Cumulative Prospect Theory and Underdiversification

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Abstract

We show that an investor whose preference is described by cumulative prospect theory (CPT, Tversky and Kahneman [1992]) optimally holds a concentrated risky portfolio, composed only of the risky assets with the highest CPT-ratio (defined as a subjective performance measure). In financial market equilibrium, the CPT investor still underdiversifies and gives rise to CAPM anomalies such as a flatter security market line.

Keywords: cumulative prospect theory, portfolio choice, underdiversification, equilibrium, CAPM anomaly

JEL classification: D81, G11, G12

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1 Introduction

It is widely documented that individual investors tend to hold under-diversified portfolios. Unlike the proverb “do not put all your eggs in one basket”, many investors fail to effectively diversify idiosyncratic risk, which is suboptimal according to classical theories, such as the mean-variance theory (Markowitz 1952).

Underdiversification can be defined in different ways. A natural definition would be to hold a small number of risky assets, as described by Barber and Odean (2000). In Goetzmann and Kumar (2008), underdiversification is also measured by portfolio variance and by the deviation from the market portfolio (portfolio weights). Gaudecker (2015) uses the return loss, the distance between the expected return of an investor’s portfolio and the expected return on the efficient frontier with the same level of risk. Liu (2014) points out that using the number of assets as the measure of diversification has the advantage that there is no estimation error. In this paper, we also focus on the number of assets in an investor’s portfolio.

Non-standard preferences have long been conjectured to be potential explanations for underdiversification, for example, rank-dependent preferences in Polkovnichenko (2005) and cumulative prospect theory in Barberis and Huang (2008). However, a theoretical discussion of the link between those preferences and underdiversification is still missing in the literature. In this paper, we take up the task.

The purpose of the paper is to demonstrate in a theoretical way that cumulative prospect theory (CPT, Tversky and Kahneman 1992) can explain
underdiversification. We show that an investor whose preference is described by CPT optimally holds a concentrated risky portfolio. This portfolio is composed only of the risky assets with the highest CPT-ratio, which we define as a subjective performance measure. The result holds both in partial equilibrium and in general equilibrium. In a financial market with the presence of both a CPT investor and a mean-variance investor, the market portfolio becomes mean-variance inefficient in equilibrium and CAPM anomalies occur.

On the empirical side, Barber and Odean (2000) find in their dataset that investors on average hold four stocks. Polkovnichenko (2005) divides investors in his sample into four cohorts, according to values of their financial assets. From 1983 to 2001, while the wealthiest cohort held between six to fifteen stocks, the least wealthy two cohorts held only one to two stocks. In Goetzmann and Kumar (2008), the average number of stocks increases from four to seven during their sample period from 1991 to 1996. Measured by portfolio variance, diversification also improved in this period, however, when measured by stocks correlation, no evidence of diversification improvement is found. Gaudecker (2015) finds that underdiversification is related to an investor’s financial literacy and contacts for advice. This result may suggest behavioral reasons for underdiversification.

On the theoretical side, Uppal and Wang (2003) show that ambiguity about returns’ joint distribution and different ambiguity levels about returns’ marginal distributions can lead to underdiversification. Nieuwerburgh and Veldkamp (2010) demonstrate that investors who can collect information before investment may hold underdiversified portfolios. Roche et al. (2013)
show how investors with labor income and facing financial constraints hold concentrated portfolios. Liu (2014) argues that solvency constraint can lead to underdiversification when the investor is not wealthy.

Cumulative prospect theory (and its predecessor “prospect theory”, Kahneman and Tversky, 1979) is probably the most discussed behavioral preference in finance. In contrast to the standard expected utility theory, CPT has the following features: (1) A CPT investor does not evaluate her utility from absolute terms, e.g., absolute wealth. Instead, she cares about relative terms, e.g., the deviation of her wealth from a reference wealth level. (2) There are both a risk-averse and a risk-seeking parts in her preference, split- ted by her reference point. (3) There is a loss-averse feature in her preference. (4) Probabilities are distorted in her decision making process.

The pricing implications of CPT have been investigated in, e.g., Barberis et al. (2001) and Barberis and Huang (2008). Its implications for investors’ trading behavior are discussed in e.g., Henderson (2012). The portfolio choice problem of a CPT investor are discussed in, e.g., Kouwenberg et al. (2000), Bernard and Ghossoub (2010) and He and Zhou (2011). A few literature have already discussed the asset allocation implications of CPT. Pirvu and Schulze (2012) find that two-fund separation holds for CPT investors when risky assets follow multivariate elliptical distributions. Through portfolio choice simulations, Polkvnichenko (2005) show that rank-dependent preferences explain underdiversification better than expected utility theory does. Barberis and Huang (2008) point out that CPT is a possible cause for underdiversification. This conjecture is basically confirmed in our analysis.

