Price Dynamics of CO2 Emission Allowance and Theory of Storage

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Abstract

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Keywords: EU emission allowances; Theory of storage; Jump-diffusion process; Stochastic volatility; Stochastic convenience yield

JEL classification: G10; G13

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Abstract

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1. Introduction

Greenhouse gas emission allowances (GGEAs) emerged after Kyoto Protocol (KP) in 1997. The goal of KP is to prevent the damage due to intensive climate change by stabilizing the greenhouse gas concentration in the atmosphere at a proper level. The regulated six kinds of greenhouse gases are translated into CO₂ equivalent tons, called carbon emission. The parties to the protocol agree to reduce the carbon emission generated in their nations. To achieve the goal of reducing the carbon emission effectively and efficiently, the distributed credit of GGEAs are tradable. The trading volume of the global carbon markets was increasing continuously. In the single year of 2011, the total GGEAs trading volume grew by 11% and reached 10.3 billion tons of CO₂, equivalent to 126 billion Eurodollars\(^1\). Therefore, developing a general and consistent pricing framework for the spot, futures, and futures options of GGEAs is an essential issue.

To implement KP, European Union Emission Trading Scheme (EU ETS) initiated a cap-and-trade system in January, 2005. EU ETS plans three phases to achieve the goal specified by KP, including the first phase 2005-2007 (the trial period), the second phase 2008-2012 (corresponding to the first commitment period in KP), and the third phase 2013-2020 (corresponding to the second commitment period in KP). EU emission allowances (EUAs) are created to represent the right to produce carbon emission. EU ETS allocates issued EUAs to firms in Europe with relatively heavy carbon emissions every February. The distributed EUAs are valid during the whole period of the current phase. According to the actual amount of carbon emission in the previous calendar year, the regulated firms have to submit sufficient credit of EUAs (could be the received EUAs or EUAs bought in carbon trading markets) in April every year. If there is any deficiency in submitting EUAs, the regulated firms will be fined, and the deficient amount in EUAs will be taken out from the distributed EUAs for the subsequent year. On the contrary, if the carbon emissions produced by the regulated firms are less than the allocated EUAs, the surplus of EUAs can be sold in the carbon trading market or

\(^{1}\) See Kossoy and Guigon (2012).
stored for the future use in the same phase.

There are many studies associated with EUA spot, futures, and futures options in the literature. For example, Seifert, Uhrig-Homburg, and Wagner (2008) study the dynamic behavior of 2005-2007 EUA spot prices. They find that there is no seasonal effect, the discounted prices are consistent with the martingale property, and the volatilities vary with time and spot prices. Uhrig-Homburg and Wagner (2009) study the relationship between 2005-2007 EUA spot and futures prices. They suggest that the cost of carry theory holds, and the storage cost and convenience yield of EUA should be zero. Daskalakis, Psychoyios, and Markellos (2009) find that the jump-diffusion process can better model the EUA spot prices. They also verify that the cost-of-carry model with the zero storage cost and zero convenience yield can describe the 2005-2007 futures prices expired by the end of 2007. Nevertheless, a stochastic convenience yield model is needed for the futures issued in 2005-2007 but expired in 2008-2012, since the underlying EUA spots are no more issued after the end of 2007. They also employ the option pricing model with the jump-diffusion process and stochastic convenience yield developed by Hilliard and Reis (1998) to fit 2007 EUA futures options expired in 2008-2012. On the other hand, Chevallier (2010) studies the intraday data of 2008 EUA spot and futures prices and finds that the convenience yield is not zero. The EUA convenience yield can be described by the AR(4) model, and it varies with the volatility of EUA spot and futures prices. Pirrong (2011) believes that the EUA convenience yield is not zero and can be explained by the Theory of Storage (TOS) proposed by Working (1949). The TOS studies the relationship between futures and spots by discussing why the market participants hold the inventories of assets. Pirrong (2011) develops a structural model to describe the supply and demand of EUA spots and discusses how the issuing frequency of EUAs and the possibility of reserving EUAs for trading in the subsequent phase can influence the market participants' motives for storing EUAs.

Similar to Daskalakis, Psychoyios, and Markellos (2009), this paper provides a comprehensive study for the price behaviors of the spot, futures, and futures options of EUAs.
However, unlike their model, our framework considers not only the jump and stochastic convenience yield (SC) but also the stochastic variance (SV) to model the EUA spot prices. Hereafter, our general framework is also called the SVSCJ model. By conducting horse-race experiments to compare the performance of all possible combinations under our general framework, we can identify the best-performing model for describing the EUA price dynamics in the spot, futures, and futures options markets. In order to achieve this goal, we modify the particle filter method such that we can estimate the parameters of the SVSCJ model with EUA spot and futures prices. In addition, this paper develops respectively the pricing formulas of EUA futures and futures options under the SCSSVJ model. It is also worth noting that our analysis is consistent and complete since we always examine all combinations of stochastic processes under our general framework for pricing EUA spots, futures, and futures options in the phase of 2008-2012. In contrast, Daskalakis, Psychoyios, and Markellos (2009) examine the phase of 2005-2007, and their horse-race experiments test different sets of candidate stochastic processes in each of the three EUA markets.

Another important feature of our general framework is that it is able to examine whether the TOS can explain the relationship between EUA futures and spot prices. The TOS should theoretically hold since holding EUA inventories can bring some benefits for firms regulated by KP. For example, when a firm receives an unexpected large order and needs to produce extra carbon emission due to the overtime operation or the change in its manufacturing process, the EUA inventories can be consumed to meet this unexpected demand of EUAs without extra costs. The flexibility and elasticity of the productivity of a firm because of holding sufficient EUA inventories can thus raise its competiveness and its firm value essentially. In addition, for those firms which are not familiar with derivatives trading, holding sufficient EUAs is the simplest and most reliable way to conform the constraint for carbon emission. Moreover, by extending Brennan’s (1958) arguments, holding the inventories of EUAs can be thought of holding an option to save costs. When EUA spot prices are higher than the marginal cost needed to reduce the carbon emission, a firm can sell the inventories of EUAs and use the proceeds to pay the cost of reducing the carbon emission. Conversely, when EUA spot prices are lower than
the marginal cost, a firm will not pay higher costs to reduce the carbon emission. Instead, it simply gathers sufficient credit of EUAs through purchasing them from the market.

According to the TOS, the benefits of holding the inventories can be measured by the convenience yield implied from the futures and spot market prices. Specifically, as the total level of inventories decreases in the market, the convenience yield of assets will increase (see Brennan 1958, Fama and French 1987, Ng and Pirrong 1994, etc.). In our preliminary empirical studies (will be presented in Section 2), we verify the above the inference and thus believe that the TOS should hold in EUA markets. However, the results of our empirical study are not sufficient to make solid conclusions since we focus on only the last year in the phase of 2008-2012. In order to obtain robust evidences, we follow Nielsen and Schwartz (2004) to model the spot volatility to be positively proportional to the level of convenience yield, since we also find that the total level of EUA inventory is inversely related to the EUA spot volatility in our preliminary empirical study. If the positive relationship between the spot volatility and the level of the convenience yield is confirmed, one can conclude the tenability of the TOS. To the best of our knowledge, this paper is the first in the literature to test the TOS based on the price data from all of the three EUA relevant markets—EUA spot, futures, and futures options markets.

This paper examine the daily prices from EUA spot, futures, and futures option markets in the 2008-2012 phase, which is the only commitment period we had experienced. Our horse-race experiments show that the combination of the jump-diffusion, stochastic variance, and stochastic convenience yield processes can best fit EUA spot, futures, and futures options prices. According to the positive relationship between the spot volatility and the convenience yield in the best-performing process, we can verify that the TOS holds in all of the three EUA relevant markets.

This paper is organized as follows. In Section 2, we introduce our general SVSCJ framework for EUA prices and show the results of our preliminary empirical studies for justifying the adoption of Nielsen and Schwartz’s (2004) model to examine the TOS in EUA
markets. Section 3 presents our estimation method for the SVSCJ model under the physical probability measure. We next derive the pricing formulas of EUA futures and futures options in Section 4. Section 5 reports the empirical results, and Section 6 concludes the paper.

2. Modelling EUA Stochastic Processes

Under the physical probability measure \( P \), suppose that EUA spot price \( (S_t) \) and its variance \( (V_t) \) and convenience yield \( (\delta_t) \) satisfy the following system of stochastic differentiation equations:

\[
\begin{align*}
    dS_t &= (\mu - \delta_t - \lambda \mu_f)S_t \, dt + \sqrt{V_t}S_t \, dW_{S1} + \sqrt{\beta_1 \delta_t + \beta_2 S_t} \, dW_{S2} + J_t S_t \, dq_t, \quad (1) \\
    dV_t &= \kappa_V (\theta_V - V_t) \, dt + \sigma_V \sqrt{V_t} \, dW_V, \quad (2) \\
    d\delta_t &= \kappa_\delta (\theta_\delta - \delta_t) \, dt + \sigma_\delta \sqrt{\beta_1 \delta_t + \beta_2} \, dW_\delta, \quad (3)
\end{align*}
\]

where \( \mu \) is the instantaneous expected growth rate, \( W_i, \ i = S1, S2, V, \delta \), are standard geometric Brownian motions under the measure \( P \). \( \text{Corr}(dW_{S1}, dW_V) = \rho_{SV} \), \( \text{Corr}(dW_{S2}, dW_\delta) = \rho_{S\delta} \), and \( \text{Corr}(dW_{S1}, dW_{S2}) = \text{Corr}(dW_{S2}, dW_V) = \text{Corr}(dW_{S1}, dW_\delta) = \text{Corr}(dW_V, dW_\delta) = 0 \). The Poisson process, \( q_t, \) is with the arrival frequency \( \lambda \). \( J_t \) is the percentage jump size and \( \ln(1 + J_t) \sim N(\ln(1 + \mu_f) - 0.5 \sigma_f^2, \sigma_f^2) \), and \( q_t \) and \( J_t \) are uncorrelated with each other and with all \( W_t \). For the SV process, \( \kappa_V \) is the speed of mean reversion, \( \theta_V \) is the long-run mean, and \( \sigma_V \) is its volatility term. In the SC process, \( \kappa_\delta \) is the speed of mean reversion, \( \theta_\delta \) is the long-run mean, and \( \sigma_\delta \) is its volatility term.

To describe the dynamics of the price behavior of the EUA, we consider the SV and SC in addition to jumps in Equations (1) to (3). Since the SC models are the most popular models for spot and futures prices of commodities (see Gibson and Schwartz (1990), Schwartz (1997), Hilliard and Reis (1998)), we consider the SC in our model. The reason to include the SV is according to the finding of non-constant volatilities for EUA prices in Seifert, Uhrig-Homburg, and Wagner (2008). Due to Daskalakis, Psychoyios, and Markellos (2009) and many empirical studies confirming that the discontinuous price behavior can be observed in the prices of almost
all types of assets, we intend to test the existence of jumps in the spot and derivatives prices of the EUA. Moreover, we follow Nielsen and Schwartz (2004) to capture the effect of the TOS by introducing $\sqrt{\beta_1 \delta_t + \beta_2 S_t} dW_{S_2}$ in the dynamics of spot prices and $\sigma_{S} \sqrt{\beta_1 \delta_t + \beta_2} dW_{S}$ in the SC process.2 Nielsen and Schwartz (2004) argue that for commodity assets, the TOS holds if there is a positive value of $\beta_1$, i.e., the volatility of the spot asset is positively related to the level of the convenience yield. The underlying reason is that when the total inventories of an asset decrease in the market, in addition to the increase in the convenience yield as predicted by the TOS, the volatility of that asset also increases. To verify the existence of the above phenomenon in EUA markets, we conduct a preliminary empirical study for EUA spots and futures in the 2008-2012 phase.

