Asset Pricing with Index Investing*

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Abstract

We provide a theoretical analysis of how index investing affects capital market equilibrium. We consider a dynamic exchange economy with heterogeneous investors and two Lucas trees and find that, contrary to common beliefs, indexing can decrease the correlation between stock returns. It also decreases market volatility and interest rates, typically increases (decreases) volatilities and betas of large (small) stocks, but has almost no effect on investors’ welfare. The impact of index investing is stronger when stocks have heterogeneous fundamentals. Our analysis highlights that indexing changes not only how investors can trade but also their incentives to trade.

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1. Introduction

Starting from the 1970’s, passive index investing has been consistently gaining popularity among institutional and individual investors. According to the 2013 Investment Company Fact Book (http://www.icifactbook.org), 33 percent of households that invested in mutual funds in 2012 owned at least one index mutual fund. The proportion of index funds in all equity mutual fund assets increased from 8.7 percent in 1998 to 17.4 percent in 2012. Moreover, the funds benchmarked to the S&P 500 index managed 33 percent of all assets invested in index mutual funds. Index investing was initially promoted by proponents of the efficient market hypothesis (e.g., Malkiel, 1973; Samuelson, 1974) and has an increasing number of supporters due to the inability of money management industry as a whole to outperform the market (e.g., Malkiel, 1995; Fama and French, 2010; Lewellen, 2011) and high costs of active investment for society (e.g, French, 2008). It is blessed even by successful investors like Warren Buffett, who in his 2013 letter to Berkshire Hathaway shareholders argues that “the goal of the non-professional should not be to pick winners – neither he nor his ‘helpers’ can do that – but should rather be to own a cross-section of businesses that in aggregate are bound to do well. A low-cost S&P 500 index fund will achieve this goal.”

Despite the growing popularity of index investing, its broad economic impact is not well understood. While many academics and practitioners tout indexing as the best investment strategy for ordinary investors, others raise concerns that the proliferation of index trading can increase volatility of stock returns, make the returns more correlated, and thereby hurt market participants (e.g., Wurgler, 2011; Sullivan and Xiong, 2012). The objective of our study is to assess those concerns and provide a rigorous theoretical analysis of how index investing affects statistical properties of stock returns and investor welfare. We build a dynamic general equilibrium model of an exchange economy with two Lucas trees and two groups of investors with the constant relative risk aversion (CRRA) preferences dubbed type P investors (professional investors) and type I investors (index investors). We interpret the type P investors as professional market participants such as hedge funds, actively managed mutual funds, proprietary traders, etc., who can implement complex trading strategies that involve individual assets. The type I investors are unsophisticated market participants like individuals who manage their savings and retirement accounts and can
trade only the market portfolio of Lucas trees (index). In reality, indexing can result from the inability of ordinary investors to model stock returns, to keep track of a large number of open trading positions, to minimize transaction costs while trading individual stocks, etc. To maintain the generality of our analysis, we do not specify the reason why the type I investors are restricted to trade the index.

Consistent with our interpretation of the investors, we also assume that the type I and type P investors have different coefficients of risk aversion. Therefore, even without indexing the investors in our model would trade stocks to share risk. Indexing changes the set of trading strategies that the type I investors can implement compared to an unconstrained economy (an economy in which fundamentals are the same but all investors can trade individual stocks) and, therefore, affects the equilibrium variables. To identify the effect of indexing, we find the equilibria in the constrained and unconstrained economies and compare their characteristics including the risk-free rate, the volatilities and betas of stocks, and the correlation between stock returns.

Our analysis delivers several results. First, we find that indexing has an ambiguous effect on the correlation of stock returns, and this conclusion challenges a wide-spread belief that indexing always increases the correlation. Second, indexing decreases market volatility and the risk-free rate, although those effects are relatively weak. Third, indexing typically increases (decreases) volatilities and betas of stocks with relatively large (small) market capitalizations. Fourth, the effect of indexing is much stronger when stocks have heterogeneous fundamentals. Fifth, although indexing can substantially change the dynamic properties of stock returns, it has only a weak effect on investors’ welfare. Overall, our results imply the lack of theoretical support for the concerns that the growing popularity of index investing destabilizes financial markets and is detrimental to market participants.

To see economic intuition behind those effects, consider first an unconstrained economy in which all investors can trade all assets. When the investors have heterogeneous risk preferences, the statistical properties of stock returns are partially determined by trading induced by risk sharing. Assume, for example, that a positive cash flow shock hits one of the stocks and increases its price. Because the less risk-averse investors in equilibrium hold more stocks than those who are
more risk averse, this shock disproportionately increases their wealth. To maintain their optimal portfolio weights, the less risk-averse investors buy more shares of the affected stock from those who are more risk averse and drive its price up even further. Thus, dynamic risk sharing tends to increase the volatility of returns. Moreover, in response to a wealth shock the less risk-averse investors buy shares of all stocks and, as a result, the returns on the stocks become correlated even when their fundamentals evolve independently.\(^1\)

When some investors follow an indexing strategy, they trade only the market portfolio in response to cash flow shocks. As a result, risk sharing is less effective than in the unconstrained economy and its impact on the equilibrium is subdued. In particular, indexing decreases market volatility inflated by risk sharing. Also, the risk-free rate decreases because more stocks are held by more risk-averse investors, and less risk-averse investors borrow less from them. Those effects are stronger in the states in which the trees produce unequal fractions of the total dividend and portfolio distortions, caused by the inability of agents to trade individual stocks, are particularly pronounced.

Surprisingly, indexing can either increase or decrease the correlation of stock returns, and the net effect is determined by the relative strength of two effects that work in opposite directions. On the one hand, indexing increases the correlation because investors effectively exert price pressure on all stocks simultaneously by buying and selling the market portfolio as a whole.\(^2\) On the other hand, indexing reduces the correlation because it hampers risk sharing between investors, which is an important source of the correlation when stocks can be traded individually. The latter channel highlights that indexing changes not only how investors can trade but also their incentives to trade and challenges the perception of indexing as an unambiguous source of positive correlation between stock returns, which is shared by practitioners (e.g., Sullivan and Xiong, 2012) and appeared in popular press.\(^3\) Also note that there is no contradiction between the decrease in the correlation of stock returns produced by indexing in our model and numerous studies that document an increase in the correlation between a stock and an index when the stock is added to the index (e.g., Vijh, Xiong (2001), Kyle and Xiong (2001), Cochrane, Longstaff, and Santa-Clara (2008), Bhamra and Uppal (2009), Ehling and Heyerdahl-Larsen (2013), and Longstaff and Wang (2012) discuss how risk sharing among investors affects the dynamics of stock returns.


\(^2\)A similar effect arises in Barberis and Shleifer (2003), Basak and Pavlova (2013, 2015), and Grégoire (2014).

1994; Barberis, Shleifer, and Wurgler, 2005; Greenwood and Sosner, 2007; Boyer, 2011). Indeed, our model describes the implications of passive indexing as a broad phenomenon that can inhibit risk sharing, whereas a migration of a single stock in or out of an index has a minuscule effect on the ability of investors to share risk.

The effect of indexing on individual stock returns depends on the relative size of the stock. On the one hand, due to indexing investors rebalance their portfolios less actively in response to changes in dividends on a smaller tree, which are less aligned with returns on the market portfolio and cannot be hedged well when only the index is tradable. As a result, the volatility of returns on a smaller tree and its beta are smaller than in the unconstrained economy. On the other hand, the price of a larger stock becomes more sensitive to changes in its dividend because investors respond to all shocks by trading only the index and effectively trade the larger stock more than in the unconstrained economy. Therefore, the volatility and beta of this stock tend to be higher than in the economy without indexing.

The described effects of indexing exist even when dividends of all stocks have the same expected growth rate and the same volatility, and the heterogeneity in stocks is solely produced by different realizations of their dividends. The difference in the parameters of the dividend processes makes the impact of index investing much stronger. This result is explained by larger portfolio distortions caused by the inability of investors to trade individual stocks when stock dividends have different dynamics. For example, consider a case in which one stock has a low expected dividend growth rate and dividend volatility, whereas for the other stock both of those characteristics are relatively high. The risk-averse investors would hold relatively more shares of the first stock in the unconstrained economy, but indexing forces them to hold the same number of the shares of each stock and, hence, makes the portfolio highly suboptimal. To ensure that the new portfolio satisfies the equilibrium conditions, the expected stock returns and return volatilities substantially deviate from their values in the unconstrained economy.

Finally, we explore the welfare implications of index investing. To quantify them, we use the certainty equivalent loss (CEL) of an infinitesimally small investor and decompose it into the loss in utility produced by the inability of the investor to arbitrarily adjust portfolio weights and the
change in utility produced by distorted investment opportunities. We find that the former channel is largely responsible for the utility loss, but the size of the loss is small: in our calibration of the model the investor would not give up more than 0.15% of his wealth for the possibility to trade all assets individually. Thus, indexing does not make investors notably worse off and in the absence of other market frictions it can be a viable trading strategy.

The analysis of dynamic economies with multiple trees, heterogeneous investors, and market frictions is a challenging task, and our paper also makes a methodological contribution to the literature by demonstrating how to find an equilibrium in an economy with indexing. The idea of our approach is to characterize the equilibrium in terms of quasilinear differential equations for the price-dividend ratio of the index and the wealth-consumption ratio of the index investors, which can be solved by a fast standard numerical procedure. The approach is quite general and works well even when investors have arbitrary coefficients of risk aversion. Note that if index investors have logarithmic preferences, their optimal consumption and portfolio choice problem simplifies since there is no hedging demand and the wealth-consumption ratio is constant. Thus, in this case the equilibrium is described by only one differential equation for the index price-dividend ratio. Nevertheless, the equation cannot be solved analytically and our numerical approach still must be used to characterize the properties of the equilibrium. To the best of our knowledge, we are the first to provide exact (albeit numerical) solution to this problem even in the case when constrained investors have logarithmic preferences.

Our paper belongs to the growing literature that uses a dynamic exchange economy framework with heterogeneous investors to study equilibrium effects of various economic frictions that make financial markets incomplete.4 Such frictions include restricted stock market participation (e.g., Basak and Cuoco, 1998), short-sale and borrowing constraints (e.g., Detemple and Murthy, 1997; Basak and Croitoru, 2000; Kogan, Makarov, and Uppal, 2007; Gallmeyer and Hollifield, 2008; Chabakauri, 2014), portfolio concentration constraints (e.g., Pavlova and Rigobon, 2008), margin constraints (e.g., Gromb and Vayanos, 2002; Gårleanu and Pedersen, 2011; Chabakauri, 2013; Rytchkov, 2014; Brumm, Grill, Kubler, and Schmedders, 2015), and transaction costs (e.g., Buss, 4Dynamic exchange economies with one Lucas tree, heterogeneous investors, and complete markets are studied by Wang (1996), Chan and Kogan (2002), Weinbaum (2009, 2010), Xiouros and Zapatero (2010), Longstaff and Wang (2012), Cvitanić, Jouini, Malamud, and Napp (2012), Bhamra and Uppal (2014), among others.

The closest to our analysis is the paper by Shapiro (2002), who considers a general equilibrium model in which a fraction of investors can implement only particular trading strategies that are consistent with the investor recognition hypothesis (IRH), and the indexing strategy is one of them. In contrast to our paper, which explicitly characterizes the equilibrium and examines the volatility of returns and their correlation, Shapiro (2002) does not solve the model for the equilibrium characteristics and largely focuses on qualitative implications of portfolio constraints for interest rates and risk premia. Moreover, Shapiro (2002) assumes that the constrained investors have logarithmic preferences, which makes his analysis more tractable but less realistic.

A dynamic model with logarithmic investors and indexing is also considered by Grégoire (2014), who uses perturbation analysis to approximate the solution to the model and demonstrates that indexing increases comovement of stock returns. In contrast to Grégoire (2014), the investors in our model have heterogeneous preferences with arbitrary coefficients of risk aversion and, therefore, trade to share risk. We show that indexing can hamper risk sharing and decrease the correlation of stock returns. The model in Grégoire (2014) cannot produce this effect because of the assumed homogeneity of investors’ preferences.

Our paper is also related to the research on equilibrium effects of institutional investors whose compensation is benchmarked to a particular index and who can trade multiple risky assets (e.g., Cuoco and Kaniel, 2011; Basak and Pavlova, 2013; Buffa, Vayanos, and Woolley, 2014; Basak and Pavlova, 2015). One of the insights of this research is that in the presence of index-related incentives fund managers tilt their portfolios towards the index. Thus, indexing arises endogenously and can be partially responsible for the identified effects of institutional investors on the equilibrium. In contrast to those papers, which study implications of active money management by institutional investors for asset prices, we investigate the impact of pure passive indexing on the capital market equilibrium.