Our discussion proceeds as follows: First discuss the portfolio choice prob-
lem of a CPT investor and show that this portfolio problem can boil down to constructing a risky portfolio with the highest possible CPT-ratio. We further show that a risky portfolio with the highest CPT-ratio can indeed be formed only by a few (down to one) risky assets. Adding assets with low CPT-ratio into a high-CPT-ratio portfolio does not make the latter more attractive to the investor. We also carried out the discussion in financial market equilibrium with the presence of a CARA investor and a CPT investor, where the CPT-ratio is endogenous. Finally, we calculate the slope of the security market line in a financial market populated by both a mean-variance investor and a CPT investor. Our results lead to the following implications:

First, the reference point plays an important role in the portfolio choice of the CPT investor. When the reference return of the risky portfolio, $R_B$, goes to zero, i.e., the investor’s reference is to lose (almost) all her risky investment, then the CPT-ratio of all risky assets go to infinity and the investor is indifferent among them. Consequently, she will hold all risky assets, i.e., full diversification. In this case, the risk-seeking part of the CPT preference vanishes and the investor behaves similarly as a classical risk-averse investor. When $R_B$ goes to infinity, i.e., the investor desires a too large return from risky investment, then the optimal fraction of her initial wealth invested in risky assets, $\theta_i^*$, goes to zero, the CPT investor hardly conducts any risky investment, due to loss aversion.

Second, covariance is not a criterion in the CPT investor’s portfolio choice. She only considers the individual performances of the assets and does not care about their dependent structure. Consequently, in a financial market where CPT investors are present, covariance risk is less compensated and so is beta.
(the covariance between an asset and the market portfolio, divided by the variance of the latter).

CAPM predicts that the slope of the security market line (plotting the expected return of assets against their beta) is the expected excess return of the market portfolio (the expected return of the market portfolio minus the return of the risk-free asset). However, empirical evidences reject this prediction and show that the security market line is in fact flatter than CAPM predicts, i.e., its slope is smaller than the expected excess return of the market portfolio, see, e.g. Fama and French (1992). Our analysis of the CPT investor offers an explanation for it: because beta is less compensated when CPT investors are present, the security market line in a market populated by both mean-variance investors and CPT investors is flatter than that in a market populated only by mean-variance investors – a typical CAPM market.

Third, Barberis and Huang (2008) point out that a CPT investor may "willingly take an undiversified position in a positively skewed stock in order to add skewness to his portfolio." We find, however, no unambiguous link between the skewness of a risky asset and its CPT-ratio, the latter of which characterizes the investor’s portfolio choice in our analysis. In our case, it is the high CPT-ratio for which an investor willingly takes an undiversified position in stocks. The relationship between the skewness and the CPT-ratio turns out to be complicated. High skewness and high CPT-ratio do not necessarily correspond with one another.

The rest of the paper is organized as follows: Section 2 introduces the CPT-ratio and discusses the portfolio choice problem of a CPT investor. Section 3 shows in a partial equilibrium setting that CPT leads to underdi-
versification. Section 4 discusses the same question in a general equilibrium setting and investigates the pricing implications of CPT. Section 5 concludes.

2 Portfolio choice of a CPT investor

Consider a financial market in discrete time and single period, with time $0$ and time $1$. There is a risky portfolio with return $R_P$, which is formed by some risky assets. There is also a risk-free asset with return $R_F$ in the market.

The CPT investor’s preference is given by,

$$U_b = U_+(X) - U_-(X),$$

where $X = W - B$, with $W$ the investor’s wealth at time $1$ and $B$ the investor’s reference wealth level; and

$$U_+(X) = \int_0^{+\infty} u_+(x) d[-w_+(1 - F(x))],$$
$$U_-(X) = \int_{-\infty}^0 u_-(x) d[w_-(F(x))],$$

where $F(x)$ is the cumulative distribution function (CDF) of $X$;

$$u_+(x) = x^\alpha,$$
$$u_-(x) = kx^\beta;$$

with $k > 1$ (loss aversion parameter) and $w_+(\cdot)$, $w_-(\cdot)$ are two non-negative and strictly increasing functions (called “probability weighting functions”),
for which we need not to assume explicit functional forms.

In Tversky and Kahneman (1992), they are given by

\[
\begin{align*}
    w_+(x) &= \frac{x^{\gamma_1}}{(x^{\gamma_1} + (1-x)^{\gamma_1})^{1/\gamma_1}}, \\
    w_-(x) &= \frac{x^{\gamma_2}}{(x^{\gamma_2} + (1-x)^{\gamma_2})^{1/\gamma_2}},
\end{align*}
\]

with \( \gamma_1, \gamma_2 > 0 \). Note that with the setup (3) and (4), if \( R_P \) is log-normally distributed, \( U_b(X) \) is finite and continuous on its domain. See He and Zhou (2011).

**Assumption 2.1.** \( \alpha < \beta \).

In the literature, different estimations give different values for \( \alpha \) and \( \beta \). In Tversky and Kahneman (1992), they are the same 0.88. In Abdellaoui (2000), \( \alpha = 0.89 \) and \( \beta = 0.92 \). In Rieger et al. (2015), the result from a global survey also implies that \( \alpha \) are smaller than \( \beta \), in almost all countries in their sample. Furthermore, when \( \alpha < \beta \), the portfolio choice problem (6)-(7) later will have finite solution, i.e. it is well-posed.\(^1\)

**Assumption 2.2.** \( X \) has finite expectation and variance.

\[ E[X] < +\infty, \quad \text{Var}[X] < +\infty. \]

Let us first rewrite the CPT preference in another way.