However, it is not a simple task to identify the total level of EUA inventories. Unlike the agricultural products and precious metals studied in the literature, whose total levels of inventories can be obtained according to the inventory amount from warehouses of agricultural products or London Metal Exchange, there is no storage center for EUAs and the usage rates of EUA credits of individual firms are not publicly available. Theoretically speaking, since regulated firms produce the carbon emission in their every day operation and thus need to consume EUAs, the total level of EUA inventories decreases day by day, and thus the length of the remaining time period can be used as a proxy for the total level of EUA inventories. Nevertheless, it is not the case in practice. For each calendar year, the total EUA inventories increase in February due to the allocation of new issued EUAs, and the total EUA inventories decrease only in April since regulated firms need to submit sufficient EUA credits at this time point. As a result, the total inventory level actually fluctuates like a digital signal in a year cycle—increases in February and decreases in April. Lastly, since the yearly issued EUAs are valid during the whole phase, a firm may retain some EUA credits in preparation for meeting its extra carbon emission due to its expected growth in production in subsequent several years.

2 Note that if $\beta_1 = 0$, the stochastic convenience yield model of Nielsen and Schwartz (2004) can be rewritten to be equivalent to those in Gibson and Schwartz (1990), Schwartz (1997), and Hilliard and Reis (1998).
This precautionary behavior is hard to predict and could significantly affect the performance of using the length of the remaining time period as a proxy for the total level of EUA inventories.

Note that the digital-signal-like changes in the total level of EUAs and the precautionary saving of EUAs do not occur after February of 2012, the last time point in the phase of 2008-2012 to distribute EUAs. This is because that new EUAs for this phase will no longer be issued after this month, and the EUA credits owned by regulated firms are valid and tradable until the 2013/01/31. Therefore, we can employ the remaining time period before the end of this phase to approximate the total level of EUA inventories in the market. The shorter the remaining time period, the lower the total level of EUA inventories. Since the expiration date of the 2012 Dec. EUA futures is 2012/12/17, to derive the convenience yield, we limit the remaining time period to be the length between the examined date and 2012/12/17 rather than the actual ending date of the phase of 2008-2012. In addition, for each examined date \( t \) from 2012/03/01 to 2012/12/17, we can derive the convenience yield by

\[
\delta_t = r_t - \frac{1}{\tau} \ln \frac{F_{t,\tau}}{S_t},
\]

where \( S_t \) is the EUA spot price, \( F_{t,\tau} \) is the EUA futures price with the remaining time to maturity \( \tau \), and \( r_t \) is the Eurodollar risk-free interest rate corresponding to the time to maturity \( \tau \). In this paper, we employ the spline interpolation method to derive \( r_t \) from the Euribor term structure at date \( t \). The scatter diagram of each convenience yield \( \delta_t \) (the \( y \)-axis) and its corresponding remaining time period \( \tau \) (the \( x \)-axis) is shown in Figure 1. It can be observed that almost all of \( \delta_t \) are significantly positive, and \( \delta_t \) tends to rise up when \( \tau \) decreases. The solid line represents the best-fit exponential function \( \delta_t = 0.0382e^{-0.5300\tau} \) obtained from the ordinary least squares approach. The negative sign in front of \( \tau \) explicitly illustrates that when \( \tau \) becomes short, i.e. the whole level of EUA inventories decreases, the convenience yield \( \delta_t \) tends to rise up. In order words, this result supports that the TOS holds in EUA markets.

[Insert Figure 1 about here]
Nevertheless, to justify the argument in Nielsen and Schwartz (2004) for examining the TOS, we also need to check the existence of a negative relationship between the spot volatility and the total level of EUA inventories (approximated again by the length of the remaining time period). We obtain the monthly spot volatilities in the period of 2012/03-2013/01 and calculate the corresponding remaining time period for each month. Take 2012/03 for example, the annualized volatility of the EUA spot prices is 52.9762%, and the remaining period between this month and 2013/01 is assumed to be 10.5 months (equals to 0.875 year). Figure 2 shows the scatter diagram of the EUA spot volatility (the y-axis) versus the remaining period of time (the x-axis). The solid line is the best-fit exponential function obtained from the ordinary least squares approach: \( \sigma_t = 0.6016e^{-0.3568\tau} \). The negative sign in front of \( \tau \) verifies that when \( \tau \) becomes short, i.e. the total level of EUA inventories decreases, the spot volatility increases. Note that the increase in the spot volatility near the end of a phase can be expected. According to Pirrong (2011), when the end of a phase is approached, EUA spot prices skyrocket fiercely if the outstanding amount of EUAs is smaller than the total demand of EUAs. On the contrary, EUA spot prices decrease to zero if the outstanding amount of EUAs is larger than the total demand of EUAs. In either scenario, the violent changes in EUA spot prices will contribute to the increase of EUA spot volatility near the end of a phase. The evidences in Figures 1 and 2 form the foundation of using the argument of Nielsen and Schwartz (2004) to examine the TOS in EUA markets.

Although the results in Figures 1 and 2 can intrigue interests to study the TOS in EUA markets, they describe only the condition in the last year of the 2008-2012 phase and thus cannot represent the general behaviors of the EUA and its derivatives in the whole 2008-2012 phase. In order to obtain robust conclusions, this paper examines the TOS by investigating the performance of our model in Equations (1) to (3) in EUA spot, futures, and option futures markets for the whole 2008-2012 phase. Specifically, Section 3 develops an estimation method by modifying the particle filter method in Christoffersen et al. (2010) and Malik and Pitt (2009,
2011) to obtain the parameter values in Equations (1) to (3). For EUA futures and futures options, we develop their pricing formulas in Section 4 such that calibrating the parameter values from market prices of EUA futures and futures options becomes feasible. If the estimated or calibrated values of $\beta_1$ are all significantly positive in all these three assets, we can attest the TOS holds in EUA markets. To the best of our knowledge, this paper is the first to propose the modified particle filter method and the pricing formulas for the EUA prices under the SVSCJ model. Equipped with those contributions, it is also the first time to have opportunity to examine the TOS in EUA markets.

3. A Modified Particle Filter Method to Estimate EUA Spot Price Processes

We first apply the Itô’s Lemma to rewrite Equation (1) as follows:

$$d\ln S_t = \left(\mu - \left(1 + \frac{1}{2}\beta_1\right)\delta_t - \frac{1}{2}\beta_2 - \frac{1}{2}V_t - \lambda\mu_f\right)dt + \sqrt{V_t}dW_{S1}$$

$$+ \sqrt{(\beta_1\delta_t + \beta_2)dW_{S2}} + \ln(1 + f_t)dq_t.$$  \hspace{1cm} (5)

By defining $y_t = \ln(S_t)$, we can discretize the system of stochastic differentiation equations (Equations (2), (3), and (5)) as follows:

$$y_{t+\Delta t} - y_t = \left(\mu - \left(1 + \frac{1}{2}\beta_1\right)\delta_t - \frac{1}{2}\beta_2 - \frac{1}{2}V_t - \lambda\mu_f\right)\Delta t + \sqrt{V_t}\Delta z_{S1}$$

$$+ \sqrt{(\beta_1\delta_{t+\Delta t} + \beta_2)\Delta z_{S2}} + \ln(1 + f_t)\sigma_e,$$  \hspace{1cm} (6)

$$V_{t+\Delta t} - V_t = \kappa_V(\theta_V - V_t)\Delta t + \sigma_V\sqrt{V_t}\Delta z_V,$$  \hspace{1cm} (7)

$$\delta_{t+\Delta t} - \delta_t = \kappa_\delta(\theta_\delta - \delta_t)\Delta t + \sigma_\delta\sqrt{(\beta_1\delta_t + \beta_2)\Delta z_\delta},$$  \hspace{1cm} (8)

where $z_i \sim N(0,1), \quad i = S1, S2, V, \delta$, $\text{Corr}(z_{S1}, z_V) = \rho_{SV}$, $\text{Corr}(z_{S2}, z_\delta) = \rho_{S\delta}$, and $\text{Corr}(z_{S1}, z_{S2}) = \text{Corr}(z_{S1}, z_\delta) = \text{Corr}(z_{S2}, z_V) = \text{Corr}(z_\delta, z_\delta) = 0$. Here, we follow Ball and Torous (1983) and Das (2002) to approximate the jump component, $dq_t$, using a Bernoulli approximation, $\sigma_e \sim Ber(\lambda)$, i.e., $\sigma_e = 1$ if the jump occurs and $\sigma_e = 0$, otherwise.

Since the asset variance is unobservable, one cannot use a normal statistical estimation method, e.g., the maximum likelihood method, to determine parameter values of a stochastic
variance process. Several numerical methods are proposed for solving this issue in the literature, such as the quasi maximum likelihood method (Harvey et al., 1994), the spectral generalized method of moments (Chacko and Viceira, 2003), the efficient method of moments (Andersen et al., 1999), and the Markov chain Monte Carlo method (Jacquier et al., 1994, Eraker, 2001 and Eraker 2004). A more efficient method, the particle filter method, is recently advocated in, for example, Christoffersen et al. (2010) and Malik and Pitt (2009, 2011). In Christoffersen et al. (2010), they examine the dynamics S&P 500 index returns by using the particle filter method to estimate stochastic variance models with or without jumps (hereafter SVJ or SV model, respectively). Malik and Pitt (2009, 2011) employ the particle filter method to estimate stochastic log-variance model with or without jumps for S&P 500 index returns. However, there is no attempt in the literature to estimate parameters of a SVSCJ model. We fill this gap by extending the particle filter method to solve this estimation problem. The following four-step method is proposed:

**Step 1. Generating \( V_0^j \):**

We draw the random numbers of the initial variance \( V_0^j \) from its unconditional distribution, 
\[
\Gamma(2\kappa \theta / \sigma^2_v, \sigma^2_v / 2\kappa_v),
\]
where \( j = 1, 2, 3 \ldots, M \), and \( M \) is the number of particles for sampling \( z_{S1} \) and \( z_{S2} \) later.

