event. The advantage of the SD approach in these cases is that there is no need to assume simultaneous equilibrium in both underlying asset and derivative markets, especially in view of evidence that such equilibrium may be misspecified.

In summary, therefore, SD has been shown to be able to deal with cases where conventional arbitrage either is not able to derive results, as with frictions, or must rely on assumptions that are hard to verify or do not hold. The results of this paper show that it deserves to be accepted as an alternative paradigm in the pricing of derivative assets.

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23 See Oancea and Perrakis (2010), and Perrakis and Boloorforoosh (2013).
Appendix

Proof of Lemma 1

We assume without loss of generality that the distribution \( P(z_{t+\Delta} \mid S_t) \) is continuous on a compact support. Let \( \Omega(x_t + y_t \mid S_t) \) denote the value function or expected utility function of a trader and \( C \) denote the market price of a call option. We show that if \( C \) does not lie within the bounds given by (2.3)-(2.4) then there exist portfolios involving the option and the two assets that yield a higher expected utility than \( \Omega(x_t + y_t \mid S_t) \).

By definition, we have

\[
\Omega(x_t + y_t \mid S_t) = \max_{y_t} E[\Omega((x_t - v_t)(1 + R) + (y_t + v_t)(1 + z_{t+\Delta}) \mid S_t)]
\]

\[
\equiv E[\Omega(x_t '(1 + R) + y_t '(1 + z_{t+\Delta}) \mid S_t)]
\]

where \( v_t \) denotes the optimal portfolio revision or stock purchase from the riskless account. To derive an upper bound on \( C \) suppose that we short a call and invest \( \alpha C, \; \alpha < 1 \) in the riskless asset and \( (1 - \alpha)C \) in the stock account. Let \( \Omega^C(x_t + \alpha C + y_t + (1 - \alpha)C \mid S_t) \) denote the expected utility of this portfolio, and

\[
\Delta_t \equiv \Omega^C(x_t + \alpha C + y_t + (1 - \alpha)C \mid S_t) - \Omega(x_t + y_t \mid S_t).
\]

We show that \( \Delta_t \geq 0 \) unless \( C \) lies below the upper bound in (2.3).

Replacing the definitions of the expected utilities, we have

\[
\Delta_t \geq E[\Omega^C(x_t '(1 + R) + \alpha C(1 + R) + y_t '(1 + z_{t+\Delta}) + (1 - \alpha)C(1 + z_{t+\Delta})) \mid S_t] - E[\Omega(x_t '(1 + R) + y_t '(1 + z_{t+\Delta})) \mid S_t] \geq 0
\]

\[
- E[\Omega(x_t '(1 + R) + y_t '(1 + z_{t+\Delta})) \mid S_t] \geq 0
\]

\[
- \Omega(x_t '(1 + R) + y_t '(1 + z_{t+\Delta})) \mid S_t) \geq 0
\]

\[
E[\Omega_i[\alpha C(1 + R) + (1 - \alpha)C(1 + z_{t+\Delta}) - C_{t+\Delta}(S_t(1 + z_{t+\Delta})]\mid S_t]\]

(A.1)

(A.1) holds because the short call portfolio acts suboptimally when it uses the optimal portfolio revision of the portfolio without the call and closes the open short position at the end of one period. \( \Omega_i \) denotes the derivative of \( \Omega \) with respect to its argument and we have used the concavity of \( \Omega \) in the last part of the relation. Consider the function

\[
\alpha C(1 + R) + (1 - \alpha)C(1 + z_{t+\Delta}) - C_{t+\Delta}(S_t(1 + z_{t+\Delta})) \equiv H(\alpha, C, z_{t+\Delta})
\]

(A.2)
It is clear from (A.2) that \( E[\overline{H}(\alpha, C, z_{t+\Delta})]\{S_t\} \) is an increasing function of \( C \) and a decreasing function of \( \alpha \) given that \( \hat{z}_{n+\Delta} \geq R \). Similarly, \( \overline{H}(\alpha, C, z_{t+\Delta}) \) is concave in \( z_{t+\Delta} \) for any \( C \) and \( \alpha \) by the convexity of \( C_{t+\Delta}(S_t(1+z_{t+\Delta})) \). Thus, it is always possible to choose \( \alpha = \bar{\alpha} \) and \( C = \bar{C} \) so that \( \overline{H}(\alpha, C, z_{\min}) = 0 \) and \( E[\overline{H}(\alpha, C, z_{t+\Delta})]\{S_t\} = 0 \), implying that the function \( \overline{H}(\alpha, C, z_{t+\Delta}) \) is initially increasing and positive and then decreasing and eventually negative and has exactly one zero, at \( z_{t+\Delta} = \bar{z} \). In such a case we have from (A.1), using the fact that \( \Omega_i \) is a decreasing function of \( z_{t+\Delta} \), that

\[
\Delta_i \geq E[\Omega_i(\alpha\bar{C})(1+R) + (1-\alpha)\bar{C}(1+z_{t+\Delta}) - C_{t+\Delta}(S_t(1+z_{t+\Delta}))]\{S_t\} \geq \Omega_i(\bar{z})E[\overline{H}(\alpha, C, z_{t+\Delta})]\{S_t\} = 0
\]  