\(^1\)By assuming \( \alpha < \beta \), our paper distinguishes itself from the literature, e.g., Barberis and Huang (2008), where \( \alpha = \beta \) is assumed and CAPM is shown to hold with normally distributed asset returns. If the assumption of normality is relaxed, the author show that the equilibrium have nonunique global optima.
Lemma 2.3. Alternative form of CPT

\[ U_+(X) = \int_{0}^{+\infty} w_+(1 - F(x))du_+(x), \]
\[ U_-(X) = -\int_{-\infty}^{0} w_-(F(x))du_-(x). \] (5)


The portfolio choice problem of the investor is,

\[ \max_{\theta_b} U_b, \] (6)

with the budget constraint,

\[ W = (\theta_b R_P + (1 - \theta_b) R_f) W_0, \] (7)

where \( W_0 \) is the investor’s wealth at time 0 and \( \theta_b \) is the fractions of the investor’s initial wealth invested into the risky portfolio at time 0.

For explorative simplicity, we impose short-sale constraint for the investor, which is not uncommon in real life, especially with individual investors.

Assumption 2.4. Short selling is impossible for the investor, i.e., \( \theta_b \geq 0 \).

Let \( R_B > 0 \) denote the investor’s (exogenous) reference return of the risky portfolio, namely, when the investor decides to invest into a risky portfolio at time 0, she has a reference level of return \( R_B \) for it. Accordingly, her
reference level for final wealth is determined by

\[ B = (\theta_b R_B + (1 - \theta_b) R_f) W_0. \]  

(8)

Then, \( X = W - B = \theta_b (R_P - R_B) \). This introduction of \( R_B \) is both an economic and a technical consideration, so that later in the discussion we will have elegant expressions. Natural choices for \( R_B \) are 1, \( R_f \) or \( E[R_P] \) (the expected return of the risky portfolio). Unless clearly stated, we assume an investor’s \( R_B \) is constant and asset-independent, i.e., \( R_B \) is the same for all risky assets / portfolios.

Now we are ready to introduce the CPT-ratio.

**Definition 2.5. (CPT-ratio)**

*Define the CPT-ratio \( \Omega \) of a return \( R \) by*

\[ \Omega(R) = \frac{\gamma(R)}{\rho(R)}, \]  

(9)

*with*

\[ \gamma(R) = (U_+(R - R_B)^\beta) = \left( \int_{R_B}^{+\infty} (x - R_B)^\alpha d[-w_+(1 - F(x))] \right)^\beta, \]

\[ \rho(R) = (U_-(R - R_B)^\alpha) = \left( k \int_0^{R_B} (R_B - x)^\beta d[w_-(F(x))] \right)^\alpha, \]  

(10)

*where \( F(x) \) is the CDF of \( R \).*

We see that the CPT-ratio \( \Omega(R) \) is a performance measure, it can also be seen as a generalized partial moments ratio, similar to Farinelli and Tibiletti (2008). Furthermore, it is investor-specific: different investors may have dif-
different decision parameters $\alpha$, $\beta$ and $k$, different probability weighting functions $w_+$ and $w_-$, as well as different reference return $R_B$. The function $\gamma(R)$ and $\rho(R)$ can respectively be seen as a subjective reward measure and a subjective risk measure for an asset / portfolio with return $R$.

**Remark 2.1. (CPT-ratio and moments of return)**

Assume at the moment that $w_+(x) = w_-(x) = x$, i.e., no probability weighting.

Let $f(x)$ denote the PDF of $R$. According to Newton’s binomial theorem, we have,

$$
\gamma(R) = \int_{R_B}^{+\infty} (x - R_B)^\alpha f(x)dx = \sum_{i=0}^{\infty} \binom{\alpha}{i} UPM_{\alpha-i,R_B}(-1)^i R_B^i
$$

$$
= UPM_{\alpha,R_B} - UPM_{\alpha-1,R_B} R_B + \frac{\alpha(\alpha-1)}{2!} UPM_{\alpha-2,R_B} R_B^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} UPM_{\alpha-3,R_B} R_B^3 + \ldots,
$$

where $UPM_{i,R_B}$ is the $i$-th upper partial moment of $R$ above $R_B$, given by

$$
UPM_{i,R_B} = \int_{R_B}^{+\infty} x^i f(x)dx.
$$

Similarly,

$$
\rho(R) = \int_{0}^{R_B} (R_B - x)^\beta f(x)dx = \sum_{i=0}^{\infty} \binom{\beta}{i} R_B^{\beta-i} (-1)^i LPM_{i,R_B}
$$

$$
= R_B^\beta - \beta R_B^{\beta-1} LPM_{1,R_B} + \frac{\beta(\beta-1)}{2!} R_B^{\beta-2} LPM_{2,R_B} - \frac{\beta(\beta-1)(\beta-2)}{3!} R_B^{\beta-3} LPM_{3,R_B} + \ldots,
$$
where \( LPM_{i,R_B} \) is the \( i \)-th lower partial moment of \( R \) below \( R_B \), given by

\[
LPM_{i,R_B} = \int_0^{R_B} x^i f(x) dx. \tag{14}
\]

Take log-normal distribution for example. Let \( R \) be log-normally distributed, \( R \sim \log \mathcal{N}(\mu, \sigma^2) \). Namely, the expectation, the variance and the skewness of \( R \) are \( e^{\mu + \frac{\sigma^2}{2}}, (e^{\sigma^2} - 1)e^{2\mu + \sigma^2} \) and \( (e^{\sigma^2} + 2)\sqrt{e^{2\sigma^2} - 1} \), respectively.