**Step 2. Generating \( z_{S2}^j \):**

The basic idea of this step is to employ the observable daily convenience yields to generate the corresponding daily innovations for \( z_\delta \) and next draw random samples of \( z_{S2}^j \). The method in Equation (4) is employed to calculate the series of convenience yields on each trading day in the 2008-2012 phase. However, there are several time series of convenience yield on each trading day because the EUA futures expired at the end of each year in this phase are traded simultaneously in the market. To ensure the liquidity and the quality of the quotations of futures prices, we employ the nearest-maturity EUA futures prices to generate the convenience yield

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on each trading day. Specifically, at each time point \( t \), \( F_{t,\tau} \) is the EUA futures price with the shortest time-to-maturity \( \tau \), and \( r_t \) is the risk-free interest rate corresponding to the maturity \( \tau \), the convenience yield can be obtained from \( \delta_t = r_t - \frac{1}{\tau} \ln \frac{F_{t,\tau}}{s_t} \). Equipped with the time series of \( \delta_t \) in the whole phase, we can express the corresponding realized \( z_\delta \) based on Equation (8) as follows

\[
z_\delta = \frac{\delta_{t+\Delta t} - \delta_t - \kappa_\delta (\theta_{\delta} - \delta_t) \Delta t}{\sigma_\delta \sqrt{(\beta_1 \delta_t + \beta_2) \Delta t}}.
\]

To capture the correlation between \( z_\delta \) and \( z_{s2} \), we draw samples of \( z_{s2}^j \) using

\[
z_{s2}^j = \rho_{s\delta} z_\delta + \sqrt{1 - \rho_{s\delta}^2} \varepsilon_{\delta}^j,
\]

where \( \varepsilon_{\delta}^j \sim \mathcal{N}(0,1) \), for \( j = 1, 2, 3, \ldots, M \). Note that \( \varepsilon_{\delta}^j \) and \( z_{s2}^j \) correspond to the \( j \)-th sample of \( V_t^j \).

**Step 3. Generating \( z_{s1}^j \):**

This step extends the particle filter method proposed by Malik and Pitt (2009) for the SVJ model to draw random samples of \( z_{s1}^j \), for \( j = 1, 2, 3, \ldots, M \). The main idea of this step focuses on how to derive the conditional density of \( z_{s1} \) on \( y_{t+\Delta t}, \delta_{t+\Delta t}, z_{s2}^j \) and \( V_t^j \). Once equipped with the conditional density of \( z_{s1} \), the task to draw the random samples \( z_{s1}^j \) corresponding to \( V_t^j \) becomes straightforward.

First, at each examined time point \( t \), we calculate the probability of a jump occurring conditional on \( y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, \) and \( z_{s2}^j \) according to the Bayes' rule

\[
Pr_{s}^j \equiv Pr(\omega_t = 1 | y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{s2}^j) = \lambda_\Delta t \cdot f_{2} \left( y_{t+\Delta t}, \delta_{t+\Delta t}, \mu_{y_{t+\Delta t}}^j + \sqrt{(\beta_1 \delta_t + \beta_2) \Delta t z_{s2}^j} + \alpha, \mu_{\delta_{t+\Delta t}}^j (\theta_{\delta}^j)^2 + \sigma_{\delta}^2, 0 \right) /
\]
\[
\left( \lambda \Delta t \cdot f_2 \left( y_{t+\Delta t}, \delta_{t+\Delta t}, \mu_{y_{t+\Delta t}}^j, + \sqrt{\left( \beta_1 \delta_t + \beta_2 \right) \Delta t z_{S_2}^j} + \alpha, \mu_{\delta_{t+\Delta t}}^j, (\theta_y^j)^2, \theta_\delta^2, 0 \right) \\
+ (1 - \lambda \Delta t) f_2 \left( y_{t+\Delta t}, \delta_{t+\Delta t}, \mu_{y_{t+\Delta t}}^j, + \sqrt{\left( \beta_1 \delta_t + \beta_2 \right) \Delta t z_{S_2}^j}, \mu_{\delta_{t+\Delta t}}^j, (\theta_y^j)^2, \theta_\delta^2, 0 \right), \right)
\] (9)

where \( f_2(x, y, \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho_{xy}) \) is a bivariate normal density function, and \( \mu_{y_{t+\Delta t}}^j = y_t + \left( \mu \left( 1 + \frac{1}{2} \beta_1 \right) \Delta t - \beta_2 \frac{1}{2} V_t^j \right) \Delta t \), \( \alpha = E[\ln(1 + J_t)] = \ln(1 + \mu) - 0.5 \sigma_j^2 \), \( \mu_{\delta_{t+\Delta t}} = \delta_t + \kappa_\delta (\theta_\delta - \delta_t) \Delta t \), \( \theta_y^j = \sqrt{V_t^j \Delta t} \), \( \theta_\delta = \sigma_\delta \sqrt{\left( \beta_1 \delta_t + \beta_2 \right) \Delta t} \). Note that when \( z_{S_2}^j \) is given, since the stochastic dynamics of \( y_{t+\Delta t} \) depends only on \( \sqrt{V_t^j \Delta t \delta_{t+\Delta t}} \) and \( \ln(1 + J_t) \sigma_t \), both of which are independent of \( \delta_t \) and thus \( \delta_{t+\Delta t} \), the correlation is 0 in \( f_2(\cdot) \).

Second, the conditional density of \( z_{S_1} \), \( g(z_{S_1}|y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j) \), should be a mixture density depending on the occurrence of the jump event, i.e.,

\[
g(z_{S_1}|y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j) = g(z_{S_1} = z_{S_1}^j|\sigma_t = 0; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j)(1 - Pr_j^j) \\
+ g(z_{S_1} = z_{S_1}^j|\sigma_t = 1; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j) Pr_j^j,
\]

where \( g(z_{S_1} = z_{S_1}^j|\sigma_t = 0; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j) \) is the normal density of \( z_{S_1} \), conditional on no jump occurring, and \( z_{S_1}^j \) is defined as

\[
z_{S_1}^j = \frac{Y_t^j - \mu_{y_{t+\Delta t}}^j - \sqrt{(\beta_1 \delta_t + \beta_2 \delta_{t+\Delta t}) \Delta \tau_{S_2}^j}}{\sqrt{V_t^j \Delta t}}.
\]

As for the density conditional on jump occurring, since

\[
g(z_{S_1} = z_{S_1}^j|\sigma_t = 1; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j)
\]

\[
\propto g(y_{t+\Delta t}, \delta_{t+\Delta t}|\sigma_t = 1; V_t^j, z_{S_1}, z_{S_2}^j) g(z_{S_1} = z_{S_1}^j),
\]

we can obtain

\[
g(z_{S_1} = z_{S_1}^j|\sigma_t = 1; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S_2}^j)
\]

13
\[
\alpha f_2 \left( y_{t+\Delta t}, \delta_{t+\Delta t}, \mu_{y_{t+\Delta t}}^j, \beta_1, \beta_2, \Delta_t \sigma_{z_{S1}}^2 + \sqrt{\beta_1^2 \Delta_t \sigma_{z_{S2}}^2 + \alpha \mu_{\delta_{t+\Delta t}}, \sigma_\delta^2, \phi^2, \theta^2} \right).
\]

\[
f(z_{S1}^j, 0, 1).
\]

where \( f(x, \mu_x, \sigma_x^2) \) is the normal density function. Given \( z_{S1}^j \) and \( z_{S2}^j \) being known, since the stochastic dynamics of \( y_{t+\Delta t} \) counts solely on \( \ln(1 + J_t)\sigma_t \), which is independent of \( z_\delta \) and thus \( \delta_{t+\Delta t} \), the correlation in \( f_2(\cdot) \) should be zero. Taking logarithm on the both sides of the above equation yields

\[
\log g(z_{S1} = z_{S1}^j | \sigma_t = 1; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S2}^j) = \text{const} - \frac{1}{2} \left( \frac{(y_{t+\Delta t} - \mu)^2}{\sigma^2} \right) - \frac{1}{2} \left( \frac{\delta_{t+\Delta t} - \mu_{\delta_{t+\Delta t}}}{\sigma^2} \right) - \frac{1}{2} \left( \frac{z_{S1}^j}{\sigma_{z_{S1}}^2} \right)^2.
\]

Finally, we can obtain a quadratic (pseudo normal density) form as follows.

\[
\log g(z_{S1} = z_{S1}^j | \sigma_t = 1; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S2}^j) = \text{const} - \frac{1}{2} \left( \frac{z_{S1}^j - \mu_t^j}{\sigma_t^2} \right)^2,
\]

where

\[
\mu_t^j = \frac{\sqrt{\beta_1^2 \Delta_t \sigma_{z_{S2}}^2 + \alpha \mu_{\delta_{t+\Delta t}}, \sigma_\delta^2, \phi^2, \theta^2}}{\sqrt{\beta_1^2 \Delta_t \sigma_{z_{S1}}^2 + \alpha \mu_{\delta_{t+\Delta t}}, \sigma_\delta^2, \phi^2, \theta^2}}.
\]

and

\[
(\sigma_{z_{S1}}^j)^2 = \frac{\sigma_j^2}{\sqrt{\beta_1^2 \Delta_t \sigma_{z_{S1}}^2 + \alpha \mu_{\delta_{t+\Delta t}}, \sigma_\delta^2, \phi^2, \theta^2}}.
\]

Note that the above equation is obtained because \( z_{S1}^j \) does not appear in \( \left( \delta_{t+\Delta t} - \mu_{\delta_{t+\Delta t}} \right)^2 / (2\sigma_\delta^2) \). If there is any estimation method which intends to maximize the conditional density \( g(z_{S1} = z_{S1}^j | \sigma_t = 1; y_{t+\Delta t}, \delta_{t+\Delta t}, V_t^j, z_{S2}^j) \), it is equivalent to maximize the quadratic form in Equation (10).

Third, following Malik and Pitt (2009), we can draw random samples of \( z_{S1}^j \) by the following rules:
where \( \hat{u}_t \sim \text{uniform}(0,1) \), and \( \Phi(\cdot) \) and \( \Phi^{-1}(\cdot) \) are respectively the cumulative distribution function and inverse cumulative distribution function of the standard normal distribution.

**Step 4. Generating** \( V_{t+\Delta t}^j \) **and the corresponding likelihood value:**

To obtain the simulated values of the variance at time \( t + \Delta t \), i.e., \( V_{t+\Delta t}^j \) for \( j = 1, 2, 3 \ldots, M \), we need to generate random samples for \( z_V \) in Equation (7). Since \( \text{Corr}(z_{S1}, z_V) = \rho_{SV} \), we can obtain

\[
z_V^j = \rho_{SV} z_{S1}^j + \sqrt{1 - \rho_{SV}^2} \varepsilon_V^j, \text{ for } j = 1, 2, 3 \ldots, M.
\]

For each simulated values of \( z_{S1}^j \) generated in **Step 3**, we draw a random sample of \( \varepsilon_V^j \) from the standard normal distribution. Equipped with \( z_V^j \) for \( j = 1, 2, 3 \ldots, M \), a set of simulated values of \( V_{t+\Delta t}^j \) can be derived based on Equation (7). We then calculate the likelihood from the Bernoulli mixture density for all \( j \):

\[
\tilde{W}_{t+\Delta t}^j = \lambda \Delta t \cdot f_2(y_{t+2\Delta t}, \delta_{t+2\Delta t}, \mu_{y_{t+2\Delta t}}^j + \alpha, \mu_{\delta_{t+2\Delta t}}^j, V_{t+\Delta t}^j \Delta t + (\beta_1 \delta_t + \beta_2) \Delta t + \sigma_\delta^2, \theta_\delta^2, \rho_{SS})
\]

\[
+ (1 - \lambda \Delta t) \cdot f_2(y_{t+2\Delta t}, \delta_{t+2\Delta t}, \mu_{y_{t+2\Delta t}}^j, \mu_{\delta_{t+2\Delta t}}^j, V_{t+\Delta t}^j \Delta t + (\beta_1 \delta_t + \beta_2) \Delta t, \theta_\delta^2, \rho_{SS}),
\]

where \( \mu_{y_{t+2\Delta t}} = y_{t+\Delta t} + \left( \mu - \left( 1 + \frac{1}{2} \beta_1 \right) \delta_{t+\Delta t} - \frac{1}{2} \beta_2 - \frac{1}{2} V_{t+\Delta t} - \lambda \mu_j \right) \Delta t \) and \( \mu_{\delta_{t+2\Delta t}} = \delta_{t+\Delta t} + \kappa_\delta(\theta_\delta - \delta_t) \Delta t \). The likelihood at time \( t + \Delta t \) can be approximated as

\[
\frac{1}{M} \sum_{j=1}^{M} \tilde{W}_{t+\Delta t}^j.
\]
Note that after calculating the likelihood, $V_{t+\Delta t}^i$ must be resampled by the smoothing algorithm used in Christoffersen et al. (2010) and Malik and Pitt (2009, 2011). The details can refer to Appendices A1 to A3 in Malik and Pitt (2011).