Finally, our paper builds upon the literature on dynamic equilibria in exchange economies...
with multiple Lucas trees and homogeneous investors (e.g., Menzly, Santos, and Veronesi, 2004; Cochrane, Longstaff, and Santa-Clara, 2008; Martin, 2013). As in those papers, the time variation in the dividend shares of individual trees in our model spills over into equilibrium characteristics. However, when investors are identical, they hold the market portfolio and indexing is irrelevant. This does not happen in our model because we combine the multiple tree framework, which is necessary for studying the effects of indexing, with the heterogeneity in the investors’ preferences.

2. Model

2.1. Assets

There are three assets in the economy: a risk-free short-term bond in zero net supply and two risky stocks. The supply of each stock is normalized to one share, which is a claim to a stream of dividends produced by a Lucas tree. The dividends $D_{1t}$ and $D_{2t}$ follow geometric Brownian motions

$$\frac{dD_{it}}{D_{it}} = \mu_{Di}dt + \Sigma_{Di}dB_t, \quad i = 1, 2,$$

where $\mu_{Di}$ are constant expected dividend growth rates, $\Sigma_{Di}$ are constant $1 \times 2$ matrices of diffusions, and $B_t$ is a $2 \times 1$ vector of independent Brownian motions. The rate of return on the bond $r_t$ as well as the stock prices $S_{1t}$ and $S_{2t}$ are determined in the equilibrium. The excess return on each stock $i$ is defined as

$$dQ_{it} = \frac{dS_{it} + D_{it}dt}{S_{it}} - r_tdt$$

and the vector $Q_t = [Q_{1t} \ Q_{2t}]'$ follows a diffusion process

$$dQ_t = \mu_{Qt}dt + \Sigma_{Qt}dB_t,$$

where the matrix of the risk premia $\mu_{Qt} = [\mu_{Q1t} \ \mu_{Q2t}]'$ and the matrix of the diffusions $\Sigma_{Qt} = [\Sigma'_{Q1t} \ \Sigma'_{Q2t}]'$ are also determined in the equilibrium.

Taken together, the stocks form a market portfolio (index), which pays the aggregate dividend $D_t = D_{1t} + D_{2t}$ and has the price $S_t = S_{1t} + S_{2t}$. Using Itô’s lemma and equation (1), the dynamics
of the dividend $D_t$ can be written as

\[ \frac{dD_t}{D_t} = \mu_d dt + \Sigma_d dB_t, \]  

(3)

where $\mu_d = u_t \mu_1 + (1 - u_t) \mu_2$, $\Sigma_d = u_t \Sigma_1 + (1 - u_t) \Sigma_2$, and $u_t = D_{1t}/D_t$. The excess return on the index is defined as

\[ dQ_{It} = \frac{dS_t + D_t dt}{S_t} - r_t dt \]

and using equation (2) its dynamics can be described as

\[ dQ_{It} = \mu_{It} dt + \Sigma_{It} dB_t, \]  

(4)

where $\mu_{It} = (\mu_{Q1t} S_{1t} + \mu_{Q2t} S_{2t})/S_t$ and $\Sigma_{It} = (\Sigma_{Q1t} S_{1t} + \Sigma_{Q2t} S_{2t})/S_t$. By construction, the index is value-weighted and its expected returns and diffusions are value-weighted averages of expected returns and diffusions of the individual stocks.

### 2.2. Agents

The economy is populated by two groups of competitive agents dubbed type P investors (professional investors) and type I investors (index investors). Each group consists of a unit mass of identical investors who have CRRA preferences. The investors differ across the groups in two respects. First, they have different coefficients of risk aversion, which are $\gamma_P$ and $\gamma_I$ for the type P and type I investors, respectively. Second, the trading strategies that the investors can implement depend on their type: the type P investors can trade all assets individually, whereas the type I investors are constrained and can trade only the risk-free bond and the market portfolio. Specifically, the type P investors form an arbitrary portfolio of the stocks $\omega_{Pt} = [\omega_{P1t} \, \omega_{P2t}]'$, where $\omega_{P1t}$ and $\omega_{P2t}$ are the fractions of their wealth $W_{Pt}$ allocated to stocks 1 and 2, respectively, and invest the rest of their wealth $\alpha_{Pt} = 1 - \omega_{P1t} - \omega_{P2t}$ in the bond. In contrast, the type I investors allocate their wealth $W_{It}$ between the index and the bond with the weights $\omega_{It}$ and $\alpha_{It} = 1 - \omega_{It}$, respectively.
The types of investors admit a natural interpretation. The type P investors can be thought of as professional traders such as hedge funds, actively managed mutual funds, proprietary traders, etc., who are relatively risk tolerant and can implement sophisticated trading strategies that involve individual assets. The type I investors are unsophisticated market participants such as individual investors who manage their savings and retirement accounts. They are more risk averse than professional investors and trade only the index, not individual stocks. In practice, indexing can be an optimal response to various factors such as information processing costs, organizational and management costs, transaction costs, etc. For example, investors with limited attention may allocate their learning capacity to macroeconomic factors rather than to firm-specific information (e.g., Peng and Xiong, 2006) and trade only the market portfolio. Investors may prefer to categorize assets in particular classes and invest in indices because this simplifies asset choice (e.g., Barberis and Shleifer, 2003). Even mutual fund and pension fund managers, whose compensation is related to index performance directly or indirectly through response of the fund flows to the fund performance, may find it optimal to partially allocate assets under management to index portfolios (e.g., Basak and Pavlova, 2013). We do not specify the reason why the type I investors can trade only the index because this preserves the generality of our analysis and allows us to study implications of pure passive indexing that is not contaminated by other economic frictions.

The optimization problem of the investors has the standard form: each investor $j = P, I$ chooses a consumption stream $C_{jt}$ and portfolio weights $\omega_{jt}$ that maximize the expected CRRA utility

$$U_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\beta_s} \frac{C_{js}^{1-\gamma_j}}{1-\gamma_j} ds \right]$$

subject to a budget constraint, which is

$$dW_{Pt} = (r_t W_{Pt} - C_{Pt}) dt + W_{Pt} \omega_{Pt} (\mu_{Qt} dt + \Sigma_{Qt} dB_t)$$

for the type P investors and

$$dW_{It} = (r_t W_{It} - C_{It}) dt + W_{It} \omega_{It} (\mu_{It} dt + \Sigma_{It} dB_t)$$

for the type I investors.
2.3. **State variables**

The model has two Lucas trees and two types of investors. Therefore, it is natural to assume that the state of the economy is described by two variables. The first measures relative consumption of each investor type, and we choose it to be the consumption share of the type I investors $s_t = C_{I_t}/D_t$. Obviously, the values of $s_t$ are bounded between 0 and 1. In general, $s_t$ follows a diffusion process

$$
d s_t = \mu_{st} \, dt + \Sigma_{st} \, dB_t, \quad (8)$$

where the scalar $\mu_{st}$ and the $1 \times 2$ matrix $\Sigma_{st}$ are determined by equilibrium conditions.\(^5\)

The second state variable measures the relative share of the dividend on the first stock in the aggregate dividend: $u_t = D_{1t}/D_t$.\(^6\) The stochastic equation for $u_t$ follows from applying Itô’s lemma to the definition of $u_t$ and using equations (1) and (3):

$$
d u_t = \mu_{ut} \, dt + \Sigma_{ut} \, dB_t, \quad (9)$$

where the drift $\mu_{ut}$ and the diffusion $\Sigma_{ut}$ are determined by exogenous model parameters as

$$
\mu_{ut} = u_t(1 - u_t)(\mu_{D1} - \mu_{D2} - (\Sigma_{D1} - \Sigma_{D2})(u_t\Sigma_{D1} + (1 - u_t)\Sigma_{D2})), \quad (10)
$$

$$
\Sigma_{ut} = u_t(1 - u_t)(\Sigma_{D1} - \Sigma_{D2}). \quad (11)
$$

By construction, the variable $u_t$ takes values in the range from 0 to 1. When $0.5 < u_t < 1$, the first tree contributes to the aggregate dividend more than the second tree, so we refer to the former as a larger tree and to the latter as a smaller tree. The terminology is opposite when $0 < u_t < 0.5$.

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\(^5\)The consumption share of one of the agents is often used as a state variable in economies with heterogeneous agents (e.g., Bhamra and Uppal, 2009, 2014; Longstaff and Wang, 2012; Chabakauri, 2013; Rytchkov, 2014).

\(^6\)This state variable is standard in the models with multiple Lucas trees (e.g., Menzly, Santos, and Veronesi, 2004; Cochrane, Longstaff, and Santa-Clara, 2008; Martin, 2013).
2.4. Equilibrium

We define the equilibrium in the model as a set of processes for the risk-free rate \( r_t \), expected excess returns \( \mu_{Qt} \), diffusions of returns \( \Sigma_{Qt} \), consumption streams \( C_{jt}, j = P, I \), and portfolio strategies \( \omega_{jt}, j = P, I \), such that

1. \( C_{jt} \) and \( \omega_{jt} \) solve the utility optimization problem of investor \( j \);
2. the aggregate consumption is equal to the aggregate dividend: \( C_{It} + C_{Pt} = D_t \);
3. the markets for the stocks and bond clear:

\[
\omega_{Pi} W_{Pt} + \omega_{Ii} W_{It} = S_{it}, \quad i = 1, 2, \tag{12}
\]

\[
\alpha_{Pi} W_{Pt} + \alpha_{It} W_{It} = 0, \tag{13}
\]

where \( \omega_{Iit} = \omega_{It} S_{it}/(S_{It} + S_{Pt}) \) is the fraction of the type I investors’ wealth allocated to stock \( i \) through investing in the index.

Assuming that the state of the economy is fully described by the two variables \( s_t \) and \( u_t \), we look for the equilibrium processes \( r_t, \mu_{Qt}, \Sigma_{Qt}, \mu_{It}, \) and \( \Sigma_{It} \) as functions of those state variables: \( r_t = r(s_t, u_t), \mu_{Qt} = \mu_Q(s_t, u_t), \Sigma_{Qt} = \Sigma_Q(s_t, u_t), \mu_{It} = \mu_I(s_t, u_t), \) and \( \Sigma_{It} = \Sigma_I(s_t, u_t) \). The same representation should exist for the drift and diffusion of \( s_t: \mu_{st} = \mu_s(s_t, u_t), \Sigma_{st} = \Sigma_s(s_t, u_t) \).

For the characterization of the equilibrium, it is convenient to introduce i) the price-dividend ratios of the index and individual stocks as functions of the state variables: \( S_{t}/D_t = f(s_t, u_t), \) \( S_{it}/D_{it} = f_i(s_t, u_t), i = 1, 2, \) and ii) the wealth-consumption ratios of the type I and type P investors as functions of the state variables: \( W_{It}/C_{It} = h(s_t, u_t) \) and \( W_{Pt}/C_{Pt} = h_P(s_t, u_t) \). Finally, we introduce the risk aversion of a representative investor as

\[
\Gamma_t = \left( \frac{s_t}{\gamma_I} + \frac{1 - s_t}{\gamma_P} \right)^{-1}. \tag{14}
\]

The following proposition characterizes the equilibrium and describes how to compute various equilibrium characteristics. To simplify notation, in the rest of the paper we omit the subscript \( t \).
for all variables as well as the arguments $s$ and $u$ of all functions.

**Proposition 1** The equilibrium in the model is characterized by the functions $r$, $\mu_s$, $\Sigma_s$, $\Sigma_I$, $f$, and $h$ that solve a system of algebraic and differential equations (A1) – (A6) from Appendix. The market price of risk $\eta$ and the expected excess returns on the index $\mu_I$ are given by equation (A7). The price-dividend ratio $f_i$ of stock $i = 1, 2$ solves equation (A8). The expected excess returns on individual stocks $\mu_{Qi}$, $i = 1, 2$, and return diffusions $\Sigma_{Qi}$, $i = 1, 2$, are given by equation (A9). The optimal portfolio weights $\omega_I$ and $\omega_P$ and the numbers of the shares held by the type I and type $P$ investors $N_{Ii}$ and $N_{Pi}$, $i = 1, 2$, are given by equations (A10), (A11), and (A13).