The \( i \)-th moment (about zero) of \( R \) is given by

\[
\mu_i = \int_0^{+\infty} x^i f(x) dx = \exp \left( i\mu + \frac{i^2}{2}\sigma^2 \right). \tag{15}
\]

The \( i \)-th lower partial moment of \( R \) below \( R_B \) is given by

\[
LPM_{i,R_B} = \int_0^{R_B} x^i f(x) dx = \mu_i \Phi \left( \frac{\log R_B - \mu - i\sigma}{\sigma} \right), \tag{16}
\]

where \( \Phi(\cdot) \) is the CDF of a standard normal variable.

The \( i \)-th upper partial moment of \( R \) above \( R_B \) is given by

\[
UPM_{i,R_B} = \int_{R_B}^{+\infty} x^i f(x) dx = \mu_i \Phi \left( \frac{\mu + i\sigma - \log R_B}{\sigma} \right). \tag{17}
\]

We can see, the relationships between \( \sigma \) (or \( \mu \)) and \( \gamma(R) \), \( \rho(R) \) are complicated and not unambiguous. Because the skewness of \( R \) is an increasing function of \( \sigma \), the relationship between \( \Omega(R) \) and the skewness of \( R \) are also not unambiguous. We are unable to tell if a high skewness corresponds with a high CPT-ratio. This result should also hold for other common distributions.

In next proposition, we solve the portfolio choice problem of the CPT.
Proposition 2.6. The investor’s optimal portfolio \( \theta_b^* \) and value function (maximal utility function) \( V_b \) are given by

\[
\theta_b^* = W_0^{-1} \left( \frac{\alpha U_+(R_P - R_B)}{\beta U_-(R_P - R_B)} \right)^{\frac{1}{\beta - \alpha}},
\]

(18)

\[
V_b = U_b(\theta_b^*) = \left( \frac{\alpha}{\beta} \right)^{\alpha - \alpha} - \left( \frac{\alpha}{\beta} \right)^{\beta - \alpha} \left( \frac{(U_+(R_P - R_B))^\beta}{U_-(R_P - R_B)} \right)^{\frac{1}{\beta - \alpha}}
\]

(19)

\[
= \left( \frac{\alpha}{\beta} \right)^{\alpha - \alpha} - \left( \frac{\alpha}{\beta} \right)^{\beta - \alpha} \Omega(R_P)^{\frac{1}{\beta - \alpha}}
\]

Proof. See Appendix A2.

Because \( \alpha < \beta \), we can see that the investor’s maximal utility \( V_b \) is increasing in \( \Omega(R_P) \). Her portfolio choice problem thus boils down to constructing a risky portfolio with the highest possible CPT-ratio \( \Omega(R_P) \). Clearly, the CPT-ratio \( \Omega(R_P) \) depends on the reference return \( R_B \). When \( R_B \) goes to infinity, \( U_+ \), the risk-averse part of the CPT preference vanishes and \( \Omega(R_P) \) (as well as \( \theta_b^* \)) goes to zero; when \( R_B \) goes to zero, then \( U_- \), the risk-seeking part of the preference vanishes and \( \Omega(R_P) \) (as well as \( \theta_b^* \)) goes to infinity.

In other words, if the investor desires a too large return from the risky investment, she can hardly find any attractive risky asset and consequently hardly conducts any risky investment. This can be explained by the loss-averse feature of the CPT preference; if the investor compares her investment result with the case of losing all risky investment (zero reference return), then
any risky asset is very attractive to her and she will invest as much as possible into risky assets.

**Remark 2.2. (Wealth effect)**

It appears that the optimal dollar amount invested into risky assets, $\theta^*_bW_0 = \left( \frac{\alpha \gamma(R_P)^{\frac{1}{\beta}}}{{\beta \rho(R_P)^{\frac{1}{\alpha}}}} \right)^{\frac{1}{\beta - \alpha}}$, does not depend on the investor’s wealth level $W_0$, a similar result as obtained from a CARA investor. However, wealth effect may arise, e.g., through the reference return $R_B$. Assume that $R_B$ is decreasing in $W_0$, i.e., the wealthier the CPT investor is, the lower the return she desires from investing into risky assets. Because $\frac{U_1(R_P - R_B)}{U_2(R_P - R_B)}$ is decreasing in $R_B$ in (18), we may have that $\theta^*_bW_0$ is increasing in $W_0$, i.e., the wealthier the CPT investor is, the more she invests into risky assets, more like a CRRA investor. We find, however, no empirical literature that discusses the relationship between the wealth level and the reference level of a CPT investor.

### 3 Underdiversification

Now we turn to the asset allocations part of the investor’s portfolio problem. Before we proceed, let us borrow the definition of underdiversification adopted in [Liu (2014)].

**Definition 3.1. (Underdiversification)**

A risky portfolio is underdiversified, if it is only a proper subset of available risky assets.

Consider $n$ risky assets in the market, Asset 1, ..., Asset $n$. Denote their returns by $R_1, R_2, \ldots, R_n$ and the CDF of their returns by $F_1, F_2, \ldots, F_n$. 
Their CPT-ratios are \( \Omega(R) = (\Omega(R_1), \Omega(R_2), \ldots, \Omega(R_n))^T \). The return of a risky portfolio is \( R_P = \lambda^T R \), where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \in \mathbb{R}_+^n \) is the weight of each risky asset in the risky portfolio, with \( \sum_{i=1}^n \lambda_i = 1 \), \( R = (R_1, R_2, \ldots, R_n)^T \) and the superscript \( T \) denotes the transpose.

Next lemma discusses whether it is normal that all risky assets have the same CPT-ratio.