Repeating **Steps 2 to 4** for all observation time points, the SVSCJ parameters can be estimated by maximizing

$$
\sum_{n=\Delta t}^T \log \left( \frac{1}{M} \sum_{j=1}^M W_{n}^j \right).
$$

To measure the estimating performance of our modified particle filter method, we conduct a Monte Carlo experiment provided that the true EUA prices follow the SVSCJ (Equations (6) to (8)) model. Since the particle filter method is based on Monte Carlo simulation, we repeat the estimation process of the particle filter method for 20 times and employ the average and standard deviation of these 20 repetitions as the final estimation results and the corresponding standard error. The results presented in Appendix A demonstrate that the proposed modified particle filter method is capable of generating accurate estimations for our SVSCJ model.

4. Pricing EUA Futures and Futures Options

Following Heston (1993), Pan (2002), and Nielsen and Schwartz (2004), we can derive that counterparts of Equations (1), (2), and (3) under the risk-neutral probability measure $Q$:

$$
dS_t = (r - \delta_t - \lambda^* \mu^*_S)S_t dt + \sqrt{V_t}S_t dW_t^* + \sqrt{\beta^*_1 \delta_t + \beta^*_2} dW_{S_2}^* + J_t^* S_t dq_t^*,
$$

$$
dV_t = \kappa^*_V (\theta^*_V - V_t) dt + \sigma^*_V \sqrt{V_t} dW_V^*,
$$

$$
d\delta_t = \kappa^*_\delta (\theta^*_\delta - \delta_t) dt + \sigma^*_\delta \sqrt{\beta^*_1 \delta_t + \beta^*_2} dW^*_\delta,
$$

where $r$ is the constant risk-free interest rate, $W_i^*$, for $i = S_1, S_2, V, \delta$, are standard geometric Brownian motions under the measure $Q$, $\text{Corr}(dW_{S_1}^*, dW_{V}^*) = \rho_{SV}^*$, $\text{Corr}(dW_{S_2}^*, dW_{\delta}^*) = \rho_{S\delta}^*$, and $\text{Corr}(dW_{S_1}^*, dW_{V}^*) = \text{Corr}(dW_{S_2}^*, dW_{V}^*) = \text{Corr}(dW_{S_2}^*, dW_{\delta}^*) = \text{Corr}(dW_{S_1}^*, dW_{\delta}^*)$.

---

3 Bakshi et al. (1997) find that when the SVJ model is considered, stochastic interest rates show little impact on the performance of derivatives pricing. Following Bakshi et al. (1997), we adopt a constant interest rate for pricing EUA futures and futures options.
Corr\( (dW^*_S, dW^*_\delta) = Corr(dW^*_\nu, dW^*_\delta) = 0 \). The jump component remains in the same form as that under the physical probability measure but with a new arrival frequency and a new mean of the jump size. The Poisson process, \( q^*_t \), is with the arrival frequency \( \lambda^* \), \( \ln(1 + J^*_t) \sim N \left( \ln(1 + \mu^*_j) - 0.5(\sigma^*_j)^2, (\sigma^*_j)^2 \right) \). \( J^*_t \) and \( q^*_t \) are uncorrelated with each other or with all \( W^*_i \). For the SV process, \( \kappa^*_\nu \), \( \theta^*_\nu \), and \( \sigma^*_\nu \) are the speed of mean reversion, long-run mean, volatility term, respectively. For the SC process, \( \kappa^*_\delta \), \( \theta^*_\delta \), \( \sigma^*_\delta \) are the speed of mean reversion, long-run mean, volatility term, respectively. The term \( \sqrt{\beta^*_1 \delta_t + \beta^*_2} \) is introduced for examining the TOS. By the Ito’s Lemma and defining \( X_t = \ln S_t \), Equation (11) can be rewritten as

\[
dX_t = \left( r - \left( 1 + \frac{1}{2} \beta^*_1 \right) \delta_t - \frac{1}{2} \beta^*_2 \delta_t - \frac{1}{2} V_t - \lambda^* \mu^*_j \right) dt + \sqrt{\lambda^*} dW^*_S + \sqrt{\beta^*_1 \delta_t + \beta^*_2} dW^*_\delta + \ln(1 + J^*_t) dq^*_t.
\]

4.1 Pricing Formula for EUA Futures

For pricing EUA futures, denote \( F(t, \tau) \equiv F(t, \tau, X_t, V_t, \delta_t) \) as the futures price at time \( t \) with a time to maturity \( \tau \). Following Cox, Ingersoll, and Ross (1981), we can derive that \( F(t, \tau) \) must satisfy following partial integro-differential equation:

\[
[r - (1 + 0.5 \beta^*_1 \delta_t - 0.5 \beta^*_2 \delta_t - 0.5 V_t - \lambda^* \mu^*_j)]F_x + \kappa^*_\nu (\theta^*_\nu - V_t) F_v \\
+ \kappa^*_\delta (\theta^*_\delta - \delta_t) F_{\delta} + 0.5 (V_t + \beta^*_1 \delta_t + \beta^*_2) F_{xx} + \sigma^*_\nu V_t \rho^*_\delta F_{vy} \\
+ \sigma^*_\delta (\beta^*_1 \delta_t + \beta^*_2 \delta_t) \rho^*_\delta F_{x\delta} + 0.5 (\sigma^*_\nu)^2 V_t F_{vv} + 0.5 (\sigma^*_\delta)^2 (\beta^*_1 \delta_t + \beta^*_2 \delta_t) F_{\delta \delta} - F_t \\
+ \lambda^* E \left[ F(t, \tau, X_t + \ln(1 + J^*_t), V_t, \delta_t) - F(t, \tau, X_t, V_t, \delta_t) \right] = 0
\]

subject to the boundary condition \( F(t + \tau, 0) = S^\tau \). Following Hilliard and Reis (1998) (see Section 6) and Yan (2002) (see Section 1.4), there is no impact of the jump component on pricing futures because the expected spot price, \( E[S^\tau] \), is independent of the jump component. Consequently, we ignore the last term in Equation (15) and conjecture the

\[
F(t, \tau) \equiv F(t, \tau, X_t, V_t, \delta_t) = S^\tau.
\]

\[\footnotemark\]

\footnotetext{This is because the jump component is independent of other processes, and the term \( -(\lambda^* \mu^*_j \delta_t) \) cancels out the effect of \( J^*_t d q^*_t \) when calculating the expected spot price.}
solution form of futures pricing formula to be

\[ F(t, \tau) = \exp(X_t + \alpha_0(\tau) + \alpha_V(\tau)V_t + \alpha_\delta(\tau)\delta_t). \]  

(16)

By substituting

\[ F_X = F, \quad F_{XX} = F, \quad F_V = \alpha_V(\tau)F, \quad F_{VV} = \alpha_V(\tau)^2F, \quad F_\delta = \alpha_\delta(\tau)F, \]

\[ F_{\delta\delta} = \alpha_\delta(\tau)^2F, \quad F_{X\delta} = \alpha_\delta(\tau)F, \quad F_{X\delta} = \alpha_\delta(\tau)F, \quad F_{V\delta} = \alpha_V(\tau)\alpha_\delta(\tau)F, \]

\[ F_\tau = (\alpha'_0(\tau) + \alpha'_V(\tau)V_t + \alpha'_\delta(\tau)\delta_t)F \]

into (15) and collecting respectively the terms including \( V_t, \delta_t \) and others, we can obtain the following three ordinary differential equations (ODEs):

\[ \alpha'_V(\tau) = 0.5(\sigma_V^2\alpha_V(\tau)^2 + (\sigma_V\rho_{SV}^\tau - \kappa_V^\tau)\alpha_V(\tau), \]  

(17)

\[ \alpha'_\delta(\tau) = 0.5(\sigma_\delta^2\beta_\delta^2(\alpha_\delta(\tau)^2 + (\sigma_\delta^2\beta_\delta^2\rho_{S\delta}^\tau - \kappa_\delta^\tau)\alpha_\delta(\tau) - 1, \]

(18)

\[ \alpha'_0(\tau) = r + \kappa_\nu^\tau a\nu^\tau a_\nu^\tau(\kappa_\nu^\tau + \sigma_\nu^2\beta_\nu^2(\alpha_\delta(\tau) + 0.5(\sigma_\delta^2\beta_\delta^2\alpha_\delta(\tau)^2. \]  

(19)

Note that \( \alpha_0(0) = \alpha_V(0) = \alpha_\delta(0) = 0 \) should hold to match the boundary condition.

Both Equations (17) and (18) can be classified as Riccati equations. Furthermore, we can infer \( \alpha_V(\tau) = 0 \) since it is a Riccati equation without the constant term. The solutions of Equations (18) and (19) are presented as follows:

\[ \alpha_\delta(\tau) = w + (ae^{-D\tau} + b)^{-1}, \]  

(20)

where

\[ w = -\frac{(\sigma_\delta^2\beta_\delta^2\rho_{S\delta}^\tau - \kappa_\delta^\tau - D)}{(\sigma_\delta^2\beta_\delta^2)}, \quad a = \frac{(\sigma_\delta^2\beta_\delta^2(\sigma_\delta^2\beta_\delta^2\rho_{S\delta}^\tau - \kappa_\delta^\tau + D))}{2D(\sigma_\delta^2\beta_\delta^2\rho_{S\delta}^\tau - \kappa_\delta^\tau - D)}, \quad \text{and} \quad b = -\frac{(\sigma_\delta^2\beta_\delta^2)}{2D}, \]

and

\[ \alpha_0(\tau) = \left[ r + (\kappa_\nu^\tau a\nu^\tau + \sigma_\nu^2\beta_\nu^2\rho_{S\nu}^\tau)\left(w + \frac{1}{b}\right) + \frac{1}{2}(\sigma_\delta^2\beta_\delta^2\left(w + \frac{1}{b}\right)^2\right) \tau \]

\[ + \frac{(\sigma_\delta^2\beta_\delta^2)}{2Db}\left(\frac{1}{a+b} - \frac{1}{ae^{-D\tau}+b}\right) \]

\[ - \frac{\kappa_\nu^\tau a\nu^\tau + \sigma_\nu^2\beta_\nu^2\rho_{S\nu}^\tau + (\sigma_\delta^2\beta_\delta^2/2)(w+2b)^{-1})}{Db}\ln \left| \frac{a+b}{ae^{-D\tau}+b} \right|. \]  

(21)
The derivation details can refer to Appendix B. To verify the accuracy of the pricing formula for EUA futures in Equations (16), Appendix B also compares the pricing results based on the proposed analytic formula and Monte Carlo simulation. The satisfactory accuracy verifies the correctness of our pricing formula for EUA futures.