**Proof.** See Appendix.

To identify the effects of index investing, we compare the equilibrium from Proposition 1 with the equilibrium in an identical unconstrained economy, that is, an economy in which the fundamentals are the same but all investors can trade all individual assets. The equilibrium in the unconstrained economy with heterogeneous agents and two Lucas trees is described by Proposition 2 in Chabakauri (2013), which shows that the differential equations for the price-dividend ratio and wealth-consumption ratio as well as the equations for the risk-free rate $r$, expected excess stock returns $\mu_{Qi}$, diffusions $\Sigma_{Qi}$, drift $\mu_s$, and market prices of risk $\eta$ appear to be identical in the economies with and without indexing. Thus, the difference in the dynamics of the constrained and unconstrained economies results solely from the change in the diffusion of the consumption share $s$. Specifically, in the unconstrained economy the closed-form solution for the diffusion $\Sigma_s$ is

$$\Sigma_{s}^{unc} = \frac{\gamma_P - \gamma_I}{\gamma_P \gamma_I} s(1 - s) \Gamma \Sigma_D, \quad (15)$$

where $\Gamma$ is defined in equation (14). In contrast, in the economy with indexing it is given by equation (A3), which can be written as

$$\Sigma_s = \Sigma_{s}^{unc} \Pi_I - \frac{s}{h + sh_s} (h \Sigma_D + h_u \Sigma_u)(I_2 - \Pi_I), \quad (16)$$
where $\Pi_I = \frac{(\Sigma_I^* \Sigma_I)}{(\Sigma_I \Sigma_I^*)}$ is the projection operator on the space of index returns (on the vector of diffusions $\Sigma_I$) and $I_2$ is a $2 \times 2$ identity matrix.

Equation (16) deserves several comments. First, it highlights the role of market incompleteness for the type I investors, who face two-dimensional uncertainty associated with the shocks $dB_1$ and $dB_2$ but can trade only one risky asset. Equation (15) implies that in the unconstrained economy the variation in the consumption share $s$ reflects the variation in the total dividend $D$, which is the source of the aggregate risk in the economy. However, the composition of $D$ in terms of the individual dividends $D_1$ and $D_2$ does not matter because all investors trade all risky assets and thereby perfectly hedge shocks to the relative dividend shares.\(^7\) In contrast, equation (16) shows that in the presence of index investors the diffusion $\Sigma_s$ contains two terms. The first is the projection of $\Sigma_{unc}^s$ on the index returns; it represents the variation in the total dividend that can be shared by investors using index as the only tradable risky asset. The second term in equation (16) contains the projector $I_2 - \Pi_I$, so it is orthogonal to the space of index returns and describes the variation in $s$ produced by the variation in the total dividend that cannot be hedged by the investors. Effectively, the type I investors have additional exposure to the unhedgeable component of fundamental shocks and absorb it by changing their consumption.\(^8\)

Second, equation (16) shows how the magnitude of the variation in the state variable $s$ is affected by indexing. On the one hand, because some changes in the total dividend are unspanned by index returns, indexing hampers risk sharing between investors and the volatility of $s$ produced by it decreases (the projection of $\Sigma_{unc}^s$ on the index in the first term of equation (16) is smaller than $\Sigma_{unc}^s$). On the other hand, the state variable $s$ is affected by the unspanned part of the fundamental shocks (as indicated by the second term in equation (16)) and the volatility of $s$ increases. Which effect dominates depends on various factors including the fundamentals of the assets, the magnitude of portfolio distortions caused by indexing, etc.

Third, equations (15) and (16) show why it is more difficult to find the equilibrium in the economy with indexing than in the unconstrained economy. Because $\Sigma_{unc}^s$ does not depend on

\(^7\)Note that the dividend share $u$ is still a state variable because it affects the expected growth rate and volatility of the total dividend.

\(^8\)Loosely speaking, the projection operator on the space of tradable assets $\Pi_I$ becomes the identity operator in the unconstrained economy and equation (16) reduces to equation (15).
the price-dividend and wealth-consumption ratios, the differential equations for those ratios in the unconstrained economy are linear, decoupled, and easy to solve. In contrast, equation (A4) implies that the projection operator $\Pi_I$ in equation (16) is determined by the values of $f$ and $h$, so in the presence of indexing the dynamics of the state variable $s$ are entangled with the dynamics of the price-dividend and wealth-consumption ratios. As a result, the differential equations (A5) and (A6) are quasilinear, not linear, and do not have a closed-form solution.

The formulas from Proposition 1 also reveal several technical tricks that help us simplify the description of the equilibrium. In general, the computation of an equilibrium in an economy with two trees and two types of investors involves the solution of three differential equations: two of them are for the price-dividend ratios of the stocks and the third is for the wealth-consumption ratio of one of the investor types. In the unconstrained economy, those equations can be solved independently from each other, but this is not the case in an economy with restrictions on portfolio weights, in which the equations for the ratios become entangled and should be solved simultaneously (e.g., Chabakauri, 2013). Proposition 1 implies that in the economy with index investors the computation of the equilibrium can be simplified by sequentially solving two sets of equations: the first is a pair of quasilinear differential equations for the price-dividend ratio of the index and the wealth-consumption ratio of the type I investors; the second is a pair of linear differential equations for the price-dividend ratios of the individual stocks. This simplification occurs because the projection operator $\Pi_I$, which modifies the dynamics of the state variable $s$, depends only on the price-dividend ratio of the index, not individual stocks. The latter immediately follows from the definition of $\Pi_I$ and equation (A4).

Indexing changes the equilibrium because it distorts portfolios of the type I investors. Therefore, the directions and magnitudes of the effects of indexing on the equilibrium variables can be interpreted by comparing the numbers of the shares of each stock held by each type of the investors in the benchmark economy and the economy with indexing. Equation (A13) shows that those numbers can be inferred from the investors’ portfolio weights and wealth-consumption ratios. We use them in the next section to quantify portfolio distortions produced by indexing.
3. Numerical results

3.1. Model parameters

In our numerical analysis, we consider two specifications for the Lucas trees. In the first one, the dividend growth rates and volatilities of the dividend processes are identical and set as $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \ 0]$, and $\Sigma_{D2} = [0 \ 0.045]$. We refer to this specification as a model with homogeneous trees and use it to identify the effects of indexing that exist only due to the difference in the relative size of the trees. In the second specification, the expected dividend growth rates and volatilities of dividends are different: $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01 \ 0]$, and $\Sigma_{D2} = [0 \ 0.08]$. This is a model with heterogeneous trees, and it allows us to explore the consequences of constructing an index from stocks with different fundamentals. We follow previous studies (e.g., Basak and Cuoco, 1998; Dumas and Lyasoff, 2012; Chabakauri, 2013) and identify the aggregate dividend with the aggregate consumption, and the chosen parameter values are in the ballpark of the estimated mean and volatility of the consumption growth rate in the United States. In the both specifications the dividends of the trees are uncorrelated.

Because we interpret the type P investors as financial professionals and the type I investors as individual investors, we set $\gamma_P = 1$ and $\gamma_I = 5$, and this choice is consistent with individual investors being more risk averse than professionals. In contrast to the vast majority of the papers that study equilibria in incomplete markets, we do not assume that constrained investors have logarithmic preferences. On the one hand, this complicates the analysis because hedging demand of such investors affects the properties of the equilibrium and should be taken into account. On the other hand, the choice of $\gamma_I > 1$ makes the analysis more realistic. The time preference parameter $\beta$ is 0.03 for all investors.

\[9\] In the Internet Appendix we explore the robustness of our results to the assumption that the index investors are more risk averse than the unconstrained investors. In particular, we solve the model assuming that $\gamma_I = 1$ and $\gamma_P = 5$ and find that our main conclusions still hold.
3.2. Numerical technique

As follows from Proposition 1, all equilibrium processes in our model can be expressed in terms of the price-dividend ratio $f$ and the wealth-consumption ratio $h$, which satisfy the system of quasilinear differential equations (A5) and (A6). To solve the equations, we use the standard finite-difference approach, which prescribes to approximate an infinite-horizon economy by an economy with a large finite horizon $T$, discretize the time interval $[0, T]$ and domains of state variables, and solve the discretized equations backward as a sequence of systems of linear algebraic equations (e.g., Lapidus and Pinder, 1999).

Specifically, we introduce a vector of functions $F = [f \ h]'$, denote the first and second partial derivatives of $F$ with respect to the state variables $s$ and $u$ as $F_s, F_u, F_{ss}, F_{uu},$ and $F_{us}$, and write the system of equations (A5) and (A6) adjusted for a finite horizon economy as

$$
A_{ss}(F, F_s, F_u, s, u)F_{ss} + A_{uu}(F, F_s, F_u, s, u)F_{uu} + A_{us}(F, F_s, F_u, s, u)F_{us} + A_s(F, F_s, F_u, s, u)F_s
$$
$$
+ A_u(F, F_s, F_u, s, u)F_u + A(F, F_s, F_u, s, u)F + 1 + \frac{\partial F}{\partial t} = 0, \quad (17)
$$

where $A_{ss}, A_{uu}, A_{us}, A_s, A_u,$ and $A$ are diagonal $2 \times 2$ matrices with elements that correspond to the coefficients of differential equations (A5) and (A6). Note that equation (17) includes the time derivative $\partial F/\partial t$, which appears as an additional term in Itô’s lemma applied to the time-dependent price-dividend ratio and indirect utility function in the derivation of equations (A5) and (A6) presented in Appendix.

Next, we set $T = 500$ and using a backward recursion solve equation (17) at the discrete moments $t = T, T - \Delta t, \ldots, \Delta t, 0$ and in the discrete states $s = 0, \Delta s, 2\Delta s, \ldots, 1,$ and $u = 0, \Delta u, 2\Delta u, \ldots, 1,$ where $\Delta t = 0.1, \Delta s = 0.01,$ and $\Delta u = 0.01$. In particular, the time $t$ solution $F(t)$ is found by solving discretized equation (17) in which all derivatives of $F(t)$ are replaced with their finite-difference approximations and the equation coefficients are computed using the solution $F(t+\Delta t)$ at time $t+\Delta t$ obtained in the previous step. Thus, the coefficients of the discretized equation do not depend on the time $t$ solution, and $F(t)$ solves a system of linear algebraic equations. Because the time horizon $T$ is large, the sequence $F(t), t = T, T - \Delta t, \ldots, \Delta t, 0,$ converges to a
time-independent solution $F$, which describes an equilibrium in the infinite-horizon economy. We verify the convergence by observing that the discrete approximation of the derivative $\partial F/\partial t$ has the order of magnitude $10^{-7}$ at $t = 0$.

The iteration procedure starts from the terminal solution $F(T) = [\Delta t \Delta t]'$. Indeed, the index price and the type I investors’ wealth at the terminal date are equal to $S_T = D_T \Delta t$ and $W_{IT} = C_{IT} \Delta t$, respectively, so the price-dividend ratio and wealth-consumption ratio at time $T$ are $f(T) = \Delta t$ and $h(T) = \Delta t$. The spacial boundary conditions for the discretized version of equation (17) are obtained by taking the limits $s \to 0$, $u \to 0$, $s \to 1$, and $u \to 1$ in equation (17). The computation of the boundary conditions is incorporated directly into the numerical algorithm. Appendix B in Chabakauri (2013) provides further details.

Having solved equation (17) and obtained $f$ and $h$, we find $r$, $\mu_s$, $\Sigma_s$, and $\Sigma_I$ as functions of the state variables using equations (A1) – (A4). Also, we compute $\eta$ and $\mu_I$ from equation (A7). To find the price-dividend ratios $f_i$, we solve differential equations (A8). Note that those equations are linear because their coefficients are known functions of the state variables. Therefore, they are solved using the finite-difference approximation that no longer requires a backward recursion.

The remaining equilibrium variables are obtained from equations (A9) – (A13).

To find the equilibrium in the benchmark economy without indexing, we also use the finite-difference approximation. However, in this case the differential equations for the price-dividend ratios and wealth-consumption ratios are linear and decoupled, so each of them is solved individually without a backward recursion. Those computations closely follow Chabakauri (2013).

### 3.3. Benchmark: economy without index investing

Consider first an unconstrained economy in which all investors can trade all assets individually. The equilibrium in this economy is characterized by Chabakauri (2013), and we use it as a benchmark for identifying and quantifying the impact of indexing. The equilibrium variables in the unconstrained economy with homogeneous trees are presented in Figure 1.

**FIGURE 1 IS HERE**
Figure 1 demonstrates that the volatilities of returns on the individual stocks and index tend to be higher than the volatilities of the corresponding dividends, and this is a consequence of dynamic risk sharing among investors with different risk preferences (e.g., Bhamra and Uppal, 2009; Longstaff and Wang, 2012). Indeed, a positive cash flow shock to one of the stocks not only increases the stock price but also disproportionately increases wealth of the type P investors, who are less risk averse and invest a higher fraction of their wealth in stocks. To maintain their optimal portfolio weights, the type P investors buy more stocks from the more risk-averse type I investors and drive the price up even further, thereby increasing its volatility. Figure 1 also shows that this effect is stronger for the larger stock (e.g., the first stock when \( u > 1/2 \)) since this stock is traded more actively when the type P investors rebalance their portfolios. Also, the larger stock has a higher beta and shocks to its dividend have a higher price of risk. This is not surprising because the larger stock is a better proxy for the whole market and the risk associated with it has a bigger effect on the investors’ consumption. Because the type P investors trade both stocks in response to a shock to one of them, the stock returns are positively correlated even though the correlation between dividends is zero (e.g., Cochrane, Longstaff, and Santa-Clara, 2008; Ehling and Heyerdahl-Larsen, 2013).