**Lemma 3.2.**

\[
\Omega(R_1) = \Omega(R_2) = \ldots = \Omega(R_n), \text{ for any } R_B \in (0, +\infty), \tag{20}
\]

if and only if

\[
F_1 = F_2 = \ldots = F_3, \text{ almost everywhere.} \tag{21}
\]

*Proof.* See Appendix A3. \[\square\]

While it may hold for one investor that \( \Omega(R_1) = \Omega(R_2) = \ldots = \Omega(R_n) \), it is unlikely to hold for other investors. We see, it is an extreme case that all risky assets have the same CPT-ratio, if we allow \( R_B \) to take any value from \((0, +\infty)\). Without loss of generality, let

\[
\Omega(R_1) \geq \Omega(R_2) \geq \ldots \geq \Omega(R_n), \tag{22}
\]

and at least one inequality is strict.

The followings lemma compares the CPT-ratio of a portfolio with the CPT-ratio of each asset in the portfolio.
Lemma 3.3. If $\Omega(R_i) < \Omega(R_j)$, $i \neq j$, then

$$\Omega(\lambda_i R_i + (1 - \lambda_i) R_j) < \Omega(R_j), \text{ for any } 0 < \lambda_i \leq 1.$$  (23)

Proof. See Appendix A4.

Hence, the CPT-ratio of any two-assets portfolio is no larger than the maximum of each asset’s CPT-ratio in the portfolio. Consequently, Asset 1 should always be included in the CPT investor’s portfolio, because it has the highest CPT-ratio.

Now we have our main proposition in this section.

Proposition 3.4. For any $\lambda \in \mathbb{R}^n_+$,

$$\Omega(R_1) \geq \Omega(R_P).$$  (24)

This follows directly from Lemma 3.3. Equation (23) can be simply extended to portfolios with more than two assets by letting Asset $j$ be a portfolio instead of a single asset. Hence, the CPT-ratio of any portfolio is no larger than the maximum of each asset’s CPT-ratio in the portfolio. Adding new assets with low CPT-ratio into a portfolio whose CPT-ratio is already high, does not increase the latter’s CPT-ratio. Consequently, holding only Asset 1 can already give the highest CPT-ratio among all portfolios. The proposition has the following implications:

First and obviously, we see that a CPT investor does not necessarily hold all risky assets in the market. More likely, she will hold a concentrated portfolio. Her optimal portfolio can even include only one single risky asset,
when the first inequality in (22) is strict.

Second, it is easy to see when $R_B$ goes to zero, i.e., the investor’s reference is to lose (almost) all her risky investment, then the CPT-ratio of all risky assets go to infinity and the investor is indifferent among them. Consequently, she will hold all risky assets, i.e., full diversification. Note that in this case, the risk-seeking part of the CPT preference vanishes and the investor behaves similarly to a classical risk-averse investor. The fraction of her initial wealth in risky assets, $\theta^*_b$, also goes to infinity in this case, i.e., the investor allocates as much as possible her wealth into risky assets. When $R_B$ goes to infinity, $\theta^*_b$ goes to zero, the CPT investor hardly conducts any risky investment, due to loss aversion.

Third, covariance or correlation is not a criterion for the investor to include a risky asset in her portfolio. She only considers the individual performances of the assets and does not care about their dependent structure. This may have direct pricing implications which we discuss in next section.

Furthermore, as can be seen from its definition in (9) and (10), the CPT-ratio $\Omega(R)$ depends on the distribution of $R$. When a substantial amount of investors in the market underdiversify, say, hold only Asset 1, then the price of Asset 1 would be driven up and its expected return would be brought down. Consequently, $\Omega(R_1)$ may change and Asset 1 may become not as attractive as it was to the investors. In next section, we will discuss whether underdiversification can occur in financial market equilibrium.

Remark 3.1. (Portfolio choice of a CARA investor)

Consider a constant absolute risk aversion (CARA) investor as a “ratio-
nal” counterpart, whose utility is given by

$$U_r = -e^{-\delta W}$$

$$= -e^{-\delta (\pi^T(R-1R_f)+R_f)W_0},$$

(25)

where $\delta > 0$, 1 is a length-$n$ column vector of ones and $\pi_r = (\pi_{r,1}, \pi_{r,n}, \ldots, \pi_{r,n})^T \in \mathbb{R}^n_+$ is the investor’s fraction of initial wealth invested into each risky asset with $\pi_r 1 \leq 1$.

Let the return of the risky assets be independently Gamma distributed, $R_i \sim \Gamma(\rho_i, \upsilon_i)$, with $\rho_i > 0$, $\upsilon_i > 0$, $i = 1, 2, \ldots, n$. Namely, the PDF of $R_i$ is given by

$$f_i(x) = \frac{x^{\rho_i - 1}e^{-\frac{x}{\upsilon_i}}}{\upsilon_i^{\rho_i} \Gamma(\rho_i)},$$

(26)

where $\Gamma(\rho_i)$ means the gamma function evaluated at $\rho_i$. We have $E[R_i] = \rho_i \upsilon_i$ and $\text{Var}[R_i] = \rho_i \upsilon_i^2$.