### 4.2 Carbon Futures Option Pricing Formula

To price EUA futures options, we first define $\tilde{Y}_t \equiv Y(t, \tau) \equiv \ln F(t, \tau)$ and perform the Ito’s Lemma based on Equation (16) to obtain

$$dY_t = \left( r - \left( 1 + \frac{1}{2} \beta_1^2 \right) \delta_t - \frac{1}{2} \beta_2^2 V_t - \lambda^* \mu^* \delta_t - \alpha_0^* (\tau) - \alpha_0^* (\tau) \delta_t + \alpha_0^* (\tau) \kappa_0^* (\theta_0^* - \delta_t) \right) dt$$

$$+ \sqrt{V_t} dW^*_{1t} + \beta_1^* \delta_t + \beta_2^* dW_{2t}^* + \alpha_0^* \delta_t + \beta_1^* \delta_t^* + \beta_2^* dW_{2t}^* + \ln(1 + J^*) \, dq_t. \quad (22)$$

Next denote $C(t, \tilde{t}) \equiv C(t, \tilde{t}, Y_t, V_t, \delta_t, K)$ as the value of a European call written on $F(t, \tau)$ with the time to maturity $\tilde{t}$ ($\tilde{t} \leq \tau$) and the strike price $K$ at time $t$. The payoff function of this futures call is $C(t + \tilde{t}, 0) = \max(e^{Y_{t+\tilde{t}}} - K, 0)$, where $e^{Y_{t+\tilde{t}}}$ is the futures price at $t + \tilde{t}$ with the time to maturity $\tau^* = \tau - \tilde{t}$.

We next follow Heston (1993), Bates (1996), Bakshi, Cao, and Chen (1997), and Bakshi and Madan (2000) to evaluate $C(t, \tilde{t})$ with the characteristic functions of the logarithm of the futures price as follows.

$$C(t, \tilde{t}) = e^{-\tau^* [F(t, \tau) \Pi_1(t, \tilde{t}) - K \Pi_2(t, \tilde{t})]}, \quad (23)$$

where

$$\Pi_j(t, \tilde{t}) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{\exp(-i\phi \ln K) f_j(t, \tilde{t}, \phi)}{i\phi} \right] d\phi, \text{ for } j = 1, 2,$$

are the risk-neutral probabilities and can be recovered from inverting the characteristic functions $f_j(t, \tilde{t}, \phi)$. By defining $f(t, \tilde{t}, \phi) \equiv f(t, \tilde{t}, \phi, Y_t, V_t, \delta_t) = E^0[\exp(i\phi Y_{t+\tilde{t}})]$,

---

5 This paper employs the European call option as an example to present the option pricing formula. The extension of our method to price European put option is straightforward.
\( f_1(t, \bar{t}, \phi) \) and \( f_2(t, \bar{t}, \phi) \) can be expressed in terms of \( f(t, \bar{t}, \phi) \), respectively, as

\[
f_1(t, \bar{t}, \phi) = \frac{f(t, \bar{t}, \phi - i)}{f(t, \bar{t} - i, \phi)} \quad \text{and} \quad f_2(t, \bar{t}, \phi) = f(t, \bar{t}, \phi).
\]

Note that the integrals in Equation (23) need to be computed numerically, and we employ the Gauss-Legendre quadrature method with 200 abscissas to evaluate them.

We conjecture the form of characteristic function to be

\[
f(t, \bar{t}, \phi) = \exp(i\phi Y_t + \Gamma_0(\bar{t}) + \Gamma_V(\bar{t})V_t + \Gamma_\delta(\bar{t})\delta_t).
\]

In addition, the characteristic function of \( f \) must satisfy the following partial integro-differential equation:

\[
[r - (1 + 0.5\beta_1^*\delta_t - 0.5\beta_2^* - 0.5V_t - \lambda^*\mu^*_\phi - \alpha_0^*(\tau) - \alpha_0^*(\tau)\delta_t + \alpha_\delta^*(\alpha^*(\tau)\delta_t - \delta_t)]f_Y
\]

\[
+ \kappa^*_\phi V_t f_Y + \kappa^*_\delta(\theta^*_\delta - \delta_t)f_\delta + \sigma^*_0 V_t \rho^*_S V f_{YY} + \sigma^*_\delta(\beta_1^*\delta_t + \beta_2^*)[\rho^*_S + \sigma^*_\delta\alpha_\delta(\tau)]f_{\delta\delta}
\]

\[
+ 0.5[V_t + [1 + 2\rho^*_S \sigma^*_\delta \alpha_\delta(\tau) + (\sigma^*_\delta)^2\alpha_\delta(\tau)^2][\beta_1^*\delta_t + \beta_2^*]f_{YY} + 0.5(\sigma^*_\delta)^2 V_t f_{VV}
\]

\[
+ 0.5(\sigma^*_\delta)^2(\beta_1^*\delta_t + \beta_2^*)f_{\delta\delta} - f_t
\]

\[
+ \lambda^* E^0[f(t, \bar{t}, \phi, Y_t + \ln(1 + J^*_t), V_t, \delta_t) - f(t, \bar{t}, \phi, Y_t, V_t, \delta_t)] = 0,
\]

subject to the boundary condition \( f(t + \bar{t}, 0, \phi) = \exp(i\phi Y_{t+\bar{t}}) \). By substituting the following partial derivatives

\[
f_Y = i\phi f, \quad f_{YY} = -\phi^2 f, \quad f_{YYV} = i\phi \Gamma_V(\bar{t}) f, \quad f_{YY\delta} = i\phi \Gamma_\delta(\bar{t}) f,
\]

\[
f_V = \Gamma_V(\bar{t}) f, \quad f_{VVV} = \Gamma_V(\bar{t})^2 f, \quad f_{V\delta} = \Gamma_V(\bar{t}) \Gamma_\delta(\bar{t}) f,
\]

\[
f_\delta = \Gamma_\delta(\bar{t}) f, \quad f_{\delta\delta} = \Gamma_\delta(\bar{t})^2 f, \quad f_t = [\Gamma'_0(\bar{t}) + \Gamma'_V(\bar{t}) V_t + \Gamma'_\delta(\bar{t}) \delta_t] f.
\]

into Equation (25), we can obtain the following three ODEs by collecting respectively the terms including \( V_t, \delta_t \), and others:

\[
\Gamma'_V(\bar{t}) = 0.5(\sigma^*_\delta)^2 \Gamma_V(\bar{t})^2 + (\sigma^*_\delta^\phi \rho^*_S) i\phi - \kappa^*_V \Gamma_V(\bar{t}) - 0.5\phi(i + \phi),
\]

\[
\Gamma'_\delta(\bar{t}) = 0.5(\sigma^*_\delta)^2 \beta_1^* \Gamma_\delta(\bar{t})^2 + [\sigma^*_\delta^\phi \beta_1^* i\phi [\sigma^*_\delta \alpha_\delta(\tau) + \rho^*_S] - \kappa^*_\delta] \Gamma_\delta(\bar{t}) - i\phi[1 + 0.5\beta_1^*]
\]

\*

\[
+ \kappa^*_\delta \theta^*_\phi \Gamma_V(\bar{t}) + 0.5(\sigma^*_\delta)^2 \beta_2^* \Gamma_\delta(\bar{t})^2 + [\sigma^*_\delta^\phi \beta_2^* i\phi [\sigma^*_\delta \alpha_\delta(\tau) + \rho^*_S] + \kappa^*_\delta \theta^*_\phi ] \Gamma_\delta(\bar{t})
\]

\]
Similar to Equations (17) and (18), Equation (26) is a Riccati equation which can be solved analytically\(^6\). As for the ODEs in Equations (27) and (28), since their analytic solutions are not available\(^7\), we employ the fourth-order Runge-Kutta algorithm to solve them. Since the Gauss-Legendre quadrature method to evaluate Equation (23) is the most time-consuming step for pricing EUA futures options, it makes little difference in computational time to solve these two ODEs numerically. To verify the correctness and accuracy of our model, we conduct numerical experiments based on the sets of parameter values examined in Broadie and Kaya (2006) and Nielsen and Schwartz (2004). The pricing results based on our model and the Monte Carlo simulation in Appendix C show that our model can generate accurate and reliable option prices.

5. Estimation Results

In our empirical studies, we intend to estimate the parameter values of the proposed SVSCJ model and all models nested in it from EUA spot prices, futures prices, and futures options values. Specifically, we examine eight models, including geometric Brownian motion (GBM), GBMJ, SV, SVJ, SC, SCJ, SVSC, and SVSCJ processes. In addition to investigating the best-performing models for EUA spots, futures, and futures options, we also test the value of \( \beta_1 \) in these three markets. If the estimated value of \( \beta_1 \) is positive in all markets, we can conclude that the TOS holds in EUA markets.

To estimate the parameters in Equations (1) to (3) under the physical probability measure, we examine the data set of 2008-2012 EUA spot and futures prices (collected from Datastream) traded on the European Energy Exchange (EEX) over the period from 2008/02/26 to

\[ +i\phi[r - 0.5\beta_2^* - \alpha_0^*(\tau) + \kappa_0^*\theta_0^*\alpha_0^*(\tau)] - 0.5\beta_2^*\phi^2[1 + (\sigma_0^*)^2\alpha_0^*(\tau)^2 \]
\[ +2\sigma_0^*\alpha_0^*(\tau)(\rho_{\theta\theta}^*) - i\phi\lambda\mu_j^* + \lambda^*[1 + \mu_j^*]^i\phi\exp(i\phi(i\phi - 1)(\sigma_j^*)^2/2) - 1], \quad (28) \]

--

\(^6\) The analytical solution of Equation (26) is available upon request from the authors.