The total number of the shares of each stock in our economy is normalized to one and the stocks have identical dividend processes. Nevertheless, the stock prices and statistical properties of their returns are different in all states except those with \( u = 1/2 \). As a result, in equilibrium each investor holds more shares of one stock than of the other. In particular, the graph for the ratio \( N_{I2}/N_{I1} \) presented in Figure 1 demonstrates that the type I investors prefer to hold more shares of the larger stock because it provides them with a better combination of risk and return.

Figure 2 presents the equilibrium in the economy with heterogeneous trees. It confirms many observations made in the case of homogeneous trees and reveals new patterns in the equilibrium.

\[ \sigma_i = \sqrt{\Sigma Q_{i,1}^2 + \Sigma Q_{i,2}^2} \text{ and } \sigma_{ind} = \sqrt{\Sigma I_{1,1}^2 + \Sigma I_{1,2}^2}. \]

\[ \rho = (\Sigma Q_{1,1} \Sigma Q_{2,1} + \Sigma Q_{1,2} \Sigma Q_{2,2})/(\sigma_1 \sigma_2). \]
characteristics. In particular, the volatilities of both individual stocks and the index tend to be higher when the economy is dominated by the more volatile second tree. This happens because both the fundamental volatility and the volatility produced by risk sharing are higher. The most interesting observation from Figure 2 is that stock returns can be negatively correlated even though the dividends are uncorrelated. To the best of our knowledge, this possibility has not been reported in the literature, which mainly considers economies with homogeneous trees and documents only positive correlation of stock returns produced by risk sharing (as in Figure 1).

To better understand the sign of the correlation, we follow Cochrane, Longstaff, and Santa-Clara (2008) and decompose the covariance between stock returns as

\[
\text{cov}(dQ_1, dQ_2) = \text{cov}\left(\frac{dD_1}{D_1}, \frac{dD_2}{D_2}\right) + \text{cov}\left(\frac{df_1}{f_1}, \frac{df_2}{f_2}\right) + \text{cov}\left(\frac{dD_1}{D_1}, \frac{df_2}{f_2}\right) + \text{cov}\left(\frac{dD_2}{D_2}, \frac{df_1}{f_1}\right). \tag{18}
\]

Equation (18) demonstrates that the covariance depends not only on the covariances of dividends and changes in the price-dividend ratios but also on how dividends on one stock covary with changes in the price-dividend ratio of the other stock. In our economy, the first term in equation (18) is zero because dividends are uncorrelated. The second term is small, so, as in Cochrane, Longstaff, and Santa-Clara (2008), the covariance between stock returns is mainly determined by the last two terms. We find that the negative correlation between stock returns in the economy with heterogeneous trees arises because the last term in equation (18) is negative and large for a wide range of the state variables \(s\) and \(u\).

Indeed, a negative shock \(dD_2\) increases the share of the first dividend \(u\) (the negative correlation is associated with changes in the state variable \(u\) because it is observed even in an economy with one type of investors). Figure 2 shows that the price-dividend ratio \(f_1\) is an increasing function of \(u\) in many states of the economy (the pattern is particularly pronounced around \(s = 0\) and \(u = 1\)), so in those states \(\text{cov}(dD_2/D_2, df_1/f_1) < 0\). The absolute value of the covariance is large due to the large volatility \(\sigma_{D2}\), which substantially exceeds \(\sigma_{D1}\). In contrast, when the trees are
homogeneous $f_1$ is an increasing function of $u$ in a smaller region and the volatility $\sigma_{D2}$ is the same as $\sigma_{D1}$. As a result, the last two terms in equation (18) have similar magnitudes, their sum is positive, and the correlation between stock returns is positive.

It remains to explain why the price-dividend ratios $f_1$ and $f_2$ increase with $u$. Figure 2 shows that the risk-free rate $r$ is a decreasing function of $u$ around $u = 1$, and the relation is stronger than in the case of homogeneous trees because interest rates are lower in economies with lower expected dividend growth rates and $\mu_{D1} < \mu_{D2}$. As a result, the cash flows are discounted at a lower rate around $u = 1$ (where the first tree dominates the economy) and the price-dividend ratios tend to be higher. Thus, the ratios $f_i$ are increasing functions of $u$ around $u = 1$, and this gives rise to the negative correlation between stock returns. Note that the described effect crucially relies on the heterogeneity of both drifts and diffusions of the dividend processes: when either of them is homogeneous, the correlation is positive in all states of the economy.

3.4. Main analysis: economy with index investors

In this section, we study the equilibrium effects of indexing by comparing the equilibrium variables in the constrained and unconstrained economies. We separately discuss the cases with homogeneous and heterogeneous trees.

3.4.1. Homogeneous trees

Consider first the economy with homogeneous trees. Because the fundamentals of the trees follow the processes with identical parameters, indexing changes the equilibrium only because the trees have different sizes produced by different realizations of the dividends.

FIGURE 3 IS HERE

Figure 3 shows the changes in the equilibrium variables produced by indexing. For the majority of the variables we plot relative changes, but for those variables that can be equal or close to zero we present absolute changes.
First of all, the change in the ratio $N_{I2}/N_{I1}$ shows how indexing distorts the investors’ portfolios. Because only the market portfolio and the risk-free bond are held by both types of the investors in the equilibrium with indexing and the market portfolio contains an equal number of the shares of each stock, indexing implies that $N_{I2}/N_{I1} = 1$ in all states of the economy. Since in the unconstrained economy the type I investors prefer to hold more shares of the larger stock (see the discussion of Figure 1), indexing increases (decreases) the relative number of the shares of the smaller (larger) stock in their portfolios. The graphs for the changes in the numbers of the shares $N_{P1}$ and $N_{P2}$ held by the type P investors further indicate that in total the type I investors hold more stocks in the economy with indexing than in the benchmark economy because the increase in the number of the shares of the smaller stock is not offset by only a slight decrease in the number of the shares of the larger stock.

An immediate consequence of a more uniform distribution of the shares across investors is the reduction in risk sharing among them. Indeed, because the type P investors hold fewer stocks than in the benchmark economy, their incentives to rebalance portfolios in response to cash flow shocks are subdued. As a result, indexing decreases the volatility of the market portfolio $\sigma_{\text{ind}}$, which is inflated by risk sharing in the unconstrained economy. A lower volatility implies that the market portfolio is safer than in the unconstrained economy, and this is why the more risk-averse type I investors hold more equity. The effect is particularly strong when the stocks have unequal sizes and the type P investors notably decrease their holdings of the smaller stock, but it disappears as $u \to 0$ or $u \to 1$ because in those limits the market coincides with one of the stocks and indexing is irrelevant.

Figure 3 shows that the portfolio distortions caused by indexing also affect the risk-free rate $r$, which is lower in the constrained economy. This happens because the stock holdings of the less risk-averse investors, who maintain leveraged positions in stocks in many states of the economy, decrease and they borrow less to finance their portfolio. As for the index volatility, the effect is particularly strong when the stocks differ in size.

Indexing has an ambiguous effect on the correlation of stock returns. Figure 3 demonstrates that the correlation increases when the stocks have comparable sizes (around $u = 0.5$) but de-
creases when the sizes are notably different. In general, the impact of indexing on the correlation is determined by relative strength of two effects that work in opposite directions. On the one hand, indexing increases the correlation because each investor holds an equal number of the shares of each stock and trades both stocks in lockstep. On the other hand, indexing reduces the correlation because it hampers risk sharing between investors, which is the main source of the correlation when stocks can be traded individually and their dividends are uncorrelated (this is explained in Section 3.3.3). The latter effect is strong when the stocks have different sizes and it dominates, so the correlation becomes lower than in the benchmark economy. When the stocks have comparable sizes, risk sharing is almost unaffected and the former effect dominates. As a result, the correlation between stock returns is higher than without indexing.

The impact of indexing on betas and volatilities of individual stocks as well as on the market prices of risk is less straightforward. As follows from Figure 3, indexing decreases the beta and volatility of the smaller stock, but the effect is opposite for the larger stock. Also, the market price of risk associated with the shock $dB_i$, $i = 1, 2$, is higher when stock $i$ is larger than the other one.

Those observations also admit an intuitive interpretation. Because due to indexing the leveraged type P investors hold fewer shares of the smaller stock, they are more reluctant to rebalance their portfolios in response to its dividend shocks. As a result, returns on the stock become less volatile and less related to the returns on the market, that is, have a lower beta. Effectively, indexing makes the smaller stock safer and, hence, its dividend shocks have a lower price of risk. The effect is opposite for the larger stock, but it is weaker because the type P investors only slightly increase their holdings of this stock compared to the unconstrained economy. Note that the beta, volatility, and price of risk are higher for the larger stock in the unconstrained economy, so our results imply that indexing increases the cross-sectional dispersion in those characteristics.

The effects of indexing on the volatilities can also be tracked down to the changes in the dynamics of the state variable $s$. Consider equation (A9), which decomposes the diffusions $\Sigma Q_i$ into three components: one of them represents the fundamental diffusion $\Sigma D_i$ and the two others are associated with the diffusions of the state variables $s$ and $u$. The changes in $\Sigma Q_{1, 1}$ (the first element in the matrix diffusion $\Sigma Q_1$) and its components produced by indexing are presented
in the upper panels of Figure 4. Because the stocks have identical fundamental processes, we consider the volatility only of the first of them. Also, we focus only on the diffusions associated with the innovation \( dB_1 \) because the changes in the diffusions associated with \( dB_i \) are the main determinants of the changes in the volatility of stock \( i \). This fact follows from the approximation

\[
\Delta \sigma_i^2 \approx 2(\Sigma_{Q1,1}^{unc} \Delta \Sigma_{Q1,1} + \Sigma_{Q1,2}^{unc} \Delta \Sigma_{Q1,2}),
\]

where the superscript \( unc \) indicates variables from the unconstrained economy, and the inequalities \( \Sigma_{Q1,1}^{unc} > \Sigma_{Q1,2}^{unc} \) and \( \Sigma_{Q2,1}^{unc} < \Sigma_{Q2,2}^{unc} \), which hold due to the presence of large constant dividend diffusions in the components \( \Sigma_{Q1,1}^{unc} \) and \( \Sigma_{Q2,2}^{unc} \). The dominant role of \( \Delta \Sigma_{Q1,1} \) in shaping the change in the volatility of the first stock is evident from the comparison of its graph and the graph for \( \Delta \sigma_1/\sigma_1^{unc} \) in Figure 3.

**FIGURE 4 IS HERE**

Figure 4 demonstrates that the effect of indexing on the volatility is primarily determined by the second component \( (f_{1s}/f_1)\Sigma_{s,1} \) in equation (A9): the deviation from its unconstrained counterpart is an order of magnitude larger than the same deviation of the component \( (f_{1u}/f_1)\Sigma_{u,1} \) and almost perfectly coincides with \( \Delta \Sigma_{Q1,1} \). The center right panel and the bottom right panel of Figure 4 further decompose the change in \( (f_{1s}/f_1)\Sigma_{s,1} \) into two parts related to the changes in the ratio \( f_{1s}/f_1 \) and in \( \Sigma_{s,1} \) using the approximation

\[
\Delta((f_{1s}/f_1)\Sigma_{s,1}) \approx \Delta(f_{1s}/f_1)\Sigma_{s,1}^{unc} + (f_{1s}^{unc}/f_1^{unc})\Delta \Sigma_{s,1}.
\]

Comparing those graphs with the graph for the total change in \( (f_{1s}/f_1)\Sigma_{s,1} \), we conclude that the latter is largely determined by the change in \( \Sigma_{s,1} \), but the effect is also shaped by the factor \( f_{1s}^{unc}/f_1^{unc} \).

Figure 4 also shows the graphs for \( f_{1s}^{unc}/f_1^{unc} \) and \( \Sigma_{s,1}^{unc} \) in the benchmark economy and how they change in the economy with indexing. In particular, \( \Sigma_{s,1}^{unc} \) is negative, and its absolute value increases with \( u \). Indeed, because the less risk-averse type P investors hold more stocks, any positive shock \( dB_1 \) disproportionately increases their wealth and consumption. Thus, the consumption share of the more risk-averse type I investors \( s \) goes down implying that \( \Sigma_{s,1}^{unc} \) is negative. The magnitude of the effect grows with the contribution of the asset to the aggregate dividend volatility, and this explains why it is stronger for larger \( u \). Indexing decreases the absolute value of \( \Sigma_{s,1} \) for the smaller stock (the first stock when \( u \) is small) because the necessity to trade
the whole index (both stocks) makes investors less responsive to changes in its dividend. The effect is opposite but weaker for the larger tree. The ratio $f_{1s}^{unc}/f_{1}^{unc}$ is also negative because the price-dividend ratio decreases with $s$: for higher $s$ the proportion of the more risk-averse investors in the economy and the required risk premium are higher and prices are lower. As a result, the effect of indexing is stronger when the economy is dominated by the risk averse type I investors ($s$ is large).