Then,

$$\max_{\pi_r} E[U_r] = \min_{\pi_r} E[e^{-\delta (\pi^T(R-1R_f)+R_f)W_0}]$$

$$= e^{-\delta R_fW_0} \min_{\pi_r} e^{\delta \pi^T R_f W_0} E\left[e^{-\delta \pi^T R W_0}\right]$$

$$= e^{-\delta R_fW_0} \prod_{i=1}^n \min_{\pi_{r,i}} e^{\delta \pi_{r,i} R_f W_0} E\left[e^{-\delta \pi_{r,i} R_i W_0}\right]$$

$$= e^{-\delta R_fW_0} \prod_{i=1}^n \min_{\pi_{r,i}} e^{\delta \pi_{r,i} R_f W_0} \frac{\rho_i \upsilon_i}{(1 + \delta \pi_{r,i} W_0 \upsilon_i)^{\rho_i}},$$

(27)

because it can be verified by calculating the integration from the expectation, that for $a > 0$,

$$E\left[e^{-aR_i}\right] = \frac{E[R_i]}{(1 + a\upsilon_i)^{\rho_i}} = \frac{\rho_i \upsilon_i}{(1 + a\upsilon_i)^{\rho_i}}.$$
Namely,

\[ \max_{\pi_r} E[U_r] = e^{-\delta R_f W_0} \prod_{i=1}^{n} \rho_i \min_{\pi_{r,i}} \frac{e^{\delta \pi_{r,i} R_f W_0}}{(1 + \delta \pi_{r,i} W_0 \nu_i) \rho_i}. \]  

(29)

The first-order conditions imply

\[ \pi_{r,i} = \frac{\rho_i \nu_i - R_f}{\delta \nu_i W_0 R_f}. \]  

(30)

In other words, if \( E[R_i] > R_f \) for all \( i \), a long-only CARA investor under this setting will buy all risky assets in the market, i.e., she will fully diversify.  

4 Underdiversification in equilibrium

This section discusses the investor’s diversification problem in a general equilibrium setting, where the CPT-ratio is endogenously determined in the financial market. Our discussion based on CPT in last section suggests that if \( \Omega(R_i) < \Omega(R_P) \), then Asset \( i \) will not be included in the investor’s portfolio. However, when \( \Omega(R_i) = \Omega(R_P) \), CPT does not tell us how the investor makes decisions. Let us make the following assumptions before we proceed.

Assumption 4.1. The CPT investor will add a new Asset \( i \) into her portfolio, once \( \Omega(R_i) = \Omega(R_P) \).

\footnote{We choose the CARA-Gamma combination for tractability reasons. The same setup is also adopted in [Liu (2014)]. Gamma distributions have similar properties as lognormal distributions, e.g. its support is \((0, +\infty)\). Other standard preference-distribution combinations, e.g., CRRA-lognormal, can also lead to full diversification, however, closed-form solutions in discrete time are unlikely.}
**Assumption 4.2.** Risky assets in the CPT investor’s portfolio are equally weighted.

Because when $\Omega(R_i) = \Omega(R_j)$, a CPT investor is indifferent between Asset $i$ and Asset $j$, we thus think these two assumptions are plausible, especially in aggregate.

### 4.1 Financial market equilibrium

Let there be two (classes of) investors in the market, a CARA investor (subscripted with $r$) and a CPT investor (subscripted with $b$)\(^3\), each endowed with initial wealth $W_{r,0} = \eta W_0$ and $W_{b,0} = (1 - \eta) W_0$, where $W_0$ is the aggregate initial wealth in the market. Thus their respective fractions in the market at time 0 are represented by $\eta$ and $1 - \eta$, $\eta \in [0, 1]$.

There are $n$ risky assets, Asset 1, ..., Asset $n$. The per-share terminal payoff of Asset $i$, $D_i$, follows Gamma distribution, $D_i \sim \Gamma(\rho_i, \upsilon_i)$. $D_i$ is independent from $D_j$ for all $j \neq i$. There is a risk-free asset, whose return $R_f$ is constant.

For each asset $i$, there are totally $\bar{X}_i$ shares available in the market, with $\bar{X}_1 = \bar{X}_2 = \ldots = \bar{X}_n = 1$. Their time-0 prices are denoted by $S = (S_1, S_2, \ldots, S_n)^T$, which are to be determined in equilibrium. Denote the CDF of $D_i$ by $F_{D,i}$. Assume that $\rho_i$ and $\upsilon_i$ are large enough, such that it always holds $\rho_i \upsilon_i > R_f S_i$, namely $E[R_i] > R_f$, $i = 1, \ldots, n$.

At time 0, the two investors each make portfolio decision to maximize

\(^3\)The CARA investor functions mainly to guarantee the existence of the equilibrium for all risky assets, in case the CPT investor underdiversifies in equilibrium, i.e., holds only some of the risky assets.
their utilities derived from their terminal wealths:

$$\max_{\pi_r} \mathbb{E}[U_r(W_r)],$$

$$\max_{\pi_b} U_b(W_b),$$

where $U_r$ and $U_b$ are given in previous sections; subject to the budget constraint

$$W_r = \left( \pi_r^T \frac{D}{S} + (1 - \pi_r^T) R_f 1 \right) W_{r,0},$$

and

$$W_b = \left( \theta_b \lambda_b^T \frac{D}{S} + (1 - \theta_b) R_f \right) W_{b,0},$$

where $\pi_r = (\pi_{r,1}, \pi_{r,2}, \ldots, \pi_{r,n})^T \in \mathbb{R}_+^n$, with $\sum_i \pi_i \leq 1$, are fractions of the CARA investor’s initial wealth invested into the risky assets; $\lambda_b = (\lambda_{b,1}, \lambda_{b,2}, \ldots, \lambda_{b,n})^T \in \mathbb{R}_+^n$, with $\sum_i \lambda_{b,i} = 1$, are weights of each risky asset in the CPT investor’s risky portfolio (so the fraction of her initial wealth in Asset $i$ is $\pi_{b,i} = \theta_b \lambda_{b,i}$); $D = (D_1, \ldots, D_n)^T$ and $\frac{D}{S} = \left( \frac{D_1}{S_1}, \ldots, \frac{D_n}{S_n} \right)^T$.