\(^7\) Yan (2002) also tried to derive a closed-form solution for futures options under a framework similar to ours. However, his target integro-differential equation is based on the logarithmic spot price rather than the logarithmic futures price like our Equations (22) and (25).
2012/12/17. Our data set consists of totally 1215 pairs of EUA spot and convenience yield observations. Table 1 reports the estimated values and the corresponding standard errors of the parameters of these eight models. The results provides several insights. First, the inclusion of the jump and SV processes leads to the better goodness-of-fit than the counterpart models without jumps or SVs in terms of the log-likelihood values, e.g., see GBMJ vs GBM and SV vs. GBM. Second, for capturing the dynamics of the EUA spot prices, the SV appear to be a more important factor than the jump process due to larger log-likelihood increases, e.g., see GBMJ vs. SV or SCJ vs. SVSC. Third, note that the log-likelihood values of the SC and non-SC models cannot be compared directly. For non-SC models, their log-likelihood values are actually based on only the EUA spot prices, whereas the log-likelihood values of SC models depends on the bivariate distribution of the EUA spot prices and the convenience yields. Although the contribution of including the stochastic convenience yield cannot be observed by comparing the log-likelihood values of SC and non-SC models, all of the SC-related parameters, including $\kappa_\delta$, $\theta_\delta$, $\sigma_\delta$, $\beta_1$, and $\beta_2$, in SC, SCJ, SVSC, and SVSCJ models are significant by comparing their estimated values with the corresponding standard errors. These evidences demonstrate the need of incorporating the stochastic convenience yield to capture the dynamics of the EUA related prices. Moreover, the importance of the jump and SV is not affected by the introduction of the SC because the comparative levels of the log-likelihood values of SC, SCJ, SVSC, and SVSCJ are similar to those of GBM, GBMJ, SV, and SVJ. Lastly, in the best-performing SVSCJ process, the estimated value of $\beta_1$ is significantly positive, which verify that the TOS holds in the EUA spot market.

[Insert Table 1 about here]

For EUA futures, we examine the daily settlement prices of 2008-2012 EUA futures traded on EEX over the period from 2008/02/26 to 2012/12/17. For each futures contract on date $t$, we employ the spline interpolation method to derive the risk-free interest rate $r$ to match the desired time to maturity based on the Euribor term structure on that day. Since the jump component and the stochastic variance play no rule for pricing EUA futures prices as we
explain in Section 4.1, we conduct the calibration process only for the GBM and our SVSCJ models, which reduces to the SC model in this case. The root mean squared errors (RMSEs) are computed to gauge the performance of both models. The columns 3 and 4 of Table 2 show respectively the RMSEs of the examined two models and the calibrated values of $\beta_1^*$ of our SVSCJ model in parentheses. It is apparent that our SVSCJ pricing formula outperforms that based on GBM in terms of smaller RMSEs. It implies that introducing the SC can substantially reduce the pricing errors of EUA futures and thus the convenience yield should not be zero for EUAs. Finally, we also find that as expected in the TOS, there exists a positive relationship between the spot volatility and the level of the convenience yield (demonstrated by the positive value of $\beta_1^*$) in each year of the 2008-2012 phase.

For EUA futures option, we examine the daily market prices of 2008-2012 EUA futures options traded on EEX. We purchased from EEX the closing prices of EUA futures options, which are not available from Datastream. Since out-of-the-money option prices are usually more liquid and thought to carry more information for empirical studies, and there is an extremely missing data problem for out-of-the-money put prices offered by EEX, we employ only the call options for calibration. There are several criteria used to screen unqualified market prices. First, we exclude the option contracts whose market prices are less than 0.005, since they are so close to the minimum tick price 0.001 that the rounding off problem could cause significant biases. Second, we eliminate the option prices with the implied volatility based on Black’s (1976) model beyond the range from 5% to 100%. Third, the time to maturity of a qualified option contract must be longer than two weeks. The total number of examined option contracts is 18,977, and the average option price is 0.3132. The calibration results are shown in Table 3.

The first row in Table 3 compares the performance of the eight examined model in terms
of RMSEs. To fit EUA futures option prices, the SV is the most important, followed by the jump, and the SC is the last. However, the improvement brought by the SC is always tenable even conditional on the existence of the jump, the SV, or both. As a consequence, we conclude that the proposed SVSCJ model can best fit EUA futures options. In the second row of Table 3, the calibrated values of $\beta_1^*$ of the SC, SCJ, SVSC, and SVSCJ models are reported. We find that all values of $\beta_1^*$ are positive, verifying that the TOS also holds in EUA option market.

6. Conclusion

This paper proposes a general price process involving the SV, SC, and jumps to examine the price dynamics of EUA spots, futures, and futures options. Furthermore, the proposed general process can be used to examine the TOS in EUA markets. We modify the particle filter method to estimate the parameters from the series of EUA spot prices under the physical probability measure. In addition, we also develop pricing formulas for EUA futures and futures options under the risk-neutral probability measure such that calibrating parameters through EUA futures and futures options become feasible.

The empirical results for EUAs in the phase of 2008-2012 suggest that the SV, SC, and jumps exist in prices of EUA spots, futures, and futures options. In the meanwhile, the estimation or calibration results from all these three markets demonstrate that the TOS can be used to describe the relationship between the EUA spot and futures prices.

Acknowledgments

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References


Hilliard, Jimmy E., and Jorge Reis, 1998, Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot, *Journal of
Appendix A. Performance of the Proposed Particle Filter Method

A Monte Carlo experiment is conducted to access the performance of our modified particle filter method for the proposed SVSCJ model. Given the true values of the parameters, we first simulate 1,200, 2,400 and 4,800 daily returns (i.e., \( T = 1,200 \), \( T = 2,400 \), and \( T = 4,800 \)) based on the proposed SVSCJ model, and next employ the modified particle filter method to estimate the parameter values. Tables A.1 presents the estimated results and the corresponding standard error shown in the underneath parentheses. Columns 3 to 5 report these results given
different period length of the simulated time series. Note that the instantaneous expected growth rate of the spot rate, $\mu$, is estimated by annualizing the arithmetic average of the daily returns to mitigate the overidentification problem for the drift term of the spot price, and we simply fix $\mu$ at this value in our modified particle filter method. In addition, when the modified particle filter method is implemented, we fix $M$ as 500, i.e., there are 500 paths of the volatility paths are generated for each repetition. Theoretically speaking, most of the SC parameters in Equation (8) (except $\rho_{S\delta}$) can be estimated separately, based on the time series of convenience yields extracted from the spot and nearest-maturity futures. Therefore, we derive the maximum likelihood estimations of the SC parameters (except $\rho_{S\delta}$) first and use these results as the initial guess of our modified particle filter method. According to our preliminary tests, this approach can enhance both the accuracy and efficiency of our modified particle filter method. The results in Table A.1 demonstrates that our modified particle filter method is able to generate accurate and reliable estimations for the proposed SVSCJ model.

Table A.1 Estimation Results for Simulated SVSCJ series

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True value</th>
<th>$T = 1,200$</th>
<th>$T = 2,400$</th>
<th>$T = 4,800$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_\delta$</td>
<td>3.00</td>
<td>2.9482 (0.0193)</td>
<td>3.3175 (0.1429)</td>
<td>2.7447 (0.0418)</td>
</tr>
<tr>
<td>$\theta_\delta$</td>
<td>0.02</td>
<td>0.0175 (2.5516e-05)</td>
<td>0.0171 (0.0003)</td>
<td>0.0166 (0.0001)</td>
</tr>
<tr>
<td>$\sigma_\delta$</td>
<td>0.20</td>
<td>0.1669 (0.0013)</td>
<td>0.1794 (0.0025)</td>
<td>0.1506 (0.0011)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.40</td>
<td>0.5339 (0.0086)</td>
<td>0.4362 (0.0142)</td>
<td>0.6844 (0.0118)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.0060</td>
<td>0.0093 (0.0002)</td>
<td>0.0085 (0.0003)</td>
<td>0.0110 (0.0002)</td>
</tr>
<tr>
<td>$\rho_{S\delta}$</td>
<td>0.30</td>
<td>0.1870 (0.0002)</td>
<td>0.1286 (0.0019)</td>
<td>0.1498 (0.0008)</td>
</tr>
<tr>
<td>$\kappa_V$</td>
<td>2.00</td>
<td>8.1958 (0.2880)</td>
<td>3.6042 (0.7476)</td>
<td>2.8084 (0.0794)</td>
</tr>
<tr>
<td>$\theta_V$</td>
<td>0.05</td>
<td>0.0499 (0.0004)</td>
<td>0.0520 (0.0007)</td>
<td>0.0456 (0.0004)</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>0.10</td>
<td>0.0862 (0.0050)</td>
<td>0.1653 (0.0190)</td>
<td>0.1415 (0.0058)</td>
</tr>
<tr>
<td>$\rho_{SV}$</td>
<td>-0.70</td>
<td>-0.7995 (0.0092)</td>
<td>-0.5567 (0.0439)</td>
<td>-0.5802 (0.0310)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>10.00</td>
<td>6.3117 (0.0436)</td>
<td>8.5426 (0.1501)</td>
<td>7.2813 (0.0705)</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>-0.08</td>
<td>-0.0759 (0.0000)</td>
<td>-0.0981 (0.0000)</td>
<td>-0.1002 (0.0000)</td>
</tr>
</tbody>
</table>
This table reports the estimated results and their standard errors (shown in the underneath parentheses) for the proposed SVSCJ model based on the modified particle filter method.

### Appendix B. Pricing Formula for EUA Futures and its Performance

#### B.1. Solving $\alpha_{\delta}(\tau)$

Since the right hand side of Equation (18) is a quadratic polynomial of $\alpha_{\delta}(\tau)$, we can find two constant solutions for $\alpha_{\delta}(\tau)$ through solving this quadratic polynomial. One of the two solutions can be expressed as

$$w = -\frac{(\sigma_{\delta}^2 \rho_{S_{\delta}S_{\delta}}^* - \kappa_{\delta}^*)}{(\sigma_{\delta}^2)^2 \beta_1^*},$$

where $D = \sqrt{(\sigma_{\delta}^* \beta_1^* \rho_{S_{\delta}S_{\delta}}^* - \kappa_{\delta}^*)^2 + 2(\sigma_{\delta}^*)^2 \beta_1^*}$. We further use $w$ as a particular solution of the ODE in Equation (18) by conjecturing the solution of $\alpha_{\delta}(\tau)$ to be $w + 1/z$, where $z$ is a function of $\tau$. As a results, we can obtain $\alpha_{\delta}^'(\tau) = (-1/z^2)z'$ and rewrite Equation (18) as

$$-\frac{1}{z^2}z' = \frac{1}{2} (\sigma_{\delta}^*)^2 \beta_1^* \left[-\frac{(\sigma_{\delta}^* \beta_1^* \rho_{S_{\delta}S_{\delta}}^* - \kappa_{\delta}^*)}{(\sigma_{\delta}^2)^2 \beta_1^*} + \frac{1}{z} \right]^2 + (\sigma_{\delta}^* \beta_1^* \rho_{S_{\delta}S_{\delta}}^* - \kappa_{\delta}^*) \left[-\frac{(\sigma_{\delta}^* \beta_1^* \rho_{S_{\delta}S_{\delta}}^* - \kappa_{\delta}^*)}{(\sigma_{\delta}^2)^2 \beta_1^*} + \frac{1}{z} \right] = 1.$$  

Some algebraic calculation yields

$$z' + Dz = -\frac{1}{2} (\sigma_{\delta}^*)^2 \beta_1^*,$$

and then $z$ can be solved as

$$z = \frac{2Dc e^{-D\tau}}{2D},$$

where $C$ is an arbitrary constant introduced during the integration. By replacing $w$ and $z$ in $\alpha_{\delta}(\tau) = w + 1/z$, we obtain
Solving $\alpha_\delta(\tau)$ subject to $\alpha_\delta(0) = 0$ yields

$$C = \frac{(\sigma_\delta^*)^2(\sigma_\delta^*)^2(\rho_{\delta S}^2 - \kappa_{\delta}^2 + D)}{2D(\sigma_\delta^*)^2(\rho_{\delta S}^2 - \kappa_{\delta}^2 - D)}.$$ 