Finally, Figure 3 shows that the price-dividend ratios $f_i$ increase relative to the unconstrained economy when stock $i$ is smaller and this is explained by the effects of indexing on the risk-free rate $r$ and the market price of risk $\eta_i$. Indeed, the approximate Gordon formula $f_i \approx 1/(r + \eta \Sigma_{D_i} - \mu_{D_i})$ shows that the price-dividend ratio increases when both the risk-free rate and the market price of risk become lower but may decrease when the decrease in the risk-free rate is offset by an increase in the market price of risk. As follows from the discussion above, the former happens for the smaller stock and the latter may happen for the larger stock.

3.4.2. Heterogeneous trees

Although indexing changes various equilibrium characteristics, the magnitudes of the effects in the economy with homogeneous trees tend to be small. Indeed, the inability of some investors to rebalance their portfolios of individual stocks matters only when returns on the stocks have different statistical properties. When the dividends follow stochastic processes with identical parameters, only different realizations of dividend shocks and ensuing heterogeneity in stock sizes contributes to the heterogeneity in stock returns, which appears to be small. As a result, the impact of indexing is also relatively weak. However, the outcome can be substantially different when the heterogeneity in the stock sizes is accompanied by the heterogeneity in the dividend processes.

FIGURE 5 IS HERE

Figure 5 shows how indexing changes the equilibrium characteristics in the specification of the model with heterogeneous trees. It demonstrates that many effects of indexing are qualitatively
similar to those documented for the economy with homogeneous trees and presented in Figure 3. In particular, we find that indexing reduces the risk-free rate \( r \) and the volatility of index returns \( \sigma_{\text{ind}} \). It also again has an ambiguous effect on the correlation between stock returns and, in contrast to common beliefs, can decrease it. As before, the presence of index investors tends to increase (decrease) the volatility and beta of the larger (smaller) stock. Finally, the sizes of all effects tend to be large when the risk-averse type I investors consume a substantial fraction of the total dividend. Thus, our main conclusions are robust to the heterogeneity in the stock dividend process.

Nevertheless, the heterogeneity in the fundamental processes quantitatively modifies the impact of indexing on the equilibrium characteristics. In particular, the graphs in Figure 5 show that it makes many of the described effects much stronger than they are in the economy with homogeneous trees. For example, the correlation between stock returns can decrease by almost 0.15, whereas the effect does not exceed 0.01 in the economy with homogeneous trees. Similarly, the changes in the volatilities, betas, and risk-free rate reported in Figures 3 and 5 differ by an order of magnitude.

To understand the intuition behind those results, compare first the ratios \( N_{I2}/N_{I1} \) in the unconstrained economies with homogeneous and heterogeneous trees presented in Figures 1 and 2. The graphs show that the optimal portfolios are much stronger tilted towards one of the stocks when fundamentals are heterogeneous. Because the market portfolio contains an equal number of the shares of each stock, the portfolio distortion caused by indexing is much larger in the case with heterogeneous trees. This conclusion is illustrated by the graphs for the change in the ratio \( N_{I2}/N_{I1} \) presented in Figures 3 and 5. In particular, when the trees are homogeneous the absolute value of the change in the ratio does not exceed 0.3, whereas it can be as high as 1.4 in the economy with heterogeneous trees. The difference in the portfolio distortions is also evident from the graphs for \( \Delta N_{P1}/N_{P1}^{\text{unc}} \) and \( \Delta N_{P2}/N_{P2}^{\text{unc}} \), which show that in some states \( N_{P1} \) increases by almost 40% and \( N_{P2} \) decreases by almost 40% when the trees are heterogeneous, although those changes are less than 3% and 25% when the trees are homogeneous. Thus, indexing has a stronger effect on the composition of the investors’ portfolios when stocks have heterogeneous
fundamentals, and this explains why in that case it also has a stronger impact on the statistical properties of the equilibrium.

Another notable difference between Figures 3 and 5 is that in the economy with heterogeneous trees the impact of indexing on the equilibrium is particularly pronounced in the states with $u > 0.5$, whereas in the economy with homogeneous trees it is identical in the states $u$ and $1 - u$. Again, this pattern can be intuitively explained by the variation in the magnitudes of portfolio distortions caused by indexing across the states. As follows from the graph for the ratio $N_{I2}/N_{I1}$ in Figure 2, in the unconstrained economy with heterogeneous trees the optimal stock portfolios of the type I investors are much better balanced in the states with $u < 0.5$ than in the states with $u > 0.5$: in the former case the ratio $N_{I2}/N_{I1}$ is greater than 0.3, but in the latter case it can be zero and even negative indicating that the type I investors can have a long position in the first stock and a short position in the second stock. The inability to trade individual stocks implies that all investors hold an equal number of stocks in the equilibrium, and the market portfolio is further away from its unconstrained analog when $u > 0.5$. As a result, the impact of indexing is more pronounced in those states, and this is exactly what we observe in Figure 5.

3.5. Welfare analysis

Next, we investigate how indexing affects welfare of constrained investors. To quantify the effect, we use the certainty equivalent loss (CEL) of an infinitesimally small investor with the coefficient of risk aversion $\gamma_I$. The CEL is equal to the fraction of wealth that the investor in the state $(s, u)$ of the unconstrained economy would give up for not being an index investor in the same state of the constrained economy. More formally, we define the function $CEL(s, u)$ as a solution to the following equation:

$$J_{unc}((1 - CEL(s, u))W, s, u, t) = J(W, s, u, t),$$

where $J_{unc}(W, s, u, t)$ and $J(W, s, u, t)$ are the investor’s indirect utility functions in the unconstrained and constrained economies, respectively. Using equation (A24), the CEL can be written
as

\[ CEL(s,u) = 1 - \left( \frac{h(s,u)}{h_{unc}(s,u)} \right)^{\frac{\gamma_I}{1-\gamma_I}} \]  

(19)

where \( h(s,u) \) and \( h_{unc}(s,u) \) are wealth-consumption ratios of the investor in the constrained and unconstrained economies.

There are two channels through which the indexing constraints affect the investor. First, they restrict portfolio choice of the investor and make his investment policy suboptimal. Second, they distort prices and investment opportunities because in the economy with indexing many investors are constrained. To disentangle the effects of those factors on the CEL, we compute two other certainty equivalents: the first CEL dubbed \( CEL_1 \) measures the loss in utility due to constraints and the second CEL dubbed \( CEL_2 \) measures the effect of distorted investment opportunities. Specifically, we define \( CEL_1 \) and \( CEL_2 \) as solutions to

\[
\tilde{J}((1 - CEL_1(s,u))W,s,u,t) = J(W,s,u,t), \quad J_{unc}((1 - CEL_2(s,u))W,s,u,t) = \tilde{J}(W,s,u,t),
\]

where \( \tilde{J} \) is indirect utility function of an unconstrained investor with the coefficient of risk aversion \( \gamma_I \) who lives in the economy with index investors. Again, using equation (A24) we find that

\[ CEL_1(s,u) = 1 - \left( \frac{h(s,u)}{\tilde{h}(s,u)} \right)^{\frac{\gamma_I}{1-\gamma_I}}, \quad CEL_2(s,u) = 1 - \left( \frac{\tilde{h}(s,u)}{h_{unc}(s,u)} \right)^{\frac{\gamma_I}{1-\gamma_I}}, \]

where \( \tilde{h}(s,u) \) is the unconstrained investor’s wealth-consumption ratio. It solves equation (A43) in which \( \gamma_P \) is replaced with \( \gamma_I \). Note that \( 1 - CEL(s,u) = (1 - CEL_1(s,u))(1 - CEL_2(s,u)) \) and when the utility losses are small, \( CEL(s,u) \approx CEL_1(s,u) + CEL_2(s,u) \).

**FIGURE 6 IS HERE**

Figure 6 presents the certainty equivalent loses \( CEL, CEL_1, \) and \( CEL_2 \) as functions of the state variables \( s \) and \( u \) in the economies with homogeneous and heterogeneous trees for an investor.
with $\gamma_I = 5$.\textsuperscript{12} It shows that $CEL$ is positive in all states and both specifications, so a risk-averse investor unambiguously prefers to be in the unconstrained economy. The decomposition of $CEL$ into $CEL_1$ and $CEL_2$ reveals that this effect is largely produced by the inability of the investor to arbitrarily adjust portfolio weights in the economy with indexing (the component $CEL_1$), not by price distortions (the component $CEL_2$). Actually, $CEL_2$ can be either positive or negative depending on the state variables and parameters of the model, and it is predominantly negative in the economy with homogeneous trees. This observation implies that in many states indexing by other investors improves investment opportunities for unconstrained risk-averse investors, and the comparison of Figures 3, 5, and 6 suggests that the effect originates from lower stock holdings by less risk-averse investors in the economy with indexing. Indeed, they rebalance their portfolios less actively and, therefore, the volatility and correlation between assets produced by risk sharing is lower. In the states with positive $CEL_2$ (there are many of them in the case with heterogeneous trees), indexing worsens investment opportunities, and Figure 5 implies that this happens because of larger holdings of the risky assets by less risk-averse investors and ensuing increase in the volatility of the first asset.

Finally, Figure 6 shows that the magnitudes of certainty equivalent losses produced by indexing are small and stay below 0.15% of investor’s wealth in all states and specifications, and they are particularly small when the trees are homogeneous. Thus, even though indexing can noticeably change the dynamic properties of stock returns, the investors are only slightly worse off due to the constraints.

### 4. Conclusion

In this paper, we investigate the impact of index investing on various properties of capital market equilibrium. It is widely believed that trading of indices, which implies simultaneous trading of multiple individual securities, makes returns more volatile and increases the correlations between them. Our analysis reveals that this conclusion is theoretically unfounded because it ignores the

\textsuperscript{12}Because the type P investors have logarithmic preferences, their wealth-consumption ratios are equal to $1/\beta$ in the both constrained and unconstrained economies and their welfare is unaffected when indexing constraints are imposed on other investors.
equilibrium effects of index investing. We argue that indexing changes not only how investors can trade but also their investment opportunities, which determine the incentives to trade. In particular, we demonstrate that indexing hampers risk sharing among investors, which is responsible for excessive volatility of returns and makes them correlated even when the asset fundamentals are independent. As a result, indexing can decrease the volatilities of returns and correlations between them.

Our results also highlight the role of the heterogeneity in the assets’ market capitalizations and dividend processes in shaping the impact of indexing. We show that in general indexing increases (decreases) volatilities and betas of stocks with relatively large (small) market capitalizations and its impact is especially strong when stocks differ in their expected dividends and dividend volatilities. The latter case is particularly realistic and empirically relevant.

Our model is designed to illustrate the general equilibrium effects of index investing on dynamic properties of stock returns. As a benchmark setting, we use the most standard specification of an unconstrained exchange economy, which features infinitely lived investors with the CRRA preferences and geometric Brownian motions as dividend processes. Although such a model qualitatively describes the dynamic effects produced by risk sharing and market clearing conditions, it falls short in reproducing realistic moments of returns and the cyclical behavior of interest rates. Moreover, the distribution of the state variables is non-stationary, so ultimately only one type of the agents survives and the economy is dominated by one of the trees. Many of those shortcomings can be mitigated by appropriate modifications of the model. In particular, the distribution of wealth across investors can be made stationary by introducing overlapping generations (as, for example, in Gârleanu and Panageas (2015)) or “catching up with the Jones” preferences (as, for example, in Chan and Kogan (2002), Xiouros and Zapatero (2010), and Bhamra and Uppal (2014)). The distribution of dividends across Lucas trees can be made stationary by modifications of the dividend processes (as, for example, in Menzly, Santos, and Veronesi (2004)). Also, the model can better reproduce the empirical dynamics of equilibrium variables when investors are assumed to have heterogeneous Epstein-Zin preferences (as, for example, in Isaenko (2008), Chabakauri (2015), and Drechsler, Savov, and Schnabl (2015)) instead of the CRRA preferences.
We choose the simplest version of the exchange economy because those modifications are unlikely to change the intuition of our results but can substantially complicate the analysis.