As before, the reference return of the CPT investor, $R_B$, is constant and asset-independent. Let $S_i^*$ and $R_i^*$ denote Asset $i$’s price and return in equilibrium. Because $R_i^* = \frac{D_i}{S_i}$, and $S_i^*$ is no more exogenous, but endogenously determined via market clearing condition, $\Omega(R_i^*)$ is endogenous and Lemma 3.2 does not simply apply here. Next lemma is an equilibrium version of Lemma 3.2

**Lemma 4.3.** In financial market equilibrium,

$$\Omega(R_1^*) = \Omega(R_2^*) = \ldots = \Omega(R_n^*), \text{ for any } R_B \in (0, +\infty),$$

(34)
if and only if

\[ F_{D,1} = F_{D,2} = \ldots = F_{D,n}, \text{ almost everywhere.} \] (35)

Proof. See Appendix A5.

Similarly as Section 3, we see it is also an extreme case that \( \Omega(R^*_1) = \Omega(R^*_2) = \ldots = \Omega(R^*_n) \) in general equilibrium, if we allow \( R_b \) to take any value from \((0, +\infty)\). We thus let, without loss of generality, \( \Omega(R^*_1) \geq \Omega(R^*_2) \geq \ldots \geq \Omega(R^*_n) \) and at least one inequality is strict. We note that whether \( R^*_i \) is endogenous or exogenous does not affect Lemma 3.3. Hence, Proposition 3.4 still applies in general equilibrium. Consequently, we achieve the same result as the partial equilibrium case in Section 3: the CPT investor does not necessarily hold all risky assets in the market. More likely, she will hold a concentrated portfolio.

4.2 Pricing implications

As a byproduct, let us turn to the pricing implications of the CPT investor and set our focus on the capital asset pricing model (CAPM, see Fama and French 2004 for a survey). CAPM predicts that the slope of the security market line (plotting the expected return of assets against their beta) is the expected excess return of the market portfolio (the expected return of the market portfolio minus the return of the risk-free asset). However, empirical evidences reject this prediction and show that the security market line is in fact flatter than CAPM predicts, i.e., its slope is smaller than the expected excess return of the market portfolio, see, e.g. Fama and French (1992).
Our analysis of the CPT preference offers an explanation for it: because the CPT investor does not consider covariance between assets in her portfolio choice, covariance risk is less compensated in a financial market when CPT investors are present. Beta, as a measure of the covariance between an asset and the market portfolio, is thus less compensated, too. Consequently, the security market line of the financial market when CPT investors are present is flatter than when they are absent.

Consider a parsimonious case \( n = 2 \), i.e. two risky assets. Let their total numbers of shares in the market be \( X_1 = \theta \) and \( X_2 = 1 - \theta \) with \( \theta \geq 0 \). Now we impose no specific distribution for the payoff vector \( D = (D_1, D_2)^T \), except that they are independently but not identically distributed. Denote their expectations and variances by \( \mu_{D,i} \) and \( \sigma_{D,i} \), respectively, with \( \mu_{D,i} \) large enough such that \( \mu_{D,i} > R_f S_i \), namely \( \mathbb{E}[R_i] > R_f, i = 1, 2 \).

Let \( U_b \) be the CPT preference as before and \( U_r \) now be the mean-variance utility,

\[
U_r = \mathbb{E} \left[ \frac{W_r}{W_{r,0}} \right] - \frac{\delta}{2} \text{Var} \left[ \frac{W_r}{W_{r,0}} \right]
\]
\[
= \mathbb{E} \left[ \frac{\pi_r D}{S} + (1 - \pi_r^T) R_f 1 \right] - \frac{\delta}{2} \text{Var} \left[ \frac{\pi_r D}{S} + (1 - \pi_r^T) R_f 1 \right]
\]
\[
= \pi_{r,1} \frac{\mu_{D,1}}{S_1} + \pi_{r,2} \frac{\mu_{D,2}}{S_2} + (1 - \pi_{r,1} - \pi_{r,2}) R_f - \frac{\delta}{2} \left( \frac{\pi_{r,1}^2}{S_1^2} \sigma_{D,1}^2 + \frac{\pi_{r,2}^2}{S_2^2} \sigma_{D,2}^2 \right)
\]

(36)

where \( \delta > 0 \).

As before, the two investors are endowed with initial wealth \( W_{r,0} = \eta W_0 \) and \( W_{b,0} = (1 - \eta) W_0 \), where \( W_0 \) is the aggregate initial wealth in the market.
Thus their respective fractions in the market at time 0 are represented by \( \eta \) and \( 1 - \eta, \eta \in [0, 1] \). Similarly as Lemma 4.3, it can be verified that the CPT-ratio of the two risky assets \( \Omega(R_1) \neq \Omega(R_2) \), if we allow \( R_b \) to take any value from \((0, +\infty)\). We thus assume \( \Omega(R_1) > \Omega(R_2) \).