Finally, the solution of $\alpha_\delta(\tau)$ is given by

$$\alpha_\delta(\tau) = w + (ae^{-D\tau} + b)^{-1}, \quad (B.1)$$

where

$$w = -\frac{(\sigma_\delta^*)^2(\rho_{\delta S}^2 - \kappa_{\delta}^2 - D)}{(\sigma_\delta^*)^2}, \quad a = \frac{(\sigma_\delta^*)^2(\sigma_\delta^*)^2(\rho_{\delta S}^2 - \kappa_{\delta}^2 + D)}{2D(\sigma_\delta^*)^2(\rho_{\delta S}^2 - \kappa_{\delta}^2 - D)}, \quad b = -\frac{(\sigma_\delta^*)^2(\rho_{\delta S}^2)}{2D}.$$ 

**B.2. Solving $\alpha_0(\tau)$**

By replacing $\alpha_U(\tau) = 0$ and the solution of $\alpha_\delta(\tau)$ in Equation (B.1) into Equation (19), we have

$$\alpha_0'(\tau) = r + (\kappa_{\delta}^*\theta_{0}^* + \sigma_\delta^*\beta_2^*\rho_{S\delta}^*)w + 0.5(\sigma_\delta^*)^2\beta_2^*w^2 \tau$$

$$+ 0.5(\sigma_\delta^*)^2\beta_1^* \int (ae^{-D\tau} + b)^{-1} d\tau + (\sigma_\delta^*)^2\beta_2^*w \int (ae^{-D\tau} + b)^{-1} d\tau$$

$$+ 0.5(\sigma_\delta^*)^2\beta_1^* \int (ae^{-D\tau} + b)^{-2} d\tau + C_\alpha, \quad (B.2)$$

where $C_\alpha$ is a constant. Let $u = ae^{-D\tau} + b$ such that $du = -Da e^{-D\tau} d\tau$ and thus $d\tau = -1/(D(u - b)) du$. The two types of integrals in Equation (B.2) can be solved separately as

$$\int (ae^{-D\tau} + b)^{-1} d\tau = -\frac{1}{D} \int \frac{1}{u(u - b)} du = -\frac{1}{Db} \int \left(\frac{1}{u - b} - \frac{1}{u}\right) du$$

$$= -\frac{1}{Db} \ln|u - b| - \ln|u| = \frac{1}{b} \tau - \frac{1}{Db} \ln\left|\frac{a}{ae^{-D\tau} + b}\right|,$$

and

29
\[ \int (ae^{-\Delta t} + b)^{-2} d\tau = -\frac{1}{Db} \int \frac{1}{u^2(u-b)} du = \frac{1}{Db} \int \left( \frac{1}{u^2} + \frac{1/b}{u} - \frac{1/b}{u-b} \right) du \]

\[ = \frac{1}{Db} \left( -\frac{1}{u} + \frac{1}{b} \ln |u| - \frac{1}{b} \ln |u - b| \right) \]

\[ = -\frac{1}{Db} \cdot \frac{1}{ae^{-\Delta t} + b} + \frac{1}{b^2} \tau - \frac{1}{Db^2} \ln \left| \frac{a}{ae^{-\Delta t} + b} \right|. \]

Thus, Equation (B.2) can be expressed as

\[ \alpha_0(\tau) = \left[ r + (\kappa_0^* \theta_0^* + \sigma_0^* \beta_2^* \rho_{S_0}^*) \left( w + \frac{1}{b} \right) + \frac{1}{2} (\sigma_0^*)^2 \beta_2^* \left( w + \frac{1}{b} \right)^2 \right] \tau \]

\[ - \frac{\kappa_0^* \theta_0^* + \sigma_0^* \beta_2^* \rho_{S_0}^* + (\sigma_0^*)^2 \beta_2^* \left( w + \frac{1}{b} \right)^{-1}}{2Db} \cdot \ln \left| \frac{a}{ae^{-\Delta t} + b} \right| \]

\[ - \frac{(\sigma_0^*)^2 \beta_2^*}{2Db} \cdot \frac{1}{ae^{-\Delta t} + b} + C_\alpha. \]

Solve \( C_\alpha \) by matching the boundary condition \( \alpha_0(0) = 0 \):

\[ C_\alpha = \frac{(\sigma_0^*)^2 \beta_2^*}{2Db} \cdot \frac{1}{a+b} + \frac{\kappa_0^* \theta_0^* + \sigma_0^* \beta_2^* \rho_{S_0}^* + (\sigma_0^*)^2 \beta_2^* \left( w + \frac{1}{b} \right)^{-1}}{Db} \ln \left| \frac{a}{a+b} \right|. \]

Finally, the solution of \( \alpha_0(\tau) \) can be obtained as follows.

\[ \alpha_0(\tau) = \left[ r + (\kappa_0^* \theta_0^* + \sigma_0^* \beta_2^* \rho_{S_0}^*) \left( w + \frac{1}{b} \right) + \frac{1}{2} (\sigma_0^*)^2 \beta_2^* \left( w + \frac{1}{b} \right)^2 \right] \tau \]

\[ + \frac{(\sigma_0^*)^2 \beta_2^*}{2Db} \cdot \frac{1}{a+b} - \frac{1}{ae^{-\Delta t} + b} \]

\[ - \frac{\kappa_0^* \theta_0^* + \sigma_0^* \beta_2^* \rho_{S_0}^* + (\sigma_0^*)^2 \beta_2^* \left( w + \frac{1}{b} \right)^{-1}}{Db} \ln \left| \frac{a+b}{ae^{-\Delta t} + b} \right|. \]

B.3 Performance of the Proposed EUA Futures Pricing Formula

To demonstrate the correctness of our analytic formula for EUA futures under the proposed SVSCJ model, we compare the results of our analytic futures prices with those based on the Monte Carlo simulation given several posited sets of parameters. The parameter values of the SVJ are collected from Broadie and Kaya (2006), and the parameter values of the SC are collected from Nielsen and Schwartz (2004). More specifically, the first set \( BK1 + NS \) includes the SVJ parameter values from Table 6 of Broadie and Kaya (2006) and the SC parameter values from Table 1 of Nielsen and Schwartz (2004), and the second set \( BK2 + NS \) includes the SVJ parameter value from Table 7 of Broadie and Kaya (2006) and the SC...
parameters values from Table 1 of Nielsen and Schwartz (2004).

BK1 + NS:

\[ S_0 = 100, \quad r = 0.0319, \quad \Delta t = 1/250, \quad V_0 = 0.008836, \quad \kappa_V^* = 3.99, \quad \theta_V^* = 0.014, \quad \sigma_V^* = 0.27, \quad \rho_{SV}^* = -0.79, \quad \delta_0 = \theta_\delta^*, \quad \kappa_\delta^* = 2.77e-03, \quad \theta_\delta^* = 5.14e-07/\kappa_\delta^*, \quad \sigma_\delta^* = 1.9e-03, \quad \beta_1^* = 0.264, \quad \beta_2^* = 1.35e-04, \quad \rho_{S\delta}^* = 0.672, \quad \lambda^* = 0.11, \quad \mu_j^* = -0.12, \quad \sigma_j^* = 0.15. \]

BK2 + NS:

\[ S_0 = 100, \quad r = 0.0319, \quad \Delta t = 1/250, \quad V_0 = 0.007569, \quad \kappa_V^* = 3.46, \quad \theta_V^* = 0.008, \quad \sigma_V^* = 0.14, \quad \rho_{SV}^* = -0.82, \quad \delta_0 = \theta_\delta^*, \quad \kappa_\delta^* = 2.77e-03, \quad \theta_\delta^* = 5.14e-07/\kappa_\delta^*, \quad \sigma_\delta^* = 1.9e-03, \quad \beta_1^* = 0.264, \quad \beta_2^* = 1.35e-04, \quad \rho_{S\delta}^* = 0.672, \quad \lambda^* = 0.47, \quad \mu_j^* = -0.1, \quad \sigma_j^* = 0.0001. \]

Moreover, for either of the above two sets of parameter values, we test two times to maturity, \( \tau = \tau_1 = 1.2 \) and \( \tau = \tau_2 = 3 \). Therefore, there are totally four sets of parameter values to be examined: BK1 + NS + \( \tau_1 \), BK2 + NS + \( \tau_1 \), BK1 + NS + \( \tau_2 \), BK2 + NS + \( \tau_2 \).

Note that the sets of BK1 and BK2 differ only in the SVJ parameters, which should be theoretically independent of the futures prices. In order to test this argument and also generate sufficiently accurate results based on the Monte Carlo simulation, we produce the pricing results and the confidence interval based 20 repetitions of 1,000,000 paths (with \( \Delta t = 1/250 \)).

The confidence interval is defined as [average simulated futures price \( \pm \) 2 \times standard error], where the standard error is approximated by the standard deviation of the results of 20 repetitions. Table B.1 shows the pricing results of our experiments. The futures prices generated by our analytic formula are very close to the means of the prices based on the Monte Carlo simulation. For \( \tau = \tau_1 = 1.2 \), the accuracy is to the second decimal place, and for \( \tau = \tau_2 = 3 \), the accuracy is to the first decimal place. Although the stochastic variance and jump parameters are different in the sets of BK1 and BK2, the futures prices generated based on the Monte Carlo simulation are almost the same, which attest the independence of these parameters from futures prices.
This table compares the EUA futures prices generated based on our closed-form formula and the Monte Carlo simulation.

**Appendix C. Performance of the Pricing Formula for EUA Futures Options**

To verify that our option pricing formula in Equation (23) can evaluate option values accurately, we compare the pricing results based on Equation (23) with the pricing results from the Monte Carlo simulation. For each set of the parameter values in Appendix B (except that the initial spot price \( S_0 = 100 \) is replaced by the initial futures price \( F_0 = 100 \)), we further consider two time to maturities for EUA futures call options (\( \bar{t}_1 = 0.3 \) and \( \bar{t}_2 = 0.8 \)). Consequently, there are 8 sets of the parameter values being examined here: BK1 + NS + \( \tau_1 + \bar{t}_1 \), BK2 + NS + \( \tau_1 + \bar{t}_1 \), BK1 + NS + \( \tau_2 + \bar{t}_1 \), BK2 + NS + \( \tau_2 + \bar{t}_1 \), BK1 + NS + \( \tau_1 + \bar{t}_2 \), BK2 + NS + \( \tau_1 + \bar{t}_2 \), BK1 + NS + \( \tau_2 + \bar{t}_2 \), BK2 + NS + \( \tau_2 + \bar{t}_2 \). Table C.1 reports the pricing results, and for the Monte Carlo simulation, the confidence interval of the pricing result is shown in the under parentheses. It can be found that our option pricing formula can generate accurate option values.