Our analysis can be extended in several other ways. In particular, our model can accommodate alternative types of indices such as fundamental indices, which were proposed in the literature and implemented in practice (e.g., Arnott, Hsu, and Moore, 2005). Also, it would be interesting to consider a setting with multiple trees in which only a subset of all trees is included in the index. Such a model could help to investigate how the choice of assets that are included in the index affects the equilibrium properties as well as to provide a fully-fledged general equilibrium analysis of the correlations between the assets included and excluded from the index. This extension is likely to be more technically complicated than our model because of a larger number of state variables. Finally, it may be interesting to endogenize the dividend processes using a production economy framework and examine the impact of indexing on firm behavior. The analysis of how portfolio constraints affect corporate policies could be a particularly fruitful direction for future research.

Appendix. Proof of Proposition 1.

The equilibrium functions \(r, \mu_s, \Sigma_s, \Sigma_I, f,\) and \(h\) solve the following system of equations:

\[
\begin{align*}
    r &= \beta + \Gamma \left( \mu_D - \frac{1}{2} (\gamma_I + 1) s \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \Sigma_D + \frac{1}{s} \Sigma_s \right)' - \frac{1}{2} (\gamma_P + 1)(1-s) \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' \right), \\
    \mu_s &= -\Sigma_s \Sigma_D' + \frac{s(1-s)}{\gamma_I \gamma_P} \Gamma \left( \mu_D (\gamma_P - \gamma_I) + \frac{\gamma_I (\gamma_I + 1)}{2} \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \Sigma_D + \frac{1}{s} \Sigma_s \right)' - \frac{\gamma_P (\gamma_P + 1)}{2} \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' \right), \\
    \Sigma_s &= \frac{\gamma_P - \gamma_I}{\gamma_I \gamma_P} s (1-s) \Gamma \Sigma_D \Pi_I - \frac{s}{h + \delta h_s} (h \Sigma_D + h_u \Sigma_u) (I_2 - \Pi_I),
\end{align*}
\]

(A1) (A2) (A3)
The optimal portfolio weights of the type I and type P investors are

\[ \omega_I = \frac{1}{\Sigma_I \Sigma_I'} \left( \frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I' \Sigma_s' + \frac{h_u}{h} \Sigma_I' \Sigma_u' \right), \]

\[ \omega_P = (\Sigma_Q \Sigma_Q')^{-1} \left( \frac{\mu_Q}{\gamma_P} + \frac{h_{P_s}}{h_P} \Sigma_Q' \Sigma_s' + \frac{h_{P_u}}{h_P} \Sigma_Q' \Sigma_u' \right), \]

where \( \Sigma_I = \left( f \Sigma_D + f_u \Sigma_u + s_{s} \left( 1 - s \right) \Gamma \Sigma_D \right) \left( f \Sigma_D + f_u \Sigma_u - \frac{s_{u}}{h + h_s} \left( h \Sigma_D + h_u \Sigma_u \right) \right)' \times \left( f \Sigma_D + f_u \Sigma_u - \frac{s_{u}}{h + h_s} \left( h \Sigma_D + h_u \Sigma_u \right) \right) \times \left( \Sigma_D + \frac{f_u}{f} \Sigma_u - \frac{s}{f} \frac{h}{h + h_s} \left( h \Sigma_D + h_u \Sigma_u \right) \right), \) (A4)

\[ \frac{1}{2} f_{ss} \Sigma_s' \Sigma_s' + \frac{1}{2} f_{uu} \Sigma_u' \Sigma_u' + f_s \Sigma_s' \Sigma_u' + f_s \left( \mu_s + (1 - \Gamma) \Sigma_D \Sigma_s' \right) + f_u \left( \mu_u + (1 - \Gamma) \Sigma_D \Sigma_u' \right) + (\mu_D - \Gamma \Sigma_D \Sigma_D') f + 1 = 0, \] (A5)

\[ \frac{1}{2} h_{ss} \Sigma_s' \Sigma_s' + \frac{1}{2} h_{uu} \Sigma_u' \Sigma_u' + h_s \Sigma_s' \Sigma_s' + h_u \Sigma_u' \Sigma_u' + \left( \mu_s - (\gamma_I - 1) \left( \Sigma_D + \frac{s}{s} \Sigma_s' \right) \right) + h_u \left( \mu_u - (\gamma_I - 1) \left( \Sigma_D + \frac{s}{s} \Sigma_s' \right) \right) \]

\[ - \left( \frac{\gamma_I - 1}{2} \left( \Sigma_D + \frac{s}{s} \Sigma_s' \right) \left( \Sigma_D + \frac{s}{s} \Sigma_s' \right)' + \frac{\left( \gamma_I - 1 \right) r + \beta}{\gamma_I} \right) h + 1 = 0, \] (A6)

where \( \Pi_I = (\Sigma_I' \Sigma_I)/(\Sigma_I \Sigma_I') \) and the subscripts s and u of the functions \( f \) and \( h \) indicate derivatives. The market price of risk \( \eta \) and the expected returns on the index \( \mu_I \) are

\[ \eta = \gamma_P \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right), \quad \mu_I = \Gamma \Sigma_I \Sigma_D'. \] (A7)

The price-dividend ratio \( f_i \) of stock \( i = 1, 2 \) solves the following differential equation:

\[ \frac{1}{2} f_{iss} \Sigma_s' \Sigma_s' + \frac{1}{2} f_{iis} \Sigma_u' \Sigma_u' + f_i \Sigma_s' \Sigma_u' + f_s \left( \mu_s + (\Sigma_D i - \eta) \Sigma_s' \right) + \left( \mu_u + (\Sigma_D i - \eta) \Sigma_u' \right) + (\mu_D i - \ eta \Sigma_D') f_i + 1 = 0. \] (A8)

The expected excess returns on individual stocks \( \mu_{Qi} \) and return diffusions \( \Sigma_{Qi} \) are

\[ \mu_{Qi} = \gamma_P \Sigma_{Qi} \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)', \quad \Sigma_{Qi} = \Sigma_{Di} + \frac{f_{is}}{f_i} \Sigma_s + \frac{f_{iu}}{f_i} \Sigma_u. \] (A9)

The optimal portfolio weights of the type I and type P investors are

\[ \omega_I = \frac{1}{\Sigma_I \Sigma_I'} \left( \frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I' \Sigma_s' + \frac{h_u}{h} \Sigma_I' \Sigma_u' \right), \]

\[ \omega_P = (\Sigma_Q \Sigma_Q')^{-1} \left( \frac{\mu_Q}{\gamma_P} + \frac{h_{P_s}}{h_P} \Sigma_Q' \Sigma_s' + \frac{h_{P_u}}{h_P} \Sigma_Q' \Sigma_u' \right), \] (A11)
where the wealth-consumption ratio of the type P investors $h_P$ is

$$h_P = \frac{1}{1-s}(uf_1 + (1-u)f_2 - sh). \quad (A12)$$

The numbers of the shares of each stock $N_{Ii}$ and $N_{Pi}$ held by the type I and type P investors are

$$N_{Ii} = \frac{s \omega_{Ii} h}{s \omega_{Ii} h + (1-s) \omega_{Pi} h_P}, \quad N_{Pi} = \frac{(1-s) \omega_{Pi} h_P}{s \omega_{Ii} h + (1-s) \omega_{Pi} h_P}, \quad (A13)$$

where

$$\omega_{I1} = \frac{\omega_{I}uf_1}{uf_1 + (1-u)f_2}, \quad \omega_{I2} = \frac{\omega_{I}(1-u)f_2}{uf_1 + (1-u)f_2}. \quad (A14)$$

We derive equations (A1) – (A14) in several steps.

**A. Price-dividend ratios**

First, we derive equations for the price-dividend ratios $f_1$, $f_2$, and $f$. By definition, $S_i = D_i f_i$. Applying Itô’s lemma to this equation, we get

$$\frac{dS_i}{S_i} = \frac{dD_i}{D_i} + \frac{df_i}{f_i} + \frac{dDF}{D_i} f_i,$$

where

$$df_i = f_{is}(\mu_s dt + \Sigma_s dB) + f_{iu}(\mu_u dt + \Sigma_u dB) + \frac{1}{2} f_{iss}\Sigma_s^t dt + \frac{1}{2} f_{iux}\Sigma_u^t dt + f_{ius}\Sigma_s\Sigma_u^t dt.$$  

Using equation (1),

$$\frac{dS_i + D_i dt}{S_i} - rdt = \left(\mu_{Di} - r + \frac{1}{2} f_{iss}\Sigma_s^t + \frac{1}{2} f_{iux}\Sigma_u^t + f_{ius}\Sigma_s\Sigma_u^t \right) + \left(\mu_s + \Sigma_{Di}\Sigma_s^t \right) \frac{f_{is}}{f_i} + \left(\mu_u + \Sigma_{Di}\Sigma_u^t \right) \frac{f_{iu}}{f_i} + \frac{1}{2} \left(\Sigma_{Di} + \frac{f_{is}}{f_i} \Sigma_s + \frac{f_{iu}}{f_i} \Sigma_u \right) dB.$$  

This process should coincide with the process for excess returns from equation (2), so

$$\mu_{Qi} = \mu_{Di} - r + \frac{1}{2} f_{iss}\Sigma_s^t + \frac{1}{2} f_{iux}\Sigma_u^t + f_{ius}\Sigma_s\Sigma_u^t$$

$$+ \left(\mu_s + \Sigma_{Di}\Sigma_s^t \right) \frac{f_{is}}{f_i} + \left(\mu_u + \Sigma_{Di}\Sigma_u^t \right) \frac{f_{iu}}{f_i} + \frac{1}{2}, \quad (A15)$$

$$\Sigma_{Qi} = \Sigma_{Di} + \frac{f_{is}}{f_i} \Sigma_s + \frac{f_{iu}}{f_i} \Sigma_u. \quad (A16)$$
Equation (A15) is effectively a differential equation for $f_i$:

$$\frac{1}{2} f_{iss} \Sigma_s \Sigma_s' + \frac{1}{2} f_{isu} \Sigma_u \Sigma_u' + f_{isu} \Sigma_s \Sigma_u' + f_{is} (\mu_s + \Sigma_D \Sigma_s')$$

$$+ f_{iu} (\mu_u + \Sigma_D \Sigma_u') + (\mu_D - r - \mu Q_i) f_i + 1 = 0. \quad (A17)$$

By definition of the market price of risk $\eta$, $\mu Q_i = \Sigma Q_i \eta'$. Plugging this representation for $\mu Q_i$ in equation (A17) and using equation (A16), we arrive at equation (A8). The same steps applied to the index yield the differential equation for the index price-dividend ratio $f$:

$$\frac{1}{2} f_{ss} \Sigma_s \Sigma_s' + \frac{1}{2} f_{uu} \Sigma_u \Sigma_u' + f_{su} \Sigma_s \Sigma_u' + f_s (\mu_s + \Sigma_D \Sigma_s')$$

$$+ f_u (\mu_u + \Sigma_D \Sigma_u') + (\mu_D - r - \mu I) f + 1 = 0. \quad (A18)$$

The index diffusion is related to the diffusions of the state variables as

$$\Sigma_I = \Sigma_D + \frac{f_s}{f} \Sigma_s + \frac{f_u}{f} \Sigma_u \quad (A19)$$

and this equation is similar to equation (A16).