The slope of the security market line of this financial market is given in next proposition, which is our main result in this subsection.

**Proposition 4.4.** The slope of the security market line, \( \beta_{SML} \), is given by

\[
\beta_{SML} = A \cdot \left[ \mu_M - R_f - \frac{\mu_{D,1} - R_fS_1}{\theta S_1 + (1 - \theta)S_2} X_b \right],
\]

where \( \mu_M \) denotes the expected return of the market portfolio, \( X_b \) is the optimal share holdings of the CPT investor in Asset 1,

\[
X_b = \frac{\left( \alpha \int_{R_B-\frac{\lambda}{x^\infty}}^{+\infty} \left( \frac{R_B}{x^\infty} \right) d[-w_1(1-F_{D,P}(x))] \right)^{\frac{1}{\beta - \alpha}}}{\beta \int_{R_B-\frac{\lambda}{x^\infty}}^{R_B+\infty} \left( \frac{R_B}{x^\infty} \right)^{\beta} d[-w_1(1-F_{D,P}(x))]},
\]

which is zero when there is no CPT investor in the financial market, i.e., \( 1 - \eta = 0 \), and

\[
A = \frac{S_2(\theta - X_b)\theta^2 \sigma_{D,1}^2 + (1 - \theta)S_2(\theta - X_b)(1 - \theta) - S_1(\theta - X_b)^2 \sigma_{D,1}^2 \sigma_{D,2}^2 - S_1(1 - \theta)\theta^2 \sigma_{D,2}^4}{S_2(\theta - X_b)\theta^2 \sigma_{D,1}^2 + (1 - \theta)(S_2(\theta - X_b)(1 - \theta) - S_1(\theta - X_b)^2 \sigma_{D,1}^2 \sigma_{D,2}^2 - S_1(1 - \theta)^3 \sigma_{D,2}^4}.\]

**Proof.** See Appendix A6. \(\square\)

It can be verified in (37) that for \( X_b > 0 \), i.e., \( 1 - \eta > 0 \),

\[
\beta_{SML} < \mu_M - R_f.
\]
Namely, with the presence of the CPT investor in the financial market, the slope of the security line is smaller than the expected excess return of the market portfolio, unlike CAPM predicts.

When \( X_b = 0 \), i.e., \( 1 - \eta = 0 \),

\[
\beta_{SML} = \mu_M - R_f. \tag{41}
\]

Without the CPT investor, the slope of the security line is the expected excess return of the market portfolio, as CAPM predicts.

Clearly, if there is no CPT investor in the financial market, the mean-variance investor holds the market and the market portfolio is thus mean-variance efficient in equilibrium. Consequently, CAPM holds in this case. When the CPT investor is present in the market, she generates an extra demand for the high-CPT-ratio asset and holds a mean-variance inefficient portfolio, while the mean-variance investor holds an efficient one. Adding the two portfolios together gives the market portfolio, which becomes mean-variance inefficient in equilibrium. Consequently, CAPM does not hold in this market and anomalies occur.

5 Concluding remarks

In this paper, we discuss the theoretical link between cumulative prospect theory (CPT) and underdiversification. We basically confirm the conjecture in Barberis and Huang (2008) that CPT is a cause for underdiversification. We show that a CPT investor’s portfolio choice is characterized by the CPT-
ratio, a subjective performance measure for assets / portfolios. The CPT investor optimally holds a concentrated risky portfolio, formed only by the assets with the highest CPT-ratio. The result holds both in partial equilibrium and in general equilibrium.

In a financial market with a mean-variance investor and a CPT investor, the latter generates mean-variance inefficient demand for risky assets and results in CAPM anomalies. In particular, because covariance is not considered by the CPT investor in her portfolio choice, beta is less compensated in the financial market when the CPT investor is present. The security market line is thus flatter than CAPM predicts.

If we can use CPT investors as a proxy for individual investors, there are some interesting topics for further investigation. In an empirical study on a complete transaction data from the Taiwan Stock Exchange, Barber et al. (2008) show that individuals lose from trading but institutions win. It would be interesting to discuss the survival of individual investors in a financial market with the presence of expected utility theory investors (as a proxy for professional investors). How are wealths reallocated between the two classes of investor over time? Will one of the two become extinct in the long run? Can individual investors be more “professional”, by reducing their references $R_B$?

It is easy to see from our analysis that the CPT investor generates larger systemic risk of the financial market, namely a higher volatility of the market portfolio at a given return level. In particular, it can be found in Appendix A6 that the covariance of Asset 2 with the market is increasing with the fraction of the CPT investor in the market (while the covariance of Asset 1
with the market is decreasing in it). In a framework probably similar to that of Basak and Pavlova (2013), it would be interesting to discuss the impact of individual investors on financial markets and on asset prices, in the presence of professional investors.

The CPT-ratio discussed in this paper may also be used to investigate the demand for derivatives. In contrast to the literature, where investors are indifferent to the presence or the absence of derivatives in their portfolios (Merton, 1971), the CPT-ratio may be able to explain the role of derivatives in investors’ portfolios. For example, a portfolio constructed by a risky asset and a put option written on this risky asset may have a higher CPT-ratio than a portfolio of the risky asset alone, because the put option with an appropriate strike can largely reduce $\rho(R_P)$ without largely reducing $\gamma(R_P)$. Option overpricing and volatility smile might thus be able to discussed, as pointed out in Barberis and Huang (2008).
References