(A.3)

This, however, implies that \( C = \bar{C} \) is a reservation write price for the call option, and any higher price \( C \) would set \( \Delta_i > 0 \), since \( E[\overline{H}(\alpha, C, z_{t+\Delta})]\{S_t\} > 0 \). Solving \( \overline{H}(\alpha, C, z_{\min}) = 0 \) and \( E[\overline{H}(\alpha, C, z_{t+\Delta})]\{S_t\} = 0 \) we find that \( C = \bar{C} \) is the upper bound given by (2.3), QED.

A similar proof also holds for the lower bound. We consider the zero net cost portfolio of purchasing a call option at the price of \( C \) by shorting an amount \( \beta S_t \), \( \beta < 1 \) of stock and investing the remainder in the riskless asset. The corresponding expected utility is

\[
\Omega_C(x_t + \beta S_t - C + y_t - \beta S_t|S_t) \geq \Omega_C(x_t(1+R) + (\beta S_t - C)(1+R) + (y_t - \beta S_t))(1+z_{t+\Delta}))|S_t| \geq E[\Omega(x_t(1+R) + (\beta S_t - C)(1+R) + (y_t - \beta S_t))(1+z_{t+\Delta})) + C_{t+\Delta}(S_t(1+\Delta_t))]|S_t|
\]  

(A.4)

where we have used arguments similar to those in (A.1). Setting now

\[
\Delta_i = \Omega_C(x_t + \beta S_t - C + y_t - \beta S_t|S_t) - \Omega(x_t + y_t|S_t), \text{ replacing the definitions of } \Omega \text{ and } \Omega_C \text{ and using (A.7) and the concavity property of the value functions we get the following relation, the equivalent of (A.1)}
\]

\[
\Delta_i \geq E[\Omega_i(z_{t+\Delta})\overline{H}(\beta, C, z_{t+\Delta})]\{S_t\}, \text{ where } \Omega_i \text{ is again the derivative of } \Omega \text{ and}
\]

\[
\overline{H}(\beta, C, z_{t+\Delta}) = (\beta S_t - C)(1+R) - \beta S_t(1+z_{t+\Delta}) + C_{t+\Delta}(S_t(1+\Delta_t)).
\]  

(A.5)

This last function is convex and has at most two zeroes, at \( z_{t+\Delta} = \bar{z} \) and \( z_{t+\Delta} = z^* \), while its expectation has a unique maximum in \( \beta \) for any \( C \). Applying the same reasoning as in relation (A.3), we have
\[ \Delta_t \geq \Pr(z_{t+\Delta t} \leq z^*) \Omega(\tilde{z}) E[H(\beta, C, z_{t+\Delta t}) \bigg| S_t, z_{t+\Delta t} \leq z^*]. \quad (A.6) \]

This is, however, positive unless the expectation in the right-hand side is negative. Replacing and maximizing with respect to \( \beta \), we get the lower bound given by (2.3)-(2.4), QED.

**Proof of Proposition 2**

Under the upper bound probability given by (2.3), the returns process becomes

\[ z_{t+\Delta t} = \begin{cases} z_t & \text{with probability } 1 - Q \\ z_{\min t+\Delta t} & \text{with probability } Q \end{cases}, \]

where \( Q \) is the following probability

\[ Q = \frac{\tilde{z} - r \Delta t}{(\tilde{z} - z_{\min t+\Delta t})} = \frac{\mu_t \Delta t - r \Delta t}{\mu_t \Delta t - (\mu_t \Delta t + \sigma_t \varepsilon_{\min} \sqrt{\Delta t})} = -\frac{\mu_t - r}{\sigma_t \varepsilon_{\min}} \sqrt{\Delta t} \]

From the definition of \( z_{t+\Delta t} \) given in (3.1) we get

\[ z_{t+\Delta t} = \mu(S_t, t) \Delta t + \sigma(S_t, t) \sqrt{\Delta t} \begin{cases} \varepsilon & \text{with probability } 1 - Q \\ \varepsilon_{\min} & \text{with probability } Q \end{cases} \quad (A.7) \]

The random component of the returns in (A.7) has a bounded discrete or continuous distribution, so the upper bound process satisfies the Lindeberg condition. The upper bound distribution (A.7) has the mean

\[ E_U[z_{t+\Delta t}] = \mu_t \Delta t + (1 + \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}} \sqrt{\Delta t}) (\sigma_t \sqrt{\Delta t}) E[\varepsilon] \]

\[ = \mu_t \Delta t - \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}} \sqrt{\Delta t} (\sigma_t \sqrt{\Delta t}) \varepsilon_{\min} = r \Delta t \]