**B. Utility maximization problem of the type P investors**

Next, consider the consumption and portfolio problem of the type P investors. Recall that they can invest in any combination of stocks, so from their perspective the market is complete. The first order conditions of their optimization problem can be interpreted as pricing equations that relate the risk-free rate $r$ and the expected excess returns $\mu Q_i$ to their discount factor $\Lambda$ (e.g., Cochrane, 2005):

$$r = -\frac{1}{dt} \mathbb{E}\left(\frac{d\Lambda}{\Lambda}\right), \quad \mu Q_i = -\frac{1}{dt} \mathbb{E}\left(\frac{d\Lambda}{\Lambda} \frac{dS_i}{S_i}\right), \quad i = 1, 2. \quad (A20)$$

Since the investors have the CRRA preferences, their discount factor is $\Lambda = \exp(-\beta t)(C_P)^{-\gamma P}$. Hence,

$$\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma P \frac{dC_P}{C_P} + \gamma P (\gamma P + 1) \left(\frac{dC_P}{C_P}\right)^2. \quad (A20)$$

Using the definition of the consumption share $s$, the consumption of the type P investors is $C_P = (1 - s)D$. Itô’s lemma applied to this equation together with equations (3) and (8) yields

$$\frac{dC_P}{C_P} = \left(\mu_D - \frac{\mu_s + \Sigma_D \Sigma_s'}{1 - s}\right) dt + \left(\Sigma_D - \frac{1}{1 - s} \Sigma_s\right) dB.$$
Therefore,
\[
\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma_P \left( \mu_D - \frac{\mu_s + \Sigma_D \Sigma'}{1 - s} - \frac{\gamma_P + 1}{2} \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)^\prime \right) dt
\]
\[
- \gamma_P \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) dB.
\]

Using equation (A20), we find the risk-free rate \( r \) and the expected excess returns \( \mu_Q \) and \( \mu_I \):
\[
r = \beta + \gamma_P \mu_D = \frac{\gamma_P}{1 - s} (\mu_s + \Sigma_s \Sigma_D') - \frac{\gamma_P (\gamma_P + 1)}{2} \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)^\prime \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right), \quad (A21)
\]
\[
\mu_Q = \gamma_P \Sigma_Q \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)^\prime, \quad \mu_I = \gamma_P \Sigma_I \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)^\prime. \quad (A22)
\]

C. Utility maximization problem of the type I investors

The type I investors maximize the CRRA utility from equation (5) subject to the budget constraint from equation (7). Because they can trade only the index and the risk-free bond, from their perspective the market is incomplete and the utility maximization problem should be solved directly. In particular, their indirect utility function \( J \) satisfies the standard Hamilton-Jacobi-Bellman (HJB) equation
\[
\max_{\{C_I,\omega_I\}} \left[ e^{-\beta t} \frac{C_I^{1-\gamma_I}}{1 - \gamma_I} + \mathcal{D} J \right] = 0, \quad (A23)
\]
where \( \mathcal{D} J = E[dJ]/dt \) is given by
\[
\mathcal{D} J = J_W (rW_I - C_I + \omega_I W_I \mu_I) + \frac{1}{2} J_{WW} W_I^2 \omega_I^2 \Sigma_I \Sigma_I' + J_{W\omega} \omega_I W_I \Sigma_I \Sigma_I' + J_{W\mu} \omega_I W_I \Sigma_I \Sigma_I' \]
\[
+ J_{s\mu} + J_{u\mu} + \frac{1}{2} J_{ss} \Sigma_s \Sigma_s' + \frac{1}{2} J_{uu} \Sigma_u \Sigma_u' + J_{us} \Sigma_s \Sigma_u' + J_{tu}
\]
and the subscripts of \( J \) denote derivatives with respect to the corresponding variable. When investors have the CRRA preferences, it is standard to look for the indirect utility in the following form:
\[
J = \frac{1}{1 - \gamma_I} W_I^{1-\gamma_I} h^\gamma_I \exp(-\beta t), \quad (A24)
\]
where the function \( h \) depends on the state variables \( s \) and \( u \). The maximization in equation (A23) with respect to \( C_I \) together with equation (A24) yields the optimal consumption:
\[
C_I = W_I h^{-1}, \quad (A25)
\]
so \( h \) is the optimal wealth-consumption ratio. Similarly, the maximization in (A23) with respect
to $\omega_I$ gives the optimal weight of the index:

$$\omega_I = \frac{1}{\Sigma_I \Sigma'_I} \left( \frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I \Sigma'_s + \frac{h_u}{h} \Sigma_I \Sigma'_u \right). \quad (A26)$$

This is equation (A10). The substitution of equations (A25) and (A26) back into equation (A23) yields a differential equation for $h$:

$$\frac{1}{2} h_{ss} \Sigma_s' + \frac{1}{2} h_{uu} \Sigma_u' + h_{us} \Sigma_s' + h_{us} \mu_s + h_u \mu_u$$

$$+ \frac{\gamma_I - 1}{2} \left( \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)' - \frac{1}{\Sigma_I \Sigma'_I} \left( \frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I \Sigma'_s + \frac{h_u}{h} \Sigma_I \Sigma'_u \right)^2 \right) h$$

$$+ \frac{1}{\gamma_I} ((1 - \gamma_I) r - \beta) h + 1 = 0. \quad (A27)$$

**D. Dynamics of the state variable $s$ and returns on the index**

Next, we find expressions for $\mu_s$, $\Sigma_s$, $\mu_I$, and $\Sigma_I$. The definition of the consumption share $s$ implies that $C_I = sD$, so using Itô’s lemma

$$\frac{dC_I}{C_I} = \mu_C dt + \Sigma_C dB, \quad \frac{dC_I^{-\gamma_I}}{C_I^{-\gamma_I}} = \left( -\gamma_I \mu_C + \frac{1}{2} \gamma_I (\gamma_I + 1) \Sigma_C \Sigma'_C \right) dt - \gamma_I \Sigma_C dB, \quad (A28)$$

where

$$\mu_C = \mu_D + \frac{\mu_s + \Sigma_s \Sigma'_D}{s}, \quad \Sigma_C = \Sigma_D + \frac{1}{s} \Sigma_s. \quad (A29)$$

Note that using $C_I = W_I h^{-1}$, the indirect utility function from equation (A24) can be rewritten as

$$J = \frac{1}{1 - \gamma_I} C_I^{-\gamma_I} W_I \exp(-\beta t).$$

Applying Itô’s lemma to this equation and taking into account equations (7) and (A28), we get

$$\frac{dJ}{J} = \left( -\beta - \gamma_I \mu_C + \frac{1}{2} \gamma_I (\gamma_I + 1) \Sigma_C \Sigma'_C + r - h^{-1} + \omega_I (\mu_I - \gamma_I \Sigma_I \Sigma'_C) \right) dt + (\omega_I \Sigma_I - \gamma_I \Sigma_C) dB. \quad (A30)$$

Alternatively, Itô’s lemma applied to equation (A24) yields

$$\frac{dJ}{J} = \frac{\mathcal{D}J}{J} dt + \left( (1 - \gamma_I) \omega_I \Sigma_I + \gamma_I \frac{h_s}{h} \Sigma_s + \gamma_I \frac{h_u}{h} \Sigma_u \right) dB. \quad (A31)$$
Noting that equations (A24) and (A25) imply that
\[ e^{-\beta t} \frac{C_I^{1-\gamma_I}}{1-\gamma_I} = J h^{-1} \]
and using the HJB equation (A23), we get \[ \mathcal{D}J = -J h^{-1} \] and rewrite equation (A31) as
\[
\frac{dJ}{J} = -h^{-1}dt + \left( (1 - \gamma_I)\omega_I + \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) dB.
\] (A32)

Matching the drifts and diffusions in equations (A30) and (A32) and using \( \mu_C \) and \( \Sigma_C \) from equation (A29), we get
\[
\frac{1 + \gamma_I}{2} \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \Sigma_D + \frac{1}{s} \Sigma_s \right)' + \frac{r - \beta}{\gamma_I} + \omega_I \frac{\mu_I}{\gamma_I} - \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \Sigma_I' = \mu_D + \frac{1}{s} (\mu_s + \Sigma_s \Sigma_D'),
\] (A33)
\[
\omega_I \Sigma_I - \frac{h_s}{h} \Sigma_s - \frac{h_u}{h} \Sigma_u = \Sigma_D + \frac{1}{s} \Sigma_s.
\] (A34)

Equation (A34) helps to derive a system of equations for \( \Sigma_I \) and \( \Sigma_s \). Plugging the optimal portfolio weight \( \omega_I \) from equation (A26) into equation (A34) yields
\[
\frac{\mu_I \Sigma_I}{\gamma_I (\Sigma_I \Sigma_I')} = \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left( I_2 - \frac{\Sigma_I' \Sigma_I}{\Sigma_I \Sigma_I'} \right) = \Sigma_D + \frac{1}{s} \Sigma_s,
\] (A35)
where \( I_2 \) is a \( 2 \times 2 \) unit matrix. Multiplying this equation by \( \Sigma_I' \), we get
\[
\mu_I = \gamma_I \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \Sigma_I',
\] (A36)
which together with the expression for \( \mu_I \) from (A22) gives
\[
\Sigma_s \Sigma_I' = \left( \frac{\gamma_I}{s} + \frac{\gamma_P}{1-s} \right)^{-1} (\gamma_P - \gamma_I) \Sigma_D \Sigma_I'.
\] (A37)

The substitution of this equation in equation (A36) yields \( \mu_I = \Gamma \Sigma_D \Sigma_I' \), where \( \Gamma \) is defined in equation (14). This expression for \( \mu_I \) is a part of equation (A7). Plugging it into equation (A35), introducing the matrix \( \Pi_I = (\Sigma_I' \Sigma_I)/(\Sigma_I \Sigma_I') \), which is a projector operator on the vector \( \Sigma_I \), and rearranging the terms, we get
\[
\Sigma_s = (\gamma_P - \gamma_I) \left( \frac{\gamma_I}{s} + \frac{\gamma_P}{1-s} \right)^{-1} \Sigma_D - s \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u + \frac{1}{\gamma_I} \Gamma \Sigma_D \right) (I_2 - \Pi_I).
\] (A38)
The resolution of this equation for \( \Sigma_s \) yields

\[
\Sigma_s = (\gamma_P - \gamma_I) \left( \frac{\gamma_I}{s} + \frac{\gamma_P}{1-s} \right)^{-1} \Sigma_D \Pi_I - \frac{s}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) (I_2 - \Pi_I).
\]  

(A39)

Using the definition of \( \Gamma \), we obtain equation (A3).

Equations (A19) and (A39) jointly determine \( \Sigma_s \) and \( \Sigma_I \). The substitution of \( \Sigma_s \) from (A19) in (A39) yields an equation for \( \Sigma_I \):

\[
\Sigma_I = \frac{f_s}{f} \left[ (\gamma_P - \gamma_I) \left( \frac{\gamma_I}{s} + \frac{\gamma_P}{1-s} \right)^{-1} \Sigma_D + \frac{s}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) \right] \Pi_I
\]

\[
+ \left[ \Sigma_D + \frac{f_u}{f} \Sigma_u - \frac{s}{f} \frac{1}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) \right].
\]  

(A40)

To solve this equation, we use the following lemma.

**Lemma 1** Consider a linear space with a scalar product \((\cdot, \cdot)\) and denote by \( \Pi_x \) the orthogonal projection on vector \( x \). Also, let \( a \) and \( b \) be two vectors and assume that \((b, b) > 0\). Then, the equation for \( x \)

\[
x = \Pi_x a + b
\]  

(A41)

has the unique solution

\[
x = \frac{(a + b, b)}{(b, b)} b.
\]

**Proof of Lemma.** The application of the operator \( \Pi_x \) to both sides of equation (A41) gives \( x = \Pi_x a + \Pi_x b \), which together with the initial equation (A41) implies that \( \Pi_x b = b \). Hence, the vector \( b \) belongs to the subspace spanned by the vector \( x \), so \( x = \lambda b \), \( \lambda \in \mathbb{R} \). The substitution of this expression in equation (A41) yields \( \lambda b = \Pi_b a + b \), which implies \( \lambda = (\Pi_b a + b, b)/(b, b) \). Finally, \((\Pi_b a, b) = (a - (I - \Pi_b)a, b) = (a, b)\), where \( I \) is the identity operator, and this completes the proof. Q.E.D.

Equation (A40) has exactly the form of equation (A41) with \( \Sigma_I \) corresponding to \( x \). Hence,

\[
\Sigma_I = \frac{\left( f \Sigma_D + f_u \Sigma_u + f_s (\gamma_P - \gamma_I) \left( \frac{\gamma_I}{s} + \frac{\gamma_P}{1-s} \right)^{-1} \Sigma_D \right) \left( f \Sigma_D + f_u \Sigma_u - \frac{s f_s}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) \right)'}{\left( f \Sigma_D + f_u \Sigma_u - \frac{s f_s}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) \right) \left( f \Sigma_D + f_u \Sigma_u - \frac{s f_s}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) \right)'} \times \left( \Sigma_D + \frac{f_u}{f} \Sigma_u - \frac{s}{f} \frac{1}{h + sh_s} (h \Sigma_D + h_u \Sigma_u) \right).
\]
This is equation (A4). To derive the expression for $\mu_s$, we use equation (A33), which together with equation (A36) yields

$$
\frac{1 + \frac{\gamma_I}{2}}{2} \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \Sigma_D + \frac{1}{s} \Sigma_s \right)' + \frac{r - \beta}{\gamma_I} = \mu_D + \frac{1}{s} (\mu_s + \Sigma_s \Sigma_D').
$$

(Equation A42)

Equations (A21) and (A42) can be viewed as a system of linear equations for $r$ and $\mu_s$. Its solution is given by equations (A1) and (A2).