**Table C.1 Simulation Tests for Pricing EUA Futures Options**

<table>
<thead>
<tr>
<th>Parameter set</th>
<th>( K = 85 )</th>
<th>( K = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closed-form solution</td>
<td>Monte Carlo Simulation</td>
<td>Closed-form solution</td>
</tr>
<tr>
<td>BK1 + NS + ( \tau_1 + \bar{t}_1 )</td>
<td>15.0407</td>
<td>14.9250</td>
</tr>
<tr>
<td>BK2 + NS + ( \tau_1 + \bar{t}_1 )</td>
<td>14.9579, 15.1000</td>
<td>14.9580, 15.1000</td>
</tr>
<tr>
<td>BK1 + NS + ( \tau_2 + \bar{t}_1 )</td>
<td>2.4074</td>
<td>2.4116</td>
</tr>
<tr>
<td>BK2 + NS + ( \tau_2 + \bar{t}_1 )</td>
<td>2.4074</td>
<td>2.4116</td>
</tr>
</tbody>
</table>
The closed-form solution columns report the pricing results based on our formula of Equation (23). The Monte Carlo simulation columns report the results of the average simulated futures option price and the corresponding confidence interval. To generate each option price, we simulate 50,000 paths of the spot prices (with $\Delta t = 1/250$) and repeat this process for 20 times. The average simulated futures option price is the average of each 20 repetitions. The confidence interval is defined as $[\text{average simulated futures option price} \pm 2 \times \text{standard error}]$, where the standard error is approximated by the standard deviation of the results of 20 repetitions.
Table 1. Estimated Results for EUA Spot Prices.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>GBM</th>
<th>GBMJ</th>
<th>SV</th>
<th>SVJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>0.3959</td>
<td>0.1892</td>
<td>(4.7041e-05)</td>
<td>(0.0428)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td></td>
<td></td>
<td>6.1819</td>
<td>7.3230</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.4360)</td>
<td>(0.4023)</td>
</tr>
<tr>
<td>$\theta$</td>
<td></td>
<td></td>
<td>0.1464</td>
<td>0.1327</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.0057)</td>
<td>(0.0052)</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td></td>
<td></td>
<td>1.0910</td>
<td>1.3083</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.0542)</td>
<td>(0.0709)</td>
</tr>
<tr>
<td>$\rho_{SV}$</td>
<td></td>
<td></td>
<td>-0.3974</td>
<td>-0.3436</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.0350)</td>
<td>(0.0258)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>190.5244</td>
<td></td>
<td>39.2833</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3.2401)</td>
<td></td>
<td>(5.5657)</td>
<td></td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>-0.0014</td>
<td></td>
<td>0.0042</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0003)</td>
<td></td>
<td>(0.0010)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_f$</td>
<td>0.0302</td>
<td></td>
<td>0.0104</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0018)</td>
<td></td>
<td>(0.0007)</td>
<td></td>
</tr>
</tbody>
</table>

Log-Likelihood Value

2640.1052  2728.5882  2835.4870  2836.9401
Table 1. (continued)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>SC</th>
<th>SCJ</th>
<th>SVSC</th>
<th>SVSCJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_\delta$</td>
<td>93.4980</td>
<td>93.4243</td>
<td>67.9910</td>
<td>74.4025</td>
</tr>
<tr>
<td></td>
<td>(0.1192)</td>
<td>(0.2576)</td>
<td>(2.9588)</td>
<td>(1.2025)</td>
</tr>
<tr>
<td>$\theta_\delta$</td>
<td>-0.0051</td>
<td>-0.0051</td>
<td>-0.0071</td>
<td>-0.0072</td>
</tr>
<tr>
<td></td>
<td>(4.7149e-05)</td>
<td>(0.0001)</td>
<td>(0.0002)</td>
<td>(0.0002)</td>
</tr>
<tr>
<td>$\sigma_\delta$</td>
<td>0.7262</td>
<td>1.0409</td>
<td>3.2303</td>
<td>3.2014</td>
</tr>
<tr>
<td></td>
<td>(0.0029)</td>
<td>(0.0048)</td>
<td>(0.0178)</td>
<td>(0.0094)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.2177</td>
<td>0.0714</td>
<td>0.7561</td>
<td>0.7698</td>
</tr>
<tr>
<td></td>
<td>(0.0031)</td>
<td>(0.0010)</td>
<td>(0.0029)</td>
<td>(0.0010)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-0.1546</td>
<td>-0.0753</td>
<td>0.0069</td>
<td>0.0070</td>
</tr>
<tr>
<td></td>
<td>(0.0006)</td>
<td>(0.0003)</td>
<td>(2.6037e-05)</td>
<td>(9.1689e-06)</td>
</tr>
<tr>
<td>$\rho_{s\delta}$</td>
<td>0.0513</td>
<td>0.0633</td>
<td>0.0603</td>
<td>0.0522</td>
</tr>
<tr>
<td></td>
<td>(0.0026)</td>
<td>(0.0027)</td>
<td>(0.0107)</td>
<td>(0.0041)</td>
</tr>
<tr>
<td>$\kappa_V$</td>
<td>5.4060</td>
<td>5.6547</td>
<td>(5.3638)</td>
<td>(5.2055)</td>
</tr>
<tr>
<td></td>
<td>(0.3638)</td>
<td>(0.2055)</td>
<td>(0.0091)</td>
<td>(0.0084)</td>
</tr>
<tr>
<td>$\theta_V$</td>
<td>0.1355</td>
<td>0.1327</td>
<td>0.1355</td>
<td>0.1327</td>
</tr>
<tr>
<td></td>
<td>(0.0951)</td>
<td>(0.0446)</td>
<td>(0.0951)</td>
<td>(0.0446)</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>1.2235</td>
<td>1.3544</td>
<td>1.2235</td>
<td>1.3544</td>
</tr>
<tr>
<td></td>
<td>(0.0951)</td>
<td>(0.0446)</td>
<td>(0.0951)</td>
<td>(0.0446)</td>
</tr>
<tr>
<td>$\rho_{SV}$</td>
<td>-0.2876</td>
<td>-0.2496</td>
<td>-0.2876</td>
<td>-0.2496</td>
</tr>
<tr>
<td></td>
<td>(0.0404)</td>
<td>(0.0311)</td>
<td>(0.0404)</td>
<td>(0.0311)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>40.5922</td>
<td>31.3867</td>
<td>40.5922</td>
<td>31.3867</td>
</tr>
<tr>
<td></td>
<td>(0.0840)</td>
<td>(1.8029)</td>
<td>(0.0840)</td>
<td>(1.8029)</td>
</tr>
<tr>
<td>$\mu_j$</td>
<td>-0.0054</td>
<td>0.0115</td>
<td>-0.0054</td>
<td>0.0115</td>
</tr>
<tr>
<td></td>
<td>(0.0006)</td>
<td>(0.0011)</td>
<td>(0.0006)</td>
<td>(0.0011)</td>
</tr>
<tr>
<td>$\sigma_j$</td>
<td>0.0397</td>
<td>0.0103</td>
<td>0.0397</td>
<td>0.0103</td>
</tr>
<tr>
<td></td>
<td>(0.0003)</td>
<td>(0.0007)</td>
<td>(0.0003)</td>
<td>(0.0007)</td>
</tr>
</tbody>
</table>

Log-Likelihood Value

This table presents the estimation results of the eight models nested in the proposed SVSCJ model. For those models without the SV, the estimates are obtained directly by maximizing the log-likelihood function, and the underneath parentheses show the standard errors retrieved from the inverse Hessian evaluated at the obtained estimates. For SV, SVJ, SVSC, and SVSCJ models, we use $M = 500$ and repeat the original or modified particle filter method for 20 times. The average and standard deviation of these 20 repetitions are employed as the final estimation result and the corresponding standard error (shown in the underneath parentheses). For all of these eight models, the instantaneous expected growth rate of the spot rate, $\mu$, is estimated by annualizing the arithmetic average of the actual daily returns.
Table 2. Calibrated Results for EUA Futures

<table>
<thead>
<tr>
<th>Year</th>
<th>N</th>
<th>GBM</th>
<th>SVSCJ = SC ($\beta_1^*$)</th>
<th>Avg. $F^A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008</td>
<td>888</td>
<td>0.7706</td>
<td>0.1910 (0.6954)</td>
<td>24.5308</td>
</tr>
<tr>
<td>2009</td>
<td>1001</td>
<td>0.3667</td>
<td>0.2500 (0.7976)</td>
<td>14.2320</td>
</tr>
<tr>
<td>2010</td>
<td>750</td>
<td>0.1150</td>
<td>0.0792 (0.2088)</td>
<td>14.9081</td>
</tr>
<tr>
<td>2011</td>
<td>499</td>
<td>0.2387</td>
<td>0.0991 (0.0106)</td>
<td>13.6097</td>
</tr>
<tr>
<td>2012</td>
<td>235</td>
<td>0.0607</td>
<td>2.3064e-05 (0.6717)</td>
<td>7.5493</td>
</tr>
</tbody>
</table>

This table reports the calibration performance of the GBM and SVSCJ models for EUA futures prices in the phase of 2008-2012. The performance is measured in terms of the RMSE defined as

$$\sqrt{\frac{\sum_{i=1}^{N}(F_i^{T} - F_i^{A})^2}{N}},$$

where $F_i^{T}$ is the theoretical futures price, $F_i^{A}$ is the corresponding actual futures price, and $N$ is the number of observations in each year. Since the stochastic variance and jump parameters are independent of the futures prices, we only report the results based on the GBM and SVSCJ (equivalent to SC for pricing EUA futures) models. The calibrated results of $\beta_1^*$ in the SVSCJ process are also presented in parentheses. The last column shows the average values of actual futures prices in each year.
Table 3. Calibrated Results for EUA Futures Options

<table>
<thead>
<tr>
<th>Model</th>
<th>GBM</th>
<th>GBMJ</th>
<th>SV</th>
<th>SVJ</th>
<th>SC</th>
<th>SCJ</th>
<th>SVSC</th>
<th>SVSCJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>0.0261</td>
<td>0.0188</td>
<td>0.0177</td>
<td>0.0165</td>
<td>0.0260</td>
<td>0.0184</td>
<td>0.0175</td>
<td>0.0162</td>
</tr>
<tr>
<td>$\beta_1^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0500</td>
<td>0.9722</td>
<td>1.0935</td>
<td>1.0840</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average prices of all futures options = 0.3132

Table 3 reports the calibration performance of each model for fitting EUA futures option prices in the phase of 2008-2012. The performance is measured in terms of the RMSE defined as $\sqrt{\sum_{t=1}^{N}(O_t^T - O_t^A)^2 / N}$, where $O_t^T$ is the theoretical futures option price, $O_t^A$ is the corresponding actual futures option price, and $N$ is the number of observations. In addition, the calibrated results of $\beta_1^*$ in the SC, SCJ, SVSC, and SVSCJ models are also represented. The last row shows the average values of actual futures option prices.
Fig. 1 Scatter diagram of the daily convenience yield $\delta_t$ versus remaining time-to-maturity $\tau$ from 2012/03/01 to 2012/12/17. The black curve is the best fitted exponential function obtained from the ordinary least square method: $\delta_t = 0.0382e^{-0.5300\tau}$. 
Fig. 2 Scatter diagram of the monthly EUA spot volatility versus the remaining time-to-maturity $\tau$ in the period of 2012/03-2013/01. The black curve is the best fitted exponential function obtained from the ordinary least square method: $\sigma_\tau = 0.6016e^{-0.3568\tau}$. 