Its variance is

\[ \text{Var}_U[z_{t+\Delta t}] = \sigma_t^2 \Delta t \left[ 1 + \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}} \sqrt{\Delta t} \right] \text{Var}[\varepsilon] + o(\Delta t) \]

\[ = \sigma_t^2 \Delta t \left[ 1 + \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}} \sqrt{\Delta t} - \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}} \sqrt{\Delta t} \varepsilon_{\min}^2 \right] + o(\Delta t) \]

\[ = \sigma_t^2 \Delta t + o(\Delta t) \]
Consequently, the upper bound process converges weakly to the diffusion (3.6), QED.

We prove the convergence of the lower bound for the case given by (2.4), under a continuous probability distribution $D$ of $\varepsilon$. A different proof applies to the discrete case; it is in an appendix available from the authors on request. The transformed returns process becomes

$$\hat{z}_{t+\Delta t} = \mu(S_t, t)\Delta t + \sigma(S_t, t)\hat{\epsilon}_{t+\Delta t}\sqrt{\Delta t},$$

where $\hat{\epsilon}_{t+\Delta t}$ is a truncated random variable $\{\hat{\epsilon}_{t+\Delta t} \mid \varepsilon \leq \bar{\varepsilon}_i\}$, with $\bar{\varepsilon}_i$ found from the condition $E^\varepsilon_{t}[\hat{z}_{t+\Delta t}] = r\Delta t$.

Since $\hat{\epsilon}_{t+\Delta t}$ is truncated from a bounded distribution the Lindeberg condition is satisfied. The risk neutrality of the lower bound distribution implies that

$$\mu_i \Delta t + \sigma_i \sqrt{\Delta t} E[\hat{\epsilon}_{t+\Delta t}] = r\Delta t,$$

and the mean of $\hat{\epsilon}_{t+\Delta t}$ is

$$E[\hat{\epsilon}_{t+\Delta t}] = -\frac{\mu_i - r}{\sigma_i} \sqrt{\Delta t} \tag{A.8}$$

Since this random variable is drawn from a distribution that is truncated from the distribution $D$ of $\varepsilon$ we get

$$E[\hat{\epsilon}_{t+\Delta t}] = \frac{1}{\text{Pr}(\varepsilon < \bar{\varepsilon}_i)} \int_{\varepsilon_{\min}}^{\bar{\varepsilon}_i} \varepsilon dD(\varepsilon) = \frac{1}{D(\bar{\varepsilon}_i)} \int_{\varepsilon_{\min}}^{\bar{\varepsilon}_i} \varepsilon dD(\varepsilon) = -\frac{\mu_i - r}{\sigma_i} \sqrt{\Delta t} \tag{A.9}$$

From (A.9) we can easily see that

$$\frac{dE[\hat{\epsilon}_{t+\Delta t}]}{d(\Delta t)} = \frac{dE[\hat{\epsilon}_{t+\Delta t}]}{d\bar{\varepsilon}_i} \frac{d\bar{\varepsilon}_i}{d(\Delta t)} < 0. \tag{A.10}$$

Since the first term in the product is clearly positive, it follows that $\frac{d\bar{\varepsilon}_i}{d(\Delta t)} < 0$. For every $\Delta t$, therefore, there exists a value $\bar{\varepsilon}_i(\Delta t)$ solving (A.8), which is a decreasing function of $\Delta t$. By assumption we have $E[\varepsilon] = 0$, implying that

$$\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon dD(\varepsilon) = \int_{\varepsilon_{\max}}^{\bar{\varepsilon}_i} \varepsilon dD(\varepsilon) + \int_{\bar{\varepsilon}_i}^{\varepsilon_{\max}} \varepsilon dD(\varepsilon) = 0, \tag{A.11}$$

with $\varepsilon_{\max} > 0$. Since $\frac{d\bar{\varepsilon}_i}{d(\Delta t)} < 0$ from (A.9), there exists a value $\Delta t = \delta$ such that $0 \leq \bar{\varepsilon}_i(\delta) \leq \bar{\varepsilon}_i \leq \varepsilon_{\max}$, for any $\Delta t < \delta$. From (A.8)-(A.11), we get
\[
\frac{\mu_t - r}{\sigma_t} \sqrt{\Delta t} = \frac{1}{\Pr(\varepsilon < \bar{\varepsilon}_t)} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon dD_D(\varepsilon)
\]
\[
\geq \frac{1}{1 - \Pr(\varepsilon > \bar{\varepsilon}_t)} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon dD_D(\varepsilon)
\]
\[
= \frac{\bar{\varepsilon}_t \Pr(\varepsilon > \bar{\varepsilon}_t)}{1 - \Pr(\varepsilon > \bar{\varepsilon}_t)} \geq \frac{\bar{\varepsilon}_t (\delta) \Pr(\varepsilon > \bar{\varepsilon}_t)}{1 - \Pr(\varepsilon > \bar{\varepsilon}_t)}.
\]