E. Differential equations for $f$ and $h$

Equation (A5) for the price-dividend ratio $f$ follows from (A18) after noting that $\mu_I = \Gamma \Sigma_D \Sigma_I'$ and $\Sigma_I$ is given by equation (A19). Equation (A6) for the wealth-consumption ratio $h$ is derived from equation (A27). Using the expression for $\mu_I$ from (A36) and noting that (A38) implies that

$$
\left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u + \Sigma_D + \frac{1}{s} \Sigma_s \right) (I_2 - \Pi_I) = 0,
$$

we get

$$
\left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)' - \frac{1}{\Sigma_I \Sigma_I'} \left( \frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I \Sigma_I' + \frac{h_u}{h} \Sigma_I \Sigma_u \right)^2
$$

$$
= \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)'
$$

$$
- \left( \Sigma_D + \frac{1}{s} \Sigma_s + \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \Pi_I \left( \Sigma_D + \frac{1}{s} \Sigma_s + \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)'
$$

$$
= \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)'
$$

$$
- \left( \Sigma_D + \frac{1}{s} \Sigma_s + \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left( \Sigma_D + \frac{1}{s} \Sigma_s + \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)'
$$

$$
= -2 \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)' - \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \Sigma_D + \frac{1}{s} \Sigma_s \right)'.
$$

This transformation allows us to eliminate the quadratic terms with $h_s$ and $h_u$ from equation (A27) and get equation (A6).

F. Optimal portfolios and numbers of shares

The optimal portfolio policy of the type I investors is given by equation (A26). The optimal portfolio of the type P investors stated in (A11) is derived from their utility optimization problem following exactly the same steps that are used to derive equation (A26) and the wealth-
consumption ratio function $h_P$, which appears in (A11), satisfies the following differential equation:

$$
\frac{1}{2}h_{Pss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{Pu}\Sigma_u\Sigma_u' + h_{Pu}\Sigma_s\Sigma_u' + h_{Ps}\mu_s + h_{Pu}\mu_u \\
+ \frac{1}{\gamma_P}\eta(h_{Ps}\Sigma_s + h_{Pu}\Sigma_u)' + \frac{1}{\gamma_P}\left(1 - \gamma_P\right)\eta'\eta + (1 - \gamma_P)r - \beta \right)h_P + 1 = 0, \quad (A43)
$$

where $\eta$ is the market price of risk from equation (A7). However, there is no need to solve this equation because $h_P$ can be found more easily from the market clearing conditions. Indeed, summing up equations (12) and (13), we get that $W_P + W_I = S_1 + S_2$. Using the definitions of the price-dividend ratios $S_i = f_iD_i$ and the wealth-consumption ratios $W_I = hC_I$, $W_P = h_PCP$, this equation can be rewritten as $(1 - s)h_P + sh = uf_1 + (1 - u)f_2$. After resolving it for $h_P$, we get (A12).

Finally, we derive the numbers of the shares $N_{Ii}$ and $N_{Pi}$. The definition of the portfolio weights implies that $N_{Ii}S_i = \omega_{Ii}W_I$ and $N_{Pi}S_i = \omega_{Pi}W_P$, so

$$
\frac{N_{Ii}}{N_{Pi}} = \frac{\omega_{Ii}W_I}{\omega_{Pi}W_P} = \frac{\omega_{Ii}C_Ih}{\omega_{Pi}CPh_P} = \frac{s\omega_{Ii}h}{(1 - s)\omega_{Pi}h_P}, \quad (A44)
$$

where the second equality uses the definition of the wealth-consumption ratio and the last equality uses the definition of the state variable $s$. Together with the market clearing condition $N_{Ii} + N_{Pi} = 1$, equation (A44) yields the expressions from (A13). The portfolio weights of the individual stocks in the index are proportional to $S_i/(S_1 + S_2)$, $i = 1, 2$, which in terms of the price-dividend ratios $f_1$ and $f_2$ are

$$
\frac{S_1}{S_1 + S_2} = \frac{uf_1}{uf_1 + (1 - u)f_2}, \quad \frac{S_2}{S_1 + S_2} = \frac{(1 - u)f_2}{uf_1 + (1 - u)f_2}.
$$

Taking into account that the type I investors allocate $\omega_I$ to the index, we get the expressions from (A14). Q.E.D.

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Figure 1. The equilibrium in the model with homogeneous trees (no indexing). This figure presents equilibrium variables in the model without indexing as functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \ 0]$, $\Sigma_{D2} = [0 \ 0.045]$, $\beta = 0.03$, $\gamma_I = 5$, $\gamma_P = 1$. 
Figure 2. The equilibrium in the model with heterogeneous trees (no indexing). This figure presents equilibrium variables in the model without indexing as functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01 \ 0]$, $\Sigma_{D2} = [0 \ 0.08]$, $\beta = 0.03$, $\gamma_I = 5$, $\gamma_P = 1$. 
Figure 3. The effect of indexing on equilibrium characteristics in the model with homogeneous trees. This figure shows how equilibrium changes due to indexing in an economy with homogeneous trees. All variables are functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 0]$, $\Sigma_{D2} = [0 0.045]$, $\beta = 0.03$, $\gamma_I = 5$, $\gamma_P = 1$. 
Figure 4. The effect of indexing on $\Sigma_{Q1,1}$ in the model with homogeneous trees. This figure presents the impact of indexing on the first component of the diffusion $\Sigma_{Q1} = \Sigma_{D1} + (f_{1s}/f_1)\Sigma_s + (f_{1u}/f_1)\Sigma_u$. All variables are functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \ 0]$, $\Sigma_{D2} = [0 \ 0.045]$, $\beta = 0.03$, $\gamma_l = 5$, $\gamma_P = 1$. 
Figure 5. The effect of indexing on equilibrium characteristics in the model with heterogeneous trees. This figure shows how equilibrium changes due to indexing in an economy with heterogeneous trees. All variables are functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01 \ 0]$, $\Sigma_{D2} = [0 \ 0.08]$, $\beta = 0.03$, $\gamma_I = 5$, $\gamma_P = 1$. 
Figure 6. Welfare analysis. This figure presents the certainty equivalent losses $CEL$, $CEL_1$, and $CEL_2$ in the models with homogeneous trees (upper row) and heterogeneous trees (bottom row) of an investor with $\gamma_I = 5$. All variables are functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. 
In the main body of the paper, we obtain numerical results under the assumption \( \gamma_P = 1 \) and \( \gamma_I = 5 \), that is, we assume that index investors are more risk averse than unconstrained investors. In the Internet Appendix, we investigate the sensitivity of our conclusions to this assumption. Specifically, we solve exactly the same model but set \( \gamma_P = 5 \) and \( \gamma_I = 1 \). Note that because now the type I investors have logarithmic preferences, they have a constant wealth-consumption ratio \( h = 1/\beta \). As a result, the system of equations that describes the equilibrium simplifies and instead of two differential equations (for the wealth-consumption ratio \( h \) and index price-dividend ratio \( f \) ) it contains only one of them (for the index price-dividend ratio \( f \) ). Nevertheless, the system does not have an analytical solution, and as in Section 3 we solve it numerically.

Because in the unconstrained case the switch in the coefficients of risk aversion is equivalent to relabeling the agents, the graphs of equilibrium variables in the unconstrained economy can be obtained from the graphs in Figures 1 and 2 by flipping them around the plane \( s = 1/2 \). Therefore, we reproduce only the graphs for the changes in the equilibrium variables, which are analogs of the graphs in Figures 3 and 5. The results are presented in Figures IA.1 and IA.2 for the economies with homogeneous and heterogeneous trees, respectively.

FIGURES IA.1 AND IA.2 ARE HERE

Consider first the effect of indexing on the equilibrium characteristics in the model with homogeneous trees. The graphs of the ratios \( \Delta N_{P_1}/N_{P_1}^{unc} \) and \( \Delta N_{P_2}/N_{P_2}^{unc} \) in Figure IA.1 unambiguously show that in the economy with the indexing constraints the more risk-averse investors (now they are the type P investors) hold more shares of each of the risky assets than in the unconstrained economy. Because the type P investors hold fewer shares of the risky assets than the less risk-averse type I investors, indexing makes the distribution of the shares across the investors more uniform. As a result, risk sharing among the investors is reduced, and this is exactly what happens in the specification with \( \gamma_P = 1 \) and \( \gamma_I = 5 \) considered in the main body of the paper. As argued in
Section 3.4.1, the reduction in risk sharing produced by indexing is responsible for lower volatility of the index (the excess volatility resulting from risk sharing is lower) and the lower risk-free rate (leveraged less risk-averse investors borrow less), and the same effects are observed in Figure IA.1.

The graph for $\Delta(N_{I2}/N_{I1})$ shows that because of indexing the type I investors hold relatively more shares of a larger tree (e.g., the first tree when $u > 1/2$), so the more risk-averse type P investors hold relatively more shares of a smaller tree. Therefore, up to relabeling of the investors, the impact of indexing on the composition of the investors’ optimal portfolios is the same as in the model with $\gamma_P = 1$ and $\gamma_I = 5$. As a result, the impact of indexing on betas and volatilities is also the same: it typically increases (decreases) the volatilities and betas of larger (smaller) stocks.

One of the most interesting results reported in Section 3 is an ambiguous effect of indexing on the correlation between stock returns: the latter can increase in some states and decrease in the others. The graph for $\Delta \rho$ in Figure IA.1 shows that the same observation holds when the less risk-averse investors are indexers. Recall that the impact of indexing on the correlation results from the reduction in risk sharing, which decreases the correlation, and the impossibility for the constrained investors to trade the stocks individually, which increases the correlation. Figure IA.1 implies that the first effect dominates when the trees have comparable cash flows, but the second effect dominates when the trees have different sizes.

Finally, Figure IA.1 shows that all effects of indexing are more pronounced when $s$ is close to 0, that is, when the more risk-averse investors dominate the economy. This result replicates the findings from Section 3, in which all effects are stronger when $s > 0.5$, that is, when the more risk-averse type I investors dominate the economy.

The comparison of Figures 3 and IA.1 also reveals an interesting difference between them. Specifically, in the economies with $\gamma_I = 1$, $\gamma_P = 5$ and $\gamma_I = 5$, $\gamma_P = 1$ indexing has opposite effects on the market prices of risk: they increase (decrease) when less (more) risk-averse investors are constrained. As a results, indexing also differently affects the valuations (price-dividend ratios) of the individual stocks in those economies: the price-dividend ratio of a small (large) stock increases when $\gamma_I = 5$, $\gamma_P = 1$ ($\gamma_I = 1$, $\gamma_P = 5$).

The effect of indexing on the equilibrium characteristics in the model with heterogeneous
trees and switched coefficients of risk aversion is shown in Figure IA.2, which is an analog of Figure 5 from the paper. Note that instead of the relative changes $\Delta N_{P1}/N_{P1}^{unc}$ and $\Delta N_{P2}/N_{P2}^{unc}$ depicted in Figure 5, we report the absolute changes $\Delta N_{P1}$ and $\Delta N_{P2}$ because in some states of the unconstrained economy the risk-averse type P investors do not hold the shares of the volatile second stock at all ($N_{P2}^{unc} = 0$).

Again, Figures 5 and IA.2 share many similarities. In particular, we observe that independently from whether the more or less risk-averse investors are constrained, indexing decreases the risk-free rate and market volatility and tends to increase (decrease) the volatility and beta of the larger (smaller) stock, although for the second stock the effect of indexing on the volatility is ambiguous. Also, indexing tends to decrease the correlation between stock returns. Finally, the magnitudes of the effects in the economies with $\gamma_I = 1$, $\gamma_P = 5$ and $\gamma_I = 5$, $\gamma_P = 1$ are comparable and much larger than in the economy with homogeneous trees.

As in the case with homogeneous trees, the switch in the coefficients of risk aversion changes how indexing affects the market prices of risk: it typically increases $\eta_1$ and decreases $\eta_2$ when the more risk-averse investors implement the indexing strategy but decreases $\eta_1$ and increases $\eta_2$ when the less risk-averse investors are indexers. A similar difference is observed for the price-dividend ratios $f_1$ and $f_2$.

Overall, the comparison of the constrained economies with $\gamma_I = 1$, $\gamma_P = 5$ and $\gamma_I = 5$, $\gamma_P = 1$ shows that our conclusions about the impact of indexing on the volatilities of returns, on the risk-free rate, and on the correlation between returns are insensitive to whether the more risk-averse or less risk-averse investors are assumed to be constrained.
Figure IA.1. The effect of indexing on equilibrium characteristics in the model with homogeneous trees. This figure shows how equilibrium changes due to indexing in an economy with homogeneous trees. All variables are functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \ 0]$, $\Sigma_{D2} = [0 \ 0.045]$, $\beta = 0.03$, $\gamma_I = 1$, $\gamma_P = 5$. 
Figure IA.2. The effect of indexing on equilibrium characteristics in the model with heterogeneous trees. This figure shows how equilibrium changes due to indexing in an economy with heterogeneous trees. All variables are functions of the consumption share $s$ of the type I investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. The model parameters are as follows: $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01\ 0]$, $\Sigma_{D2} = [0 \ 0.08]$, $\beta = 0.03$, $\gamma_I = 1$, $\gamma_P = 5$. 