From the last inequality of (A.12) we get
\[
\Pr(\varepsilon > \bar{\varepsilon}_t) \leq \frac{\mu_t - r}{\sigma_t} \sqrt{\Delta t} \frac{1}{\bar{\varepsilon}_t (\delta) + \frac{\mu_t - r}{\sigma_t} \sqrt{\Delta t}} = O(\sqrt{\Delta t}). \quad \text{(A.13)}
\]

(A.13) implies that as \( \Delta t \to 0 \) the probability that \( \varepsilon > \bar{\varepsilon}_t \) tends to zero. Therefore, the limit lower bound distribution contains all the possible outcomes of \( \varepsilon \). This result is used to compute the limit of the variance of \( \hat{\varepsilon}_{t+\Delta t} \)
\[
\lim_{\Delta t \to 0} \text{Var}(\hat{\varepsilon}_{t+\Delta t}) = \lim_{\Delta t \to 0} \left\{ E[\hat{\varepsilon}_{t+\Delta t}^2] - (E[\hat{\varepsilon}_{t+\Delta t}])^2 \right\}
\]
\[
= \lim_{\Delta t \to 0} \left\{ \frac{1}{\Pr(\varepsilon < \bar{\varepsilon}_t)} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^2 dD(\varepsilon) - \left( \frac{\mu_t - r}{\sigma_t} \right)^2 \Delta t \right\}
\]
\[
= \lim_{\Delta t \to 0} \left\{ \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^2 dD(\varepsilon) - \left( \frac{\mu_t - r}{\sigma_t} \right)^2 \Delta t \right\} = 1,
\]

where the third equality in (A.14) applies the conclusion derived from (A.10) and the last equality uses the fact that
\[
\text{Var}(\varepsilon) = \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^2 dD(\varepsilon) = 1
\]

It follows that
\[
\text{Var}[\zeta_{t+\Delta t}] = \sigma_t^2 \Delta t + O(\Delta t)^2
\]

The diffusion limit is, therefore, the process described by equation (3.6), QED.

**Proof of Proposition 3**

Since Lemma 2 holds and \( C_t(S_t) \) is convex, Lemma 3 holds as well, and the option price lies within the set of bounds given by (4.3). We then consider the limit of these bounds as \( \Delta t \to 0 \), as in the proof of Proposition 2.
Assume for simplicity that the distributions of both $\varepsilon$ and $\eta$ are continuous. It then follows from (4.3) and (4.5) that the two bounds are given by recursive expectations of the option payoff, taken with respect to the following distributions, which replace (2.4), with $U_i$ and $L_i$ the same distributions as those given by (2.4).

$$
\hat{U}(v_{t+\Delta t}) = D_v(\eta)U_i(z_{t+\Delta t}), \quad \hat{L}(v_{t+\Delta t}) = D_v(\eta)L_i(z_{t+\Delta t}).
$$

(A.15)

The rest of the proof follows along the lines of the proof of Proposition 2. It is easy to show that

$$
E^{\hat{U}}[v_{t+\Delta t}] = E^{\hat{L}}[v_{t+\Delta t}] = r\Delta t, \text{ taking into account (4.4) and (4.6). Further,}
$$

$$
Var^{\hat{U}}[v_{t+\Delta t}] = \sigma^2_r (1 - \rho^2)\Delta t + Var^{\hat{L}}(\tilde{v}(z_{t+\Delta t})), \quad Var^{\hat{L}}[v_{t+\Delta t}] = \sigma^2_r (1 - \rho^2)\Delta t + Var^{\hat{L}}(\tilde{v}(z_{t+\Delta t})) \text{ implying that}
$$

$$
Var^{\hat{U}}[v_{t+\Delta t}] = Var^{\hat{L}}[v_{t+\Delta t}] = \sigma^2_r \Delta t + o(\Delta t) \text{ as in the proof of Proposition 2, thus completing the proof.}$$
References


Perrakis, Stylianos, 1988, “Preference-free Option Prices when the Stock Returns Can Go Up, Go Down or Stay the Same”, in Frank J. Fabozzi, ed., *Advances in Futures and Options Research*, JAI Press, Greenwich, Conn.


Convergence of the option bounds – Univariate Diffusion

$S_0=100$
$K=100$
$T=0.25$
$r=0.03$
$\sigma=0.1$

$dS_t = \mu S_t dt + \sigma S_t dW_t$

Figure 4.1
Figure 3